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STOCHASTIC FITZHUGH-NAGUMO SYSTEMS WITH DELAY

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Abstract. This paper is concerned with the existence of random attractors for the stochastic FitzHugh-Nagumo systems with delay on infinite lattice. Under suitable dissipative conditions, It shows that such a system has a random attractor which is a random compact invariant set.

1. INTRODUCTION

In this paper, we consider the following stochastic FitzHugh-Nagumo lattice dynamical system with delay

$$\begin{aligned} &\dot{u}_i + (Au(t))_i + \psi_i - f_i(u_{it}) - a_i\dot{w}_{1i}(t) = 0, \\ &\dot{\psi}_i + \lambda_1\psi_i - \lambda_2 u_i - b_i\dot{w}_{2i}(t) = 0, \\ &u_{i\tau} = u_i(s+\tau),\ s \in [-\tau, 0], \end{aligned} \tag{1}$$

where $t \geq \tau$, $i \in \mathbf{Z}$, $u_{it} = u_{it}(s) = u_i(t+s)$ is the delay term with the interval of delay time $[-\tau, 0]$, and $u_{i\tau} = u_i(\tau + s)$ is the initial datum in the interval $[0, \tau]$, $\mathbf{Z}$ is the integer set, A is a linear operator defined by $(Au)_i = u_{i+1} + u_{i-1} - 2u_i$, $u = (u_i)_{i\in\mathbf{Z}} \in \ell^2$, λ_1, λ_2 and τ are positive constants, $a = (a_i)_{i\in\mathbf{Z}}, b = (b_i)_{i\in\mathbf{Z}} \in \ell^2$, f_i is a smooth function satisfying some dissipative conditions (see the hypotheses (H1) and (H2) in Section 3), $\{w_{ki}(t) : i \in \mathbf{Z}\}$ are independent two-side real valued standard Wiener processes, $k = 1, 2$.

Lattice dynamical systems (LDSs) play an important role in their potential applications such as biology [25, 9], chemical reaction [16], pattern recognition and image processing [7, 8], electrical engineering [6], laser systems [11], and material science [14], etc. One of most famous model is discrete FitzHugh-Nagumo systems describing the signal transmission across axons. In fact, this infinite lattice model describe infinite

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many cycle oscillators, $u_i(t)$ denotes the action potentials, also called spiking or firing events, in a neuron. $\psi_i(t)$ denotes a refractory period after each firing event during which the neutron cannot fire again. More details, see [17, 22]. Recently, Huang [15] study the existence of random attractor for the stochastic FitzHugh-Nagumo equation on infinite lattice.

In fact, the stochastic lattice dynamical systems with delay have gained great attention by considering that, a system in reality is usually affected by external perturbations which in many cases are of great uncertainty or random influence. These random effects are introduced not only to compensate for the defects in some deterministic models, but also to reveal the intrinsic phenomena. Some hereditary characteristics such as after-effect, time-lag and time delay can appear in the variables, this leads the delay effect in an system. The study of existence and stability of solutions for stochastic partial differential equations with delays also has attracted by many authors (see [27, 24, 20, 4, 5]). In [3], small lattices of N nearest-neighbor coupled excitable FitzHugh-Nagumo systems, with time-delayed coupling are studied and compared with systems of FitzHugh-Nagumo oscillators with the same delayed coupling. The delay terms f satisfies

$$f(0) = 0,\ f'(0) = \delta > 0.$$

Using augmented moment method, Hasegawa[13] study the stochastic ensembles with delayed couplings for the FitzHugh-Nagumo model, where the exponent delay term is considered.

The existence of global random attractor for a kind of first order dynamical systems driven by white noise on lattice $\mathbf{Z}$ first investigated by Bates et al. [2]. Then, Lv [18] extended their results to the higher dimensional lattices. Later, stochastic complex Ginzburg-Landau equation on infinite lattice were also studied by Lv [19]. Most Recently, Yan, W, etc al [26] obtained the existence of random attractors for first order stochastic retarded lattice dynamical system. In present paper, we study the existence of random attractor for the stochastic discrete FitzHugh-Nagumo systems with delay.

Throughout this paper, the inner product and norm of Hilbert space ℓ^2 are defined as

$$(u, v) = \sum_{i\in\mathbf{Z}} u_i v_i, \quad \|u\|^2 = (u, u) = \sum_{i\in\mathbf{Z}} u_i^2$$

for each $u = (u_i)_{i\in\mathbf{Z}}, v = (v_i)_{i\in\mathbf{Z}} \in \ell^2$. In addition, for the reason of delays, we take $\mathbf{X}_\tau = C([-\tau, 0]; \ell^2)$ as the phase space endowed with the norm

$$\|u\|_{\mathbf{X}_\tau} = \max_{s\in[-\tau,0]} \left(\sum_{i\in\mathbf{Z}} |u_i(s)|^2\right)^{\frac{1}{2}}.$$

This paper is organized as follows. In Section 2, we recall some basic concepts and already known results related to random dynamical systems and random attractors.

In Section 3, we first give an abstract theorem for the existence of random attractor associated with the stochastic FitzHugh-Nagumo lattice systems with delay by reformulating the result of Bates et al. [2]. Then we apply the abstract theorem to prove the existence of random attractor for system (1).

2. Definition and Preliminaries

In this section, we introduce some basic concepts related to random dynamical systems and random attractors, which are taken from [2, 10].

Let $(H, \|\cdot\|_H)$ be a Hilbert space, $(\Omega, \mathbf{F}, P)$ be a probability space.

Definition 2.1. $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$ is called a metric dynamical systems, if $\theta : \mathbf{R}\times\Omega \to \Omega$ is $(\mathcal{D}(\mathbf{R})\times\mathbf{F}, \mathbf{F})$ measurable, $\theta_0 = \mathcal{I}$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbf{R}$, and $\theta_t P = P$ for all $t \in \mathbf{R}$.

Definition 2.2. A stochastic process $\phi(t, \omega, \cdot) : \mathbf{R}^+ \times \Omega \times H \to H$ is called a random dynamical system (RDS) over $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$ if ϕ is $(\mathbf{B}(\mathbf{R}^+) \times \mathbf{F} \times \mathbf{B}(H), \mathbf{B}(H))$-measurable, and for P-a.e. $\omega \in \Omega$,

- the mapping ϕ is continuous;
- $\phi(0, \omega, \cdot) = \mathcal{I}$ on H;
- $\phi(t+s, \omega, \cdot) = \phi(t, \theta_s\omega, \cdot) \circ \phi(s, \omega, \cdot)$ for all $t, s \geq 0$.

Definition 2.3. A random bounded set $\{B(\omega)\}_{\omega\in\Omega} \subset H$ is called tempered with respect to $(\theta_t)_{t\in\mathbf{R}}$ if for P-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\epsilon t} d(B(\theta_{-t}\omega)) = 0, \; for\ all\ \epsilon > 0,$$

where $d(B) = \sup_{x\in B} \|x\|_H$.

In what follows, we shall consider the continuous random dynamical system $\phi(t, \omega, \cdot)$ over $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$.

Definition 2.4. Let $\mathcal{D}$ be a collection of random subsets of H and $\{\mathcal{K}(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then a random set $\{\mathcal{K}(\omega)\}_{\omega\in\Omega}$ is called an absorbing set in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset \mathcal{K}(\omega), \; for\ all\ t \geq t_B(\omega).$$

Definition 2.5. Let $\mathcal{D}$ be a collection of random subsets of H. Then a random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is a $\mathcal{D}$-random attractor for RDS ϕ if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,

- $\mathcal{A}(\omega)$ is compact, and $\omega \to d(x, \mathcal{A}(\omega))$ is measurable for every $x \in H$;
- $\mathcal{A}(\omega)$ is invariant, i.e., $\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t \omega)$, for all $t \geq 0$;
- $\mathcal{A}(\omega)$ attracts every set in $\mathcal{D}$, i.e., for all $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to\infty} d(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d denotes the Hausdorff semi-metric given by $d(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_H$ for any $X, Y \subset H$.

Definition 2.6. Let $\mathcal{D}$ be a collection of random subsets of H. Then ϕ is called $\mathcal{D}$-Pullback asymptotically compact in H if for P-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in H whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

The following existence result for a random attractor for a continuous RDS can be found in [2, 12].

Proposition 2.1. *Let $\mathcal{D}$ be an inclusion-closed collection of random subsets of H and ϕ be a RDS on H over $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$. Suppose that $\{\mathcal{K}(\omega)\}_{\omega\in\Omega}$ is a closed random absorbing set for a $\mathcal{D}$-pullback asymptotically compact continuous RDS ϕ. Then ϕ has a unique $\mathcal{D}$-random attractor*

$$\mathcal{A}(\omega) = \bigcap_{\kappa\geq 0} \overline{\bigcup_{t\geq\kappa} \phi(t, \theta_{-t}\omega, \mathcal{K}(\theta_{-t}\omega))},$$

which is compact in H.

3. The Existence of Random Attractors

(1) is a system for unknown u and ψ, and firstly we search for a solution $u, \psi \in \ell^2$. For convenience, we rewrite system (1) as

$$\begin{aligned} &\dot{u} + Au + \psi - f(u_t) = a\dot{w}_1(t), \\ &\dot{\psi} + \lambda_1\psi - \lambda_2 u = b\dot{w}_2(t), \\ &u_\tau = u(s+\tau),\ s \in [-\tau, 0],\ \tau > 0, \end{aligned} \tag{3.1}$$

where $u_t = u_t(s) = u(t+s)$ is the delay term with the interval of delay time $[-\tau, 0]$, $u_\tau = u(\tau + s)$ is the initial datum in the interval $[0, \tau]$, and f is a smooth function satisfying the dissipative conditions:

(H1) For any bounded set $Y \subset \ell^2$ there exists a positive constant L_f such that for any $u, v \in Y$

$$\|f(u) - f(v)\| \leq L_f \|u - v\|.$$

(H2) There exist positive constants $\alpha_1,\ \alpha_2$ such that

$$f_i(0)=0,\ f_i(u_{it})u_i \le -\alpha_1|u_i|^2+\alpha_2 \max_{s\in[-\tau,0]}|u_{it}|^2,\ \forall t\in \mathbf{R}^+.$$

Remark 1. Here, assumption (H2) about nonlinear term f in system (3.1) is a dissipative condition which making the solution of system (3.1) has the decay property about time t. The choice of parameters α_1, α_2, λ_1 and λ_2 depends on how can make the Gronwall's inequality hold.

Denote the linear operators B and B^* from ℓ^2 to ℓ^2 by $(Bu)_i = u_{i+1}-u_i$ and $(B^*u)_i = u_i - u_{i-1}$ respectively, then it is easy to see that $A = BB^* = B^*B$ and $(B^*u, v) = (u, Bv)$. Therefore, $(Au, u) \ge 0$ for all $u \in \ell^2$.

Let $W_1(t) = \sum_{i\in\mathbf{Z}} a_i w_{1i}(t)e^i,\ W_2(t) = \sum_{i\in\mathbf{Z}} b_i w_{2i}(t)e^i,\ (a_i)_{i\in\mathbf{Z}} \in \ell^2$ and $(b_i)_{i\in\mathbf{Z}} \in \ell^2$, where $\{e^i\}_{i\in\mathbf{Z}}$ denotes a complete orthonormal basis in ℓ^2. Then $W_1(\bullet)$ and $W_2(\bullet)$ are, obviously, Q-Wiener processes with $Q = diag\{\cdots, a_i^2, \cdots\}$ and $Q = diag\{\cdots, b_i^2, \cdots\}$, respectively. It is obvious that $EW_j(t) = 0$, $Cov(W_j(t)) = tQ$, for $j = 1, 2$. For details we refer to [23].

Consider the probability space

$$\Omega = \{\omega \in \mathcal{C}(\mathbf{R}, \ell^2) : \omega(0) = 0\}$$

endowed with the compact open topology [1]. We denote P to be the corresponding Wiener measure and $\mathbf{F}$ to be the P-completion of the Borel σ-algebra on Ω.

Let $\theta_t W_j(\cdot) = W_j(\cdot + t) - W_j(t),\ t \in \mathbf{R}$. Then $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$ is a metric dynamical system with the filtration $\mathbf{F}_t := \bigvee_{s\le t} \mathbf{F}_s^t,\ t \in \mathbf{R}$, where $\mathbf{F}_s^t = \sigma\{W_j(t_2) - W_j(t_1) : s \le t_1 \le t_2 \le t\}$ is the smallest σ-algebra generated by the random variable $W_j(t_2) - W_j(t_1)$ for all t_1, t_2 such that $s \le t_1 \le t_2 \le t$, see [1] for more details.

We introduce an Ornstein-Uhlenbeck process in ℓ^2 on the metric dynamical system $(\Omega, \mathbf{F}, P, \theta_t)$ given by the Wiener process:

$$y_i(t) = \int_{-\infty}^{t} e^{\lambda_{i-1}(s-t)} dW_s,\ for\ i = 1, 2,$$

where $\lambda_0, \lambda_1 > 0$. The above integral exists in the sense that for any path ω with a subexponential growth, $y_1(t)$ and $y_2(t)$ solve the following Itô equations

$$\begin{aligned} &dy_1(t) + \lambda_0 y_1(t)dt = dW_1(t),\ for\ t \ge 0,\\ &dy_2(t) + \lambda_1 y_2(t)dt = dW_2(t),\ for\ t \ge 0. \end{aligned} \tag{3.2}$$

In fact, the mapping $t \to y_k(\theta_t\omega),\ k = 1, 2,$ is an Ornstein-Uhlenbeck process. Furthermore, there exists a θ_t invariant set $\Omega' \subset \Omega$ such that:

(1) the mapping $t \to y_k(\theta_t\omega)$ is continuous for P-a.e. $\omega \in \Omega'$;

(2) the random variables $|y_k(\omega)|$ is tempered, i.e. $\lim_{t\to\pm\infty}\frac{1}{|t|}\log^+|y_k(\theta_t\omega)| = 0$, P-a.e.

Moreover, all the parameters satisfying

$$2\alpha_1 - 2\alpha_2 e^{\tau} \geq L_f + \lambda_0 + \lambda_2 + 13,\ 2\lambda_1 - 2\lambda_2 \geq 1,\ \alpha_1 - \alpha_2 \geq \lambda_2 - \frac{\lambda_1}{2} + \frac{13}{2},$$

where $\lambda_0 > 0$ are defined in (3.2).

Denote

$$\begin{aligned} v(t) &= u(t) - y_1(\theta_t\omega), \\ \varphi(t) &= \psi(t) - y_2(\theta_t\omega), \end{aligned} \tag{3.3}$$

where $(u(t), \psi(t))$ is a solution of system (3.1). Then $(v(t), \varphi(t))$ satisfies:

$$\begin{aligned} \dot{v} &= -Av - \varphi + f(v_t + y_1(\theta_{t+s}\omega)) + (\lambda_0 - A)y_1(\theta_t\omega) - y_2(\theta_t\omega), \\ \dot{\varphi} &= -\lambda_1\varphi + \lambda_2 v + \lambda_2 y_1(\theta_t\omega), \\ v_\tau &= u_\tau - y(\theta_{\tau+s}\omega). \end{aligned} \tag{3.4}$$

Remark 2. By (3.1)-(3.2) and the definition of W_1 and W_2, we get that

$$\begin{aligned} &\dot{u} + Au + \psi - f(u_t) = \frac{dy_1}{dt} + \lambda_0 y_1, \\ &\dot{\psi} + \lambda_1\psi - \lambda_2 u = \frac{dy_2}{dt} + \lambda_1 y_2, \\ &u_\tau = u(s+\tau),\ s \in [-\tau, 0],\ \tau > 0. \end{aligned}$$

Then, by (3.3), (3.4) can be derived.

In what follows, we first give an abstract theorem about the existence of random attractors associated with general first order stochastic retarded lattice system (3.1), which can be seen as the reformulation of Proposition 2.1. Then we apply it to prove the existence of random attractors for system (3.1).

Denote $\mathcal{D}$ be a collection of random subsets of $\mathcal{X}_\tau = C([-\tau, 0]; \ell^2)$ with norm $\|u\|_{\mathcal{X}_\tau} = \max_{s\in[-\tau,0]}(\sum_{i\in\mathbf{Z}}|u_i(s)|^2)^{\frac{1}{2}}$. Assume that system (3.1) has a unique solution and generates a continuous random dynamical system (RDS) ϕ_t (Theorem 3.2-3.3 will explain the assumptions).

Theorem 3.1. *The continuous random dynamical system* (*RDS*) ϕ_t (*see Theorem 3.3 for more details about* ϕ_t) *generated by system* (*3.1*) *has a unique* $\mathcal{D}$*-random attractor* $\mathcal{A}$ *in* $\mathcal{X}_\tau$ *if it satisfies, for* P*-a.e.* $\omega \in \Omega$,

($C1$) *RDS ϕ_t possesses a bounded absorbing set $\{\mathcal{K}(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$;*

($C2$) *For any $\{t_n\}_{n\in\mathbf{N}}$ with $t_n\to\infty$ as $n\to\infty$,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\max_{s\in[-\tau,0]}\sum_{|i|\geq m}(\phi_{it_n}(s,\theta_{-t_n}\omega,x_n))^2=0,$$

where $x_n\in\mathcal{K}(\theta_{-t_n}\omega)$.

Proof. According to Proposition 2.1, we only need to prove RDS ϕ_t possesses a bounded absorbing set and its $\mathcal{D}$-pullback asymptotically compact. Assumption (C1) shows that ϕ_t possesses a bounded absorbing set, the rest is to use assumption (C2) to prove that RDS ϕ_t is $\mathcal{D}$-pullback asymptotically compact.

Let

$$S=\{\phi_{t_n}(s,\theta_{-t_n}\omega,x_n):x_n\in\mathcal{K}(\theta_{-t_n}\omega),\ t_n\to\infty\ as\ n\to\infty\},$$

where $\{\mathcal{K}(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ is the bounded absorbing set of RDS ϕ_t. According to Proposition 2.1, it is only need to prove S is precompact in $\mathcal{X}_\tau\times\ell^2$.

By (C2), for any $\varepsilon>0$, there exist $m_1(\varepsilon)$ and $N(\varepsilon)$ such that

$$\max_{s\in[-\tau,0]}(\sum_{|i|\geq m_1(\varepsilon)}\phi^2_{it_n}(s,\theta_{-t_n}\omega,x_n))^{\frac{1}{2}}<\varepsilon,\ for\ n\geq N(\varepsilon). \tag{3.5}$$

Denote $\Lambda=\{\phi_{t_n}\}_{n=1}^{N(\varepsilon)}$. Noting that Λ is a finite set and for all $\phi_t\in\mathcal{X}_\tau\times\ell^2$, we have

$$\lim_{m\to\infty}\max_{s\in[-\tau,0]}(\sum_{|i|\geq m}\phi^2_{it}(s,\theta_{-t_n}\omega,x_n))^{\frac{1}{2}}=0.$$

So there exists $m_2(\varepsilon)>0$ such that

$$\max_{s\in[-\tau,0]}(\sum_{|i|\geq m_2(\varepsilon)}\phi^2_{it_n}(s,\theta_{-t_n}\omega,x_n))^{\frac{1}{2}}\leq\varepsilon,\ for\ n\leq N(\varepsilon). \tag{3.6}$$

Let $m(\varepsilon)=\max\{m_1(\varepsilon),m_2(\varepsilon)\}$. Then, by (3.5)–(3.6), we have

$$\max_{s\in[-\tau,0]}(\sum_{|i|\geq m(\varepsilon)}\phi^2_{it_n}(s,\theta_{-t_n}\omega,x_n))^{\frac{1}{2}}\leq\varepsilon,\ for\ n\geq 1. \tag{3.7}$$

Set

$$\mathcal{X}_\tau^{(m(\varepsilon))}\times\ell^2_{m(\varepsilon)}:=C([-\tau,0];\ell^2_{m(\varepsilon)}\times\ell^2_{m(\varepsilon)})\times\ell^2_{m(\varepsilon)},$$

where $\ell^2_{m(\varepsilon)}:=\{v=(v_i)_{|i|\leq m(\varepsilon)},v_i\in\mathbf{R}\}$, and rewrite (3.4) as

$$\begin{aligned} v(t)&=v(\tau)+\int_\tau^t[f(v_h+y_1(\theta_{h+s}\omega))-\varphi-Av\\ &\quad+(\lambda_0-A)y_1(\theta_t\omega)-y_2(\theta_t\omega)]dh,\ t\geq\tau>0,\\ \varphi(t)&=\varphi(\tau)+\int_\tau^t[-\lambda_1\varphi+\lambda_2v+\lambda_2y_1(\theta_t\omega)]dh,\ t\geq\tau>0,\\ v_\tau(s)&=u_\tau-y(\theta_{\tau+s}\omega). \end{aligned} \tag{3.8}$$

In what follows we will use Ascoli-Arzelá theorem to prove set $\Gamma = \{(\phi_{it_n})_{|i|\leq m(\varepsilon)} : \phi_{t_n} = (\phi_{it_n})_{i\in\mathbf{Z}} \in \Lambda\}$ is precompact in $\mathcal{X}_\tau^{(m(\varepsilon))} \times \ell^2_{m(\varepsilon)}$. For convenience, we denote any sequence in Γ by $\{(u_i(t_n), \psi_i(t_n))\}_{|i|\leq m(\varepsilon)}$. By (C1), it is obvious that $\{(u_i(t_n), \psi_i(t_n))\}_{|i|\leq m(\varepsilon)}$ is uniform bounded in $\mathcal{X}_\tau^{(m(\varepsilon))} \times \ell^2_{m(\varepsilon)}$. Then, by (H1) and (3.8), we have

$$\begin{aligned}
0 \leq\ & \lim_{\varsigma\to 0}\{\|u_i(t_n+\varsigma,\omega,u_\tau(\omega)) - u_i(t_n,\omega,u_\tau(\omega))\|_{\ell^2_{m(\varepsilon)}} \\
& +\|\psi_i(t_n+\varsigma,\omega,u_\tau(\omega)) - \psi_i(t_n,\omega,u_\tau(\omega))\|_{\ell^2_{m(\varepsilon)}}\} \\
\leq\ & \lim_{\varsigma\to 0}\{\|v_i(t_n+\varsigma,\omega,v_\tau(\omega)) - v_i(t_n,\omega,v_\tau(\omega))\|_{\ell^2_{m(\varepsilon)}} \\
& +\|\varphi_i(t_n+\varsigma,\omega,v_\tau(\omega)) - \varphi_i(t_n,\omega,v_\tau(\omega))\|_{\ell^2_{m(\varepsilon)}}\} \\
& + \lim_{\varsigma\to 0}\{\|y_1(\theta_{t_n+\varsigma}\omega) - y_1(\theta_{t_n}\omega)\|_{\ell^2_{m(\varepsilon)}} \\
& +\|y_2(\theta_{t_n+\varsigma}\omega) - y_2(\theta_{t_n}\omega)\|_{\ell^2_{m(\varepsilon)}}\} \\
=\ & \lim_{\varsigma\to 0}\|\int_{t_n}^{t_n+\varsigma}(f(v_h + y_1(\theta_{h+s}\omega)) - \varphi - Av(h,\omega,v_\tau(\omega)) \\
& +(\lambda_0 - A)y_1(\theta_h\omega) - y_2(\theta_h\omega))dh\|_{\ell^2_{m(\varepsilon)}} \\
& + \lim_{\varsigma\to 0}\|\int_{t_n}^{t_n+\varsigma}(-\lambda_1\varphi + \lambda_2 v + \lambda_2 y_2(\theta_h\omega))dh\|_{\ell^2_{m(\varepsilon)}} \\
& + \lim_{\varsigma\to 0}\{\|y_1(\theta_{t_n+\varsigma}\omega) - y_1(\theta_{t_n}\omega)\|_{\ell^2_{m(\varepsilon)}} \\
& +\|y_2(\theta_{t_n+\varsigma}\omega) - y_2(\theta_{t_n}\omega)\|_{\ell^2_{m(\varepsilon)}}\} \\
=\ & 0.
\end{aligned}$$

Then, by replacing ω in above estimate with $\theta_{-t_n}\omega$, we have

$$\begin{aligned}
&\lim_{\varsigma\to 0}\{\|u_i(t_n+\varsigma,\theta_{-t_n}\omega,u_\tau(\theta_{-t_n}\omega)) - u_i(t_n,\theta_{-t_n}\omega,u_\tau(\theta_{-t_n}\omega))\|_{\ell^2_{m(\varepsilon)}} \\
&\quad +\|\psi_i(t_n+\varsigma,\omega,u_\tau(\omega)) - \psi_i(t_n,\omega,u_\tau(\omega))\|_{\ell^2_{m(\varepsilon)}}\} = 0,
\end{aligned}$$

which implies that $\{(u_i(t_n), \psi_i(t_n))\}_{|i|\leq m(\varepsilon)}$ is equicontinuous in $\mathcal{X}_\tau^{m(\varepsilon)} \times \ell^2_{m(\varepsilon)}$. Thus, Γ is precompact in $\mathcal{X}_\tau^{m(\varepsilon)} \times \ell^2_{m(\varepsilon)}$. So, for above $\varepsilon > 0$ there exists a finite subset

$$\begin{aligned}
\Gamma^{m_3(\varepsilon)} :=\ & \{((u_i)^{(1)}_{|i|\leq m(\varepsilon)}, (\psi_i)^{(1)}_{|i|\leq m(\varepsilon)}), ((u_i)^{(2)}_{|i|\leq m(\varepsilon)}, (\psi_i)^{(2)}_{|i|\leq m(\varepsilon)}), \ldots, \\
& ((u_i)^{(m_3(\varepsilon))}_{|i|\leq m(\varepsilon)}, (\psi_i)^{(m_3(\varepsilon))}_{|i|\leq m(\varepsilon)})\} \\
\subset\ & \mathcal{X}_\tau^{(m(\varepsilon))} \times \ell^2_{m(\varepsilon)},
\end{aligned}$$

which forms a finite ε-net of Γ.

For $j = 1, 2, \ldots, m_3(\varepsilon)$, we choose

$$\bar{u}^{(j)} = (\bar{u}_i^{(j)})_{i\in\mathbf{Z}} = \begin{cases} u_i^{(j)}, & |i| \le m(\varepsilon); \\ 0, & |i| > m(\varepsilon), \end{cases}$$

and

$$\bar{\psi}^{(j)} = (\bar{\psi}_i^{(j)})_{i\in\mathbf{Z}} = \begin{cases} \psi_i^{(j)}, & |i| \le m(\varepsilon); \\ 0, & |i| > m(\varepsilon), \end{cases}$$

Obviously, $\{(\bar{u}^{(j)}, \bar{\psi}^{(j)})\}_{j=1}^{m_3(\varepsilon)} \subset \mathcal{X}_\tau \times \ell^2_{m(\varepsilon)}$. Then, for any $\phi_{t_n}(s, \theta_{-t_n}\omega, x_n) \in S$, by (3.7) we have

$$\|\phi_{t_n}(s, \theta_{-t_n}\omega, x_n) - (u^{(j)}, \psi^{(j)})\|_{\mathcal{X}_\tau \times \ell^2_{m(\varepsilon)}} \;\le\; 2\varepsilon.$$

Hence, S is precompact in $\mathcal{X}_\tau \times \ell^2_{m(\varepsilon)}$. The proof is completed. ■

Theorem 3.2. *For any $T > 0$ and $(u_\tau, \psi_\tau) \in \mathcal{X}_\tau \times \ell^2$, there is a unique solution $(u(t, u_\tau), \psi(t, \psi_\tau)) \in \mathcal{L}^2(\Omega, C([0, T]; \ell^2 \times \ell^2))$ of system (3.1), with $(u_t(\cdot, u_\tau), \psi(\cdot, \psi_\tau)) \in \mathcal{X}_\tau \times \ell^2, t \in [\tau, T)$, $(u_\tau(\cdot, u_\tau), \psi_\tau(\cdot, \psi_\tau)) = (u_\tau, \psi_\tau)$. Moreover, the mapping $(u_\tau, \psi_\tau) \to (u(\cdot, \omega, u_\tau), \psi(\cdot, \omega, \psi_\tau)) \in C([0, T]; \ell^2 \times \ell^2)$ is continuous for each $\omega \in \Omega$.*

Proof. By (H1)–(H2), it is known that equation (3.1) has a unique solution $(u, \psi) \in \mathcal{L}^2(\Omega, C([0, T]; \ell^2 \times \ell^2))$, and there exists a bounded set $Y \subset \mathcal{X}_\tau \times \ell^2$ such that $(u, \psi) \in Y \times \ell^2$. (see, for example, Theorem 2.6 in [21], and also can be obtained by the method in [2]).

Rewriting (3.1) as

$$\begin{aligned} u(t) &= u(\tau) + \int_\tau^t (f(u_h) - \psi(h) - Au(h) + \dot{W}^1(h))dh, \\ \psi(t) &= \psi(\tau) + \int_\tau^t (-\lambda_1\psi(h) + \lambda_2 u(h) + \dot{W}^2(h))dh, \\ u_\tau &= u(s + \tau), \;\; s \in [-\tau, 0], \end{aligned} \tag{3.9}$$

then, by (H1), we know that for any $(u_\tau, \psi(\tau)), (v_\tau, \varphi(\tau)) \in Y \subset \mathcal{X}_\tau \times \ell^2$ and (u, ψ), (v, φ) the corresponding solutions of system (3.9)

$$\begin{aligned} &\|u(t) - v(t)\|^2 + \|\psi(t) - \varphi(t)\|^2 \\ = \;& \|u_\tau - v_\tau\|^2 + 2\int_\tau^t (f(u_h) - f(v_h), u - v) - (\psi - \varphi, u - v) \\ &-(A(u-v), u-v)dh + \|\psi_\tau - \varphi_\tau\|^2 + 2\int_\tau^t (-\lambda_1\|\psi-\varphi\|^2 + \lambda_2(u-v, \psi-\varphi))dh \\ \le \;& \|u_\tau - v_\tau\|^2 + \|\psi_\tau - \varphi_\tau\|^2 + (9 + \lambda_2 - 2\alpha_1)\int_\tau^t \|u - v\|^2 dh \end{aligned}$$

$$
\begin{aligned}
&+2\alpha_2\int_\tau^t \max_{s\in[-\tau,0]}|u_h-v_h|^2dh+(1+\lambda_2-2\lambda_1)\int_\tau^t\|\psi-\varphi\|^2dh\\
\le\ & \|u_\tau-v_\tau\|^2+\|\psi_\tau-\varphi_\tau\|^2+(9+\lambda_2-2\alpha_1)\int_\tau^t\|u-v\|^2dh\\
&+2\alpha_2\max_{s\in[-\tau,0]}\int_{\tau+s}^{t+s}\|u(h)-v(h)\|^2dh+|\beta|(t-\tau)\\
&+(1+\lambda_2-2\lambda_1)\int_\tau^t\|\psi-\varphi\|^2dh\\
\le\ & (1+2\alpha_2\tau)\|u_\tau-v_\tau\|^2+\|\psi_\tau-\varphi_\tau\|^2+(9+\lambda_2+2\alpha_2-2\alpha_1)\int_\tau^t\|u-v\|^2dh\\
&+(1+\lambda_2-2\lambda_1)\int_\tau^t\|\psi-\varphi\|^2dh.
\end{aligned}
$$

By (H2) and Gronwall's inequality, we obtain

$$\|u(t)-v(t)\|^2+\|\psi(t)-\varphi(t)\|^2\le e^{\beta(t-\tau)}(\|u_\tau-v_\tau\|^2+\|\psi_\tau-\varphi_\tau\|^2),\quad for\ \ t\in[\tau,T],$$

where $\beta=\max\{9+\lambda_2+2\alpha_2-2\alpha_1,1+\lambda_2-2\lambda_1\}$.

Therefore

$$\sup_{t\in[\tau,T]}(\|u(t)-v(t)\|^2+\|\psi(t)-\varphi(t)\|^2)\le e^{\beta(t-\tau)}(\|u_\tau-v_\tau\|^2+\|\psi_\tau-\varphi_\tau\|^2),$$

which implies that the uniqueness and continuous dependence on the initial data. This completes the proof. ■

Theorem 3.3. *Equation* (3.1) *generates a continuous random dynamical system* $(\phi_t(s))_{t\ge\tau}$ *over* $(\Omega,\mathbf{F},P,(\theta_t)_{t\in\mathbf{R}})$, *where*

$$\phi_t(s,\omega,u_\tau)=(u_t(s,\omega,u_\tau),\psi(t,\omega,\psi_\tau)),\ \forall t\ge\tau,\ s\in[-\tau,0]\ and\ for\ all\ \omega\in\Omega.$$

Proof. The proof is similar to that of Theorem 3.2 in [2], so here it is omitted. ■

Lemma 3.1. *For* P*-a.e.* $\omega\in\Omega$, *there exists* $\{\mathcal{K}(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ *such that* $\{\mathcal{K}(\omega)\}_{\omega\in\Omega}$ *is a random bounded absorbing set for RDS* ϕ_t.

Proof. Taking the inner product of the first equation in (3.4) with $v(t)$, we have

$$
\begin{aligned}
\frac{d}{dt}\|v\|^2 =\ & -2(Av,v)-2(\varphi,v)+2(f(v_t+y_1(\theta_{t+s}\omega)),v)\\
&+2((\lambda_0-A)y_1(\theta_t\omega),v)-2(y_2,v),\ \forall\ t\ge\tau.
\end{aligned}\tag{3.10}
$$

By (H2), it is easy to see that

$$
\begin{aligned}
(f(v_t+y_1(\theta_{t+s}\omega)),v) =\ & (f(v_t+y_1(\theta_{t+s}\omega))-f(v_t),v)+(f(v_t),v),\\
(f(v_t+y_1(\theta_{t+s}\omega))-f(v_t),v) \le\ & L_f\|y_1(\theta_{t+s}\omega)\|_{\mathcal{X}_\tau}\|v\|\\
\le\ & \frac{L_f}{2}(\|y_1(\theta_{t+s}\omega)\|^2_{\mathcal{X}_\tau}+\|v\|^2),\\
(f(v_t+y_1,v(\theta_t\omega)) \le\ & -\alpha_1\sum_{i\in\mathbf{Z}}|v_i|^2+\alpha_2\max_{s\in[-\tau,0]}\sum_{i\in\mathbf{Z}}|v_{it}|^2,\\
\le\ & -\alpha_1\|v\|^2+\alpha_2\|v_t\|_{\mathcal{X}_\tau}.
\end{aligned}\tag{3.11}
$$

In addition, noting that

$$(Av, v) \geq 0, \tag{3.12}$$

$$\begin{aligned}(Ay_1, v) &= -(By_1, Bv) \leq \frac{1}{8}\|Bv\|^2 + 8\|By_1(\theta_t\omega)\|^2 \\ &\leq \frac{1}{2}\|v\|^2 + 32\|y_1(\theta_t\omega)\|^2,\end{aligned} \tag{3.13}$$

thus by (3.11)–(3.13), for $t \geq \tau$, we can obtain

$$\begin{aligned}\frac{d}{dt}\|v\|^2 \leq\ & (3 + L_f + \lambda_0 - 2\alpha_1)\|v\|^2 + \|\varphi\|^2 + 2\alpha_2\|v_t\|^2_{\mathcal{X}_\tau} \\ & +(64 + \lambda_0)\|y_1(\theta_t\omega)\|^2 + L_f\|y_1(\theta_{t+s}\omega)\|^2_{\mathcal{X}_\tau} + \|y_2(\theta_t\omega)\|^2.\end{aligned}$$

Therefore, for any given γ which specified later, we have

$$\begin{aligned}\frac{d}{dt}(e^{\gamma t}\|v\|^2) =\ & \gamma e^{\gamma t}\|v\|^2 + e^{\gamma t}\frac{d}{dt}\|v\|^2 \\ \leq\ & (3+L_f+\lambda_0+\gamma - 2\alpha_1)e^{\gamma t}\|v\|^2 + 2\alpha_2 e^{\gamma t}\|v_t\|^2_{\mathcal{X}_\tau} + e^{\gamma t}\|\varphi\|^2 \\ & +(64 + \lambda_0)e^{\gamma t}\|y_1(\theta_t\omega)\|^2 \\ & +L_f e^{\gamma t}\|y_1(\theta_{t+s}\omega)\|^2_{\mathcal{X}_\tau} + e^{\gamma t}\|y_2(\theta_t\omega)\|^2.\end{aligned} \tag{3.14}$$

Integrating (3.14) over $[\tau, t]$ with $t \geq \tau$, and by (H2), it yields that

$$\begin{aligned} & e^{\gamma t}\|v\|^2 - e^{\gamma\tau}\|v(\tau)\|^2 \\ \leq\ & (3 + L_f + \lambda_0 + \gamma - 2\alpha_1)\int_\tau^t e^{\gamma h}\|v\|^2 dh + 2\alpha_2\int_\tau^t e^{\gamma h}\|v_h\|^2_{\mathcal{X}_\tau} dh \\ & + \int_\tau^t e^{\gamma h}\|\varphi\|^2 dh + (64 + \lambda_0)\int_\tau^t e^{\gamma h}\|y_1(\theta_h\omega)\|^2 dh \\ & +L_f\int_\tau^t e^{\gamma h}\|y_1(\theta_{h+s}\omega)\|^2_{\mathcal{X}_\tau} dh + \int_\tau^t e^{\gamma h}\|y_2(\theta_h\omega)\|^2 dh \\ \leq\ & (3 + L_f + \lambda_0 + \gamma - 2\alpha_1)\int_\tau^t e^{\gamma h}\|v\|^2 dh + 2\alpha_2 e^\tau\int_{\tau+s}^{t+s} e^{\gamma h}\|v\|^2 dh \\ & + \int_\tau^t e^{\gamma h}\|\varphi\|^2 dh + (64 + \lambda_0)\int_\tau^t e^{\gamma h}\|y_1(\theta_h\omega)\|^2 dh \\ & +L_f e^\tau\int_{\tau+s}^{t+s} e^{\gamma h}\|y_1(\theta_h\omega)\|^2_{\mathcal{X}_\tau} dh + \int_\tau^t e^{\gamma h}\|y_2(\theta_h\omega)\|^2 dh \\ \leq\ & (3 + L_f + \lambda_0 + \gamma - 2\alpha_1)\int_\tau^t e^{\gamma h}\|v\|^2 dh + 2\alpha_2 e^\tau\int_0^\tau e^{\gamma h}\|v\|^2 dh \\ & +2\alpha_2 e^\tau\int_\tau^t e^{\gamma h}\|v\|^2 dh + \int_\tau^t e^{\gamma h}\|\varphi\|^2 dh + (64 + \lambda_0 + L_f e^\tau)\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&\int_\tau^t e^{\gamma h}\|y_1(\theta_t\omega)\|^2 dh \\
&+L_f e^\tau \int_0^\tau e^{\gamma h}\|y_1(\theta_h\omega)\|^2 dh + \int_\tau^t e^{\gamma h}\|y_2(\theta_h\omega)\|^2 dh \\
\leq\ & (3+L_f+\lambda_0+\gamma+2\alpha_2 e^\tau-2\alpha_1)\int_\tau^t e^{\gamma h}\|v\|^2 dh \\
&+2\alpha_2\tau e^{(\gamma+1)\tau}\max_{t\in[0,\tau]}\|v(t)\|^2 \\
&+\int_\tau^t e^{\gamma h}\|\varphi\|^2 dh + (64+\lambda_0+L_f e^\tau)\int_\tau^t e^{\gamma h}\|y_1(\theta_h\omega)\|^2 dh \\
&+L_f e^{(\gamma+1)\tau}\max_{t\in[0,\tau]}\|y_1(\theta_t\omega)\|^2 + \int_\tau^t e^{\gamma h}\|y_2(\theta_h\omega)\|^2 dh.
\end{aligned}
$$

Taking the inner product of the second equation in (3.4) with $\varphi(t)$, we have

$$
\begin{aligned}
\frac{d}{dt}\|\varphi\|^2 =\ & -2\lambda_1\|\varphi\|^2 + 2\lambda_2(v,\varphi) + 2\lambda_2(y_1(\theta_t\omega),\varphi) \\
\leq\ & 2(\lambda_2-\lambda_1)\|\varphi\|^2 + \lambda_2\|v\|^2 + \lambda_2\|y_1(\theta_t\omega)\|^2.
\end{aligned}
$$

As done in (3.15), we get

$$
\begin{aligned}
&e^{\gamma t}\|\varphi\|^2 - e^{\gamma\tau}\|\varphi(\tau)\|^2 \\
\leq\ & (\gamma+2\lambda_2-2\lambda_1)\int_\tau^t e^{\gamma h}\|\varphi\|^2 dh + \lambda_2\int_\tau^t e^{\gamma h}\|v\|^2 dh \\
&+\lambda_2\int_\tau^t e^{\gamma h}\|y_1(\theta_h\omega)\|^2 dh.
\end{aligned}
\tag{3.16}
$$

Summing up (3.15)-(3.16), for

$$
0<\gamma<\min\{2\alpha_1-L_f-\lambda_0-\lambda_2-2\alpha_2 e^\tau-3, 2\lambda_1-2\lambda_2-1\},
$$

we have

$$
\begin{aligned}
&\|v(t,\omega,v_\tau(\omega))\|^2 + \|\varphi(t,\omega,v_\tau(\omega))\|^2 \\
\leq\ & (3+L_f+\lambda_0+\lambda_2+\gamma+2\alpha_2 e^\tau-2\alpha_1)\int_\tau^t e^{\gamma(h-t)}\|v\|^2 dh \\
&+(1+\gamma+2\lambda_2-2\lambda_1)\int_\tau^t e^{\gamma(h-t)}\|\varphi\|^2 dh \\
&+2\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\|v(t)\|^2 \\
&+(64+\lambda_0+L_f e^\tau)\int_\tau^t e^{\gamma(h-t)}\|y_1\|^2 dh + L_f e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\|y_1(\theta_t\omega)\|^2 \\
&+(1+\lambda_2)\int_\tau^t e^{\gamma(h-t)}\|y_2(\theta_h\omega)\|^2 dh + e^{\gamma(\tau-t)}(\|v(\tau)\|^2+\|\varphi(\tau)\|^2)
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
&\leq 2\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\|v(t)\|^2+(64+\lambda_0+L_f e^{\tau}+\lambda_2)\int_\tau^t e^{\gamma(h-t)}\|y_1(\theta_h\omega)\|^2dh\\
&\quad +L_f e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\|y_1(\theta_t\omega)\|^2+1\int_\tau^t e^{\gamma(h-t)}\|y_2(\theta_h\omega)\|^2dh\\
&\quad +e^{\gamma(\tau-t)}(\|v(\tau)\|^2+\|\varphi(\tau)\|^2).
\end{aligned}
$$

Furthermore, since $\|y_k(\omega)\|$ $(k=1,2)$ are tempered and $y_k(\theta_t\omega)$ is continuous in t, by Proposition 4.33 in [1], there is a tempered function $r(\omega)>0$, such that

$$
\|y_k(\theta_t\omega)\|^2\leq r(\theta_t\omega)\leq r(\omega)e^{\frac{\gamma}{2}|t|}.
$$

Thus, by replacing ω in (3.17) with $\theta_{-t}\omega$, we have

$$
\begin{aligned}
&\|v(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))\|^2+\|\varphi(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))\|^2\\
&\leq 2\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\|v(t)\|^2\\
&\quad +(64+\lambda_0+L_f e^{\tau}+\lambda_2)\int_\tau^t e^{\gamma(h-t)}\|y_1(\theta_{h-t}\omega)\|^2dh\\
&\quad +L_f e^{(\gamma+1)\tau-\gamma t}\max_{s\in[0,\tau]}\|y_1(\theta_{s-t}\omega)\|^2+1\int_\tau^t e^{\gamma(h-t)}\|y_2(\theta_{h-t}\omega)\|^2dh\\
&\quad +e^{\gamma(\tau-t)}(\|v_\tau(\theta_{-t}\omega)\|^2+\|\varphi_\tau(\theta_{-t}\omega)\|^2)\\
&\leq 2\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\|v(t)\|^2+e^{-\gamma(t-\tau)}(\|v_\tau(\theta_{-t}\omega)\|^2\\
&\quad +\|\varphi_\tau(\theta_{-t}\omega)\|^2)+Cr(\omega),
\end{aligned}
\tag{3.18}
$$

where $C>0$ is a constant.

By (3.3) and (3.18), we obtain

$$
\begin{aligned}
&\|u(t,\theta_{-t}\omega,u_\tau(\theta_{-t}\omega))\|^2+\|\psi(t,\theta_{-t}\omega,u_\tau(\theta_{-t}\omega))\|^2\\
&=\|v(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))+y_1(\theta_\tau\omega)\|^2\\
&\quad +\|\varphi(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))+y_2(\theta_\tau\omega)\|^2\\
&\leq 2\|v(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))\|^2+2\|v(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))\|^2\\
&\quad +2r(\omega)e^{\frac{\gamma\tau}{2}}(\|y_1(\omega)\|^2+\|y_2(\omega)\|^2)\\
&\leq 4\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\|v(t)\|^2\\
&\quad +2e^{-\gamma(t-\tau)}(\|v_\tau(\theta_{-t}\omega)\|^2+\|\varphi_\tau(\theta_{-t}\omega)\|^2)\\
&\quad +4Cr(\omega)+4(\|y_1(\omega)\|^4+\|y_2(\omega)\|^4),\ t\geq\tau.
\end{aligned}
$$

Therefore, for $t>2\tau$, we have $t+s>\tau$ for any $s\in[-\tau,0]$, and

$$\begin{aligned}&\|u(t+s,\theta_{-t-s}\omega,u_\tau(\theta_{-t-s}\omega))\|^2+\|\psi(t,\theta_{-t}\omega,u_\tau(\theta_{-t}\omega))\|^2\\ \le\ &4\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\|v(t)\|^2+e^{-\gamma(t+s-\tau)}(\|v_\tau(\theta_{-t-s}\omega)\|^2+\|\varphi_\tau(\theta_{-t}\omega)\|^2)\\ &+4Cr(\omega)+4(\|y_1(\omega)\|^4+\|y_2(\omega)\|^4).\end{aligned}$$

By replacing ω in above estimate with $\theta_s\omega$, we obtain

$$\begin{aligned}&\|u_t(s,\theta_{-t}\omega,u_\tau(\theta_{-t}\omega))\|^2+\|\psi(t,\theta_{-t}\omega,u_\tau(\theta_{-t}\omega))\|^2\\ \le\ &4\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\|v(t)\|^2\\ &+e^{-\gamma(t-2\tau)}(\|v_\tau(\theta_{-t}\omega)\|^2+\|\varphi_\tau(\theta_{-t}\omega)\|^2)\\ &+4Cr(\omega)+4(\|y_1(\omega)\|^4+\|y_2(\omega)\|^4).\end{aligned}$$

Set

$$R(\omega)=8Cr(\omega)+8(\|y_1(\omega)\|^4+\|y_2(\omega)\|^4).$$

Then $\mathcal{K}(\omega)=\{(u,\psi)\in\mathcal{X}_\tau\times\ell^2:\|u_t\|_{\mathcal{X}_\tau}+\|\psi\|\le R(\omega)\}$ is a bounded absorbing set for RDS ϕ_t. This completes the proof. ∎

Lemma 3.2. *Assume that $(u_\tau,\varphi_\tau)\in\mathcal{K}(\omega)$. Then for any $\varepsilon>0$ and P-a.e. $\omega\in\Omega$, there exist $T(\varepsilon,\omega)>0$ and $N(\varepsilon,\omega)>0$ such that the solution u_t of system (3.8) satisfies $\forall t\ge T(\varepsilon,\omega)+\tau$,*

$$\max_{s\in[-\tau,0]}\sum_{|i|>N(\varepsilon,\omega)}|u_{it}(s,\theta_{-t}\omega,u_\tau(\theta_{-t}\omega))|^2+\sum_{|i|>N(\varepsilon,\omega)}|\varphi_i(t,\theta_{-t}\omega,u_\tau(\theta_{-t}\omega))|^2\le\varepsilon.$$

Proof. Let $\eta(x)\in\mathcal{C}(\mathbf{R}_+,[0,1])$ be a cut-off function satisfying

$$\eta(x)=0,\ for\ all\ x\in[0,1];\quad \eta(x)=1,\ for\ all\ x\in[2,+\infty),$$

and $|\eta'(x)|\le\eta_0$ (a positive constant).

Taking the inner product of the first equation in (3.4) with $a(t)=(a_i(t))_{i\in\mathbf{Z}}=(\eta(\frac{|i|}{M})v_i)_{i\in\mathbf{Z}}$ in ℓ^2, we get

$$\begin{aligned}(\dot v,a(t))\ =\ &-(Av(t),a(t))-(\varphi(t),a(t))+(f(v_t+y_1(\theta_{t+s}\omega)),a(t))\\ &+((\lambda_0-A)y_1(\theta_t\omega),a(t))-(y_2(\theta_t\omega),a(t)).\end{aligned}\tag{3.19}$$

Now we estimate the terms (3.19) one by one. First, taking into consideration that

$$(Av(t),a(t))=(Bv(t),Ba(t))$$

and

$$(Ba(t))_i = a_{i+1} - a_i = (\eta(\frac{|i+1|}{M}) - \eta(\frac{|i|}{M}))v_{i+1} + \eta(\frac{|i|}{M})v_{i+1} - \eta(\frac{|i|}{M})v_i,$$

we have

$$\begin{aligned}(Av(t), a(t)) =& \sum_{i\in\mathbf{Z}}(\eta(\frac{|i+1|}{M}) - \eta(\frac{|i|}{M}))v_{i+1}(Bv)_i \\ &+ \sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})v_{i+1}(Bv)_i - \sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})v_i(Bv)_i \\ =& \sum_{i\in\mathbf{Z}}(\eta(\frac{|i+1|}{M}) - \eta(\frac{|i|}{M}))v_{i+1}(Bv)_i + \sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(Bv)_i^2.\end{aligned} \tag{3.20}$$

Then, noting that

$$\begin{aligned}|\sum_{i\in\mathbf{Z}}(\eta(\frac{|i+1|}{M}) - \eta(\frac{|i|}{M}))v_{i+1}(Bv)_i| \le& \frac{1}{M}\sum_{i\in\mathbf{Z}}|\eta'(\xi_i)v_{i+1}(Bv)_i| \\ \le& \frac{1}{M}\sum_{i\in\mathbf{Z}}|\eta'(\xi_i)||v_{i+1}^2 - v_iv_{i+1}| \\ \le& \frac{3\eta_0}{2M}\sum_{i\in\mathbf{Z}}v_{i+1}^2 + \frac{\eta_0}{2M}\sum_{i\in\mathbf{Z}}v_i^2 \\ \le& \frac{2\eta_0}{M}\|v\|^2,\end{aligned} \tag{3.21}$$

and

$$|\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(Bv)_i^2| \le 4\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i|^2, \tag{3.22}$$

thus, by (3.20)–(3.22), it yields that

$$(Av(t), a(t)) \ge -4\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i|^2 - \frac{2\eta_0}{M}\|v\|^2. \tag{3.23}$$

Note that by (H2)

$$\begin{aligned}& f(v_t + y_1(\theta_{t+s}\omega)) - f(v_t), a) \\ =& \sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})[f_i(v_{it} + y_1(\theta_{t+s}\omega)) - f(v_{it})]v_i, \\ \le& \frac{L_f}{2}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_1(\theta_{t+s}\omega)|^2 + \frac{L_f}{2}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i|^2,\end{aligned} \tag{3.24}$$

and

$$
\begin{aligned}
(f(v_t), a) &= \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) f_i(v_{it}) v_i \\
&\le -\alpha_1 \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |v_i|^2 + \alpha_2 \max_{s\in[-\tau,0]} \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |v_{it}|^2 ds.
\end{aligned} \tag{3.25}
$$

By (3.24)-(3.25), we have

$$
\begin{aligned}
&(f(v_t + y_1(\theta_{t+s}\omega)), a) = (f(v_t + y_1(\theta_{t+s}\omega)) - f(v_t), a) + (f(v_t), a), \\
&\le (\frac{L_f}{2} - \alpha_1) \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |v_i|^2 + \alpha_2 \max_{s\in[-\tau,0]} \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |v_{it}|^2 ds \\
&\quad + \frac{L_f}{2} \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |y_1(\theta_{t+s}\omega)|^2.
\end{aligned} \tag{3.26}
$$

Furthermore, noting that

$$
\begin{aligned}
&-(Ay_1(\theta_t\omega), a(t)) \\
&= -\sum_{i\in\mathbf{Z}} (By_1)_i (\eta(\frac{|i+1|}{M}) v_{i+1} - \eta(\frac{|i|}{M}) v_i) \\
&= -\sum_{i\in\mathbf{Z}} (\eta(\frac{|i+1|}{M}) - \eta(\frac{|i|}{M}))(By_1)_i v_i - \sum_{i\in\mathbf{Z}} \eta(\frac{|i+1|}{M})(By_1)_i (Bv)_i \\
&\le \frac{1}{M} \sum_{i\in\mathbf{Z}} |\eta'(\xi_i)||(By_1)_i v_i| - \sum_{i\in\mathbf{Z}} \eta(\frac{|i+1|}{M})(By_1)_i (Bv)_i \\
&\le \frac{2\eta_0}{M} \|y_1\|^2 + \frac{\eta_0}{2M} \|v\|^2 + 2 \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |v_i|^2 + 2 \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |y_{1i}|^2,
\end{aligned} \tag{3.27}
$$

and

$$
(\dot{v}, a) = \tfrac{1}{2} \tfrac{d}{dt} \textstyle\sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |v_i|^2, \tag{3.28}
$$

thus, by (3.23)–(3.28), we have

$$
\begin{aligned}
&\frac{d}{dt} \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |v_i|^2 \\
&\le [13 + L_f + \lambda_0 - 2\alpha_1] \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |v_i|^2 + 2\alpha_2 \max_{s\in[-\tau,0]} \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |v_{it}|^2 \\
&\quad + \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |\varphi_i|^2 + \frac{4\eta_0}{M} \|y_1\|^2 + \frac{5\eta_0}{M} \|v\|^2 + (4 + \lambda_0) \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |y_{1i}|^2 \\
&\quad + \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |y_{2i}|^2 + \frac{L_f}{2} \sum_{i\in\mathbf{Z}} \eta(\frac{|i|}{M}) |y_1(\theta_{t+s}\omega)|^2,
\end{aligned} \tag{3.29}
$$

holds for $t \geq \tau$.

Thus, by (3.29), for any given γ specified later, we have

$$\begin{aligned}
&\frac{d}{dt}(e^{\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})v_i^2(t)) = \gamma e^{\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})v_i^2(t) + e^{\gamma t}\frac{d}{dt}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})v_i^2(t)\\
\leq\ & [13+L_f+\lambda_0+\gamma-2\alpha_1]e^{\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i|^2+2\alpha_2 e^{\gamma t}\max_{s\in[-\tau,0]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_{it}|^2\\
&+e^{\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|\varphi_i|^2+\frac{4\eta_0}{M}e^{\gamma t}\|y_1\|^2+\frac{5\eta_0}{M}e^{\gamma t}\|v\|^2+(4+\lambda_0)e^{\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}|^2\\
&+e^{\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{2i}|^2+\frac{L_f e^{\gamma t}}{2}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_1(\theta_{t+s}\omega)|^2.
\end{aligned} \tag{3.30}$$

Integrating (3.30) over $[\tau, t]$ and taking into consideration of (H2), we obtain

$$\begin{aligned}
&e^{\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})v_i^2(t) - e^{\gamma\tau}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})v_i^2(\tau)\\
\leq\ & [13+L_f+\lambda_0+\gamma-2\alpha_1]\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i|^2dh\\
&+2\alpha_2\max_{s\in[-\tau,0]}\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_{ih}|^2dh\\
&+\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|\varphi_i|^2dh+\frac{4\eta_0}{M}\int_\tau^t e^{\gamma h}\|y_1(\theta_h\omega)\|^2dh+\frac{5\eta_0}{M}\int_\tau^t e^{\gamma h}\|v\|^2dh\\
&+(4+\lambda_0)\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2dh\\
&+\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{2i}(\theta_h\omega)|^2dh+\frac{L_f}{2}\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_1(\theta_{h+s}\omega)|^2dh\\
\leq\ & [13+L_f+\lambda_0+\gamma-2\alpha_1]\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|v_i|^2dh\\
&+2\alpha_2 e^\tau\int_0^\tau e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i|^2dh+2\alpha_2 e^\tau\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i|^2dh\\
&+\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|\varphi_i|^2d+\frac{4\eta_0}{M}\int_\tau^t e^{\gamma h}\|y_1(\theta_h\omega)\|^2dh\\
&+\frac{5\eta_0}{M}\int_\tau^t e^{\gamma h}\|v\|^2dh+(4+\lambda_0)\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2dh
\end{aligned}$$

$$
\begin{aligned}
&+\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{2i}(\theta_h\omega)|^2dh+\frac{L_fe^\tau}{2}\int_0^\tau e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2dh\\
&+\frac{L_fe^\tau}{2}\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2dh\\
\le\ &[13+L_f+\lambda_0+\gamma-2\alpha_1+2\alpha_2e^\tau]\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|v_i|^2dh\\
&+2\alpha_2\tau e^{(\gamma+1)\tau}\max_{t\in[0,\tau]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i(t)|^2+\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|\varphi_i|^2dh\\
&+(4+\lambda_0+\frac{L_fe^\tau}{2})\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|y_{1i}(\theta_h\omega)|^2dh\\
&+\frac{4\eta_0}{M}\int_\tau^t e^{\gamma h}\|y_1(\theta_h\omega)\|^2dh+\frac{5\eta_0}{M}\int_\tau^t e^{\gamma h}\|v\|^2dh\\
&+\int_\tau^t e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{2i}(\theta_h\omega)|^2dh+\frac{L_fe^\tau}{2}\int_0^\tau e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2dh,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})v_i^2(t)\\
\le\ &[13+L_f+\lambda_0+\gamma-2\alpha_1+2\alpha_2e^\tau]\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|v_i|^2dh\\
&+2\alpha_2\tau e^{\gamma\tau}\max_{t\in[0,\tau]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i(t)|^2+\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|\varphi_i|^2dh\\
&+e^{\gamma(\tau-t)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i(\tau)|^2+\frac{4\eta_0}{M}\int_\tau^t e^{\gamma(h-t)}\|y_1(\theta_h\omega)\|^2dh\\
&+(4+\lambda_0+\frac{L_fe^\tau}{2})\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma(h-t)}|y_{1i}(\theta_h\omega)|^2dh\\
&+\frac{5\eta_0}{M}\int_\tau^t e^{\gamma(h-t)}\|v\|^2dh+\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma(h-t)}|y_{2i}(\theta_h\omega)|^2dh\\
&+\frac{L_fe^\tau}{2}\int_0^\tau e^{\gamma(h-t)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2dh.
\end{aligned}
\tag{3.31}
$$

Taking the inner product of the second equation in (3.4) with $b(t)=(b_i(t))_{i\in\mathbf{Z}}=(\eta(\frac{|i|}{M})\varphi_i)_{i\in\mathbf{Z}}$ in ℓ^2, we get

$$
(\dot\varphi,b(t))=-\lambda_1(\varphi(t),b(t))+\lambda_2(v(t),b(t))-\lambda_2(y_2(\theta_t\omega),b(t)).
\tag{3.32}
$$

By (3.32), it follows that

$$\frac{d}{dt}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|\varphi_i|^2 \le 2(\lambda_2-\lambda_1)\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|\varphi_i|^2 + \lambda_2\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i|^2+\lambda_2\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{2i}|^2. \tag{3.33}$$

As the proving process of (3.31), it holds

$$\begin{aligned}&\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|\varphi_i|^2\\ \le\ & (2\lambda_2+\gamma-2\lambda_1)\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|\varphi_i|^2dh\\ &+\lambda_2\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|v_i|^2dh\\ &+\lambda_2 e^{-\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|y_{2i}(\theta_h\omega)|^2dh+e^{\gamma(\tau-t)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|\varphi_i(\tau)|^2.\end{aligned}\tag{3.34}$$

Summing up (3.31) and (3.34), by the choose of

$$0<\gamma<\min\{2\alpha_1-L_f-\lambda_0-\lambda_2-2\alpha_2e^\tau-13,\ 2\lambda_1-2\lambda_2-1\},$$

we get

$$\begin{aligned}&\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(|v_i|^2+|\varphi_i|^2)\\ \le\ &[13+L_f+\lambda_0+\lambda_2+\gamma-2\alpha_1+2\alpha_2e^\tau]\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|v_i|^2dh\\ &+(1+2\lambda_2+\gamma-2\lambda_1)\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|\varphi_i|^2dh\\ &+2\alpha_2\tau e^{(\gamma+1)\tau}\max_{t\in[0,\tau]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i(t)|^2\\ &+e^{\gamma(\tau-t)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(|v_i(\tau)|^2+|\varphi_i(\tau)|^2)+\frac{5\eta_0}{M}e^{-\gamma t}\int_\tau^t e^{\gamma h}\|v\|^2dh\\ &+(4+\lambda_0+\frac{L_fe^\tau}{2})e^{-\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|y_{1i}(\theta_h\omega)|^2dh\\ &+\frac{4\eta_0}{M}e^{-\gamma t}\int_\tau^t e^{\gamma h}\|y_1(\theta_h\omega)\|^2dh\\ &+(1+\lambda_2)e^{-\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_\tau^t e^{\gamma h}|y_{2i}(\theta_h\omega)|^2dh\end{aligned}\tag{3.35}$$

$$
\begin{aligned}
&+\frac{L_f e^{\tau}}{2}\int_0^{\tau} e^{\gamma(h-t)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2 dh\\
\leq\ & 2\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i(t)|^2\\
&+e^{\gamma(\tau-t)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(|v_i(\tau)|^2+|\varphi_i(\tau)|^2)\\
&+\frac{5\eta_0}{M}e^{-\gamma t}\int_{\tau}^{t} e^{\gamma h}\|v\|^2 dh\\
&+(4+\lambda_0+\frac{L_f e^{\tau}}{2})e^{-\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_{\tau}^{t} e^{\gamma h}|y_{1i}(\theta_h\omega)|^2 dh\\
&+\frac{4\eta_0}{M}e^{-\gamma t}\int_{\tau}^{t} e^{\gamma h}\|y_1(\theta_h\omega)\|^2 dh\\
&+(1+\lambda_2)e^{-\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_{\tau}^{t} e^{\gamma h}|y_{2i}(\theta_h\omega)|^2 dh\\
&+\frac{L_f e^{\tau}}{2}e^{\gamma(\tau-t)}\max_{h\in[0,\tau]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2.
\end{aligned}
$$

Replacing ω in (3.35) by $\theta_{-t}\omega$, we have

$$
\begin{aligned}
&\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(v_i^2(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega)+\varphi_i^2(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))\\
\leq\ & 2\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|v_i(t)|^2\\
&+e^{-\gamma(t-\tau)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(v_i^2(\tau,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))+\varphi_i^2(\tau,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega)))\\
&+(4+\lambda_0+\frac{L_f e^{\tau}}{2})\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_{\tau}^{t} e^{\gamma(h-t)}|y_{1i}(\theta_{h-t}\omega)|^2 dh\\
&+(1+\lambda_2)e^{-\gamma t}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})\int_{\tau}^{t} e^{\gamma(h-t)}|y_{2i}(\theta_{h-t}\omega)|^2 dh\\
&+\frac{5\eta_0}{M}\int_{\tau}^{t} e^{\gamma(h-t)}\|v(h,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))\|^2 dh\\
&+\frac{4\eta_0}{M}\int_{\tau}^{t} e^{\gamma(h-t)}\|y_1(\theta_{h-t}\omega)\|^2 dh\\
&+\frac{L_f e^{\tau}}{2}e^{\gamma(\tau-t)}\max_{h\in[0,\tau]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2.
\end{aligned}
\tag{3.36}
$$

In what follows, we estimate each term in (3.36). First, it is easy to see that there exists $T_0(\varepsilon,\omega)>\tau$ such that if $t>T_0(\varepsilon,\omega)$, we have

$$2\alpha_2\tau e^{(\gamma+1)\tau-\gamma t}\max_{t\in[0,\tau]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(|v_i(t)|^2+|\varphi_i(t)|^2)\le\varepsilon.$$

Then, by (3.18), we derive

$$\begin{aligned}
& e^{-\gamma(t-\tau)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(v_i^2(\tau,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))+\varphi_i^2(\tau,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega)))\\
\le\ & 2\alpha_2\tau e^{-2\gamma(t-\tau)}\max_{t\in[0,\tau]}\|v(t)\|^2)+Ce^{-\gamma(t-\tau)}r(\omega)\\
& +e^{-2\gamma(t-\tau)}(\|v_\tau(\theta_{-t}\omega)\|^2+\|\varphi_\tau(\theta_{-t}\omega)\|^2.
\end{aligned}$$

Thus, there exists $T_1(\varepsilon,\omega)>\tau$ such that if $t>T_1(\varepsilon,\omega)$, we have

$$e^{-\gamma(t-\tau)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})(v_i^2(\tau,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))+\varphi_i^2(\tau,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega)))\le\varepsilon. \tag{3.37}$$

For the rest term in the right side of (3.36), let $T'>0$ to be determined later, we have

$$\begin{aligned}
& (4+\lambda_0+\frac{L_fe^\tau}{2})\int_\tau^t e^{\gamma(h-t)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_{h-t}\omega)|^2dh\\
=\ & (4+\lambda_0+\frac{L_fe^\tau}{2})\int_{-T'}^0 e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2dh\\
& +(4+\lambda_0+\frac{L_fe^\tau}{2})\int_{\tau-t}^{-T'} e^{\gamma h}\|y_1(\theta_h\omega)\|^2dh.
\end{aligned}$$

Choosing $T_2>\frac{2}{\gamma}\ln(\frac{8(4+\lambda_0+\frac{L_fe^\tau}{2})r(\omega)}{\gamma\varepsilon})$ and noting that

$$\|y_1(\theta_t\omega)\|^2\le r(\theta_t\omega)\le r(\omega)e^{\frac{\gamma}{2}|t|}, \tag{3.38}$$

then for $t>T_2+\tau$, we have

$$(4+\lambda_0+\frac{L_fe^\tau}{2})\int_{\tau-t}^{-T_2} e^{\gamma h}\|y_1(\theta_h\omega)\|^2dh\le\frac{8(4+\lambda_0+\frac{L_fe^\tau}{2})}{\gamma}r(\omega)e^{-\frac{\gamma T_2}{2}}<\varepsilon. \tag{3.39}$$

Furthermore, for the fixed T, by Lebesgue's theorem there is a positive constant $N_1(\varepsilon,\omega)$ such that for $M>N_1(\varepsilon,\omega)$,

$$(4+\lambda_0+\frac{L_fe^\tau}{2})\int_{-T_2}^0 e^{\gamma h}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2dh<\varepsilon,$$

which combining with (3.38) and (3.39) implies

$$(4+\lambda_0+\frac{L_f e^{\tau}}{2})\int_{\tau}^{t} e^{\gamma(h-t)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_{h-t}\omega)|^2dh\leq 2\varepsilon. \tag{3.40}$$

By the same method, there is a constant $T_3>\frac{2}{\gamma}\ln(\frac{8(1+\lambda_2)r(\omega)}{\gamma\varepsilon})$, for $t>T_3+\tau$,

$$(1+\lambda_2)\int_{\tau}^{t} e^{\gamma(h-t)}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{2i}(\theta_{h-t}\omega)|^2dh\leq 2\varepsilon. \tag{3.41}$$

By (3.18), it yields that

$$\begin{aligned}
&\frac{5\eta_0}{M}\int_{\tau}^{t} e^{\gamma(h-t)}\|v(h,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))\|^2dh\\
\leq\ &\frac{10\alpha_2\tau\eta_0}{M}(t-\tau)e^{2\gamma(\tau-t)}\max_{t\in[0,\tau]}\|v(t)\|^2\\
&+\frac{5\eta_0}{M}(t-\tau)e^{2\gamma(\tau-t)}(\|v_\tau(\theta_{-t}\omega)\|^2+\|\varphi_\tau(\theta_{-t}\omega)\|^2)+\frac{5C\eta_0 r(\omega)}{M\gamma}.
\end{aligned}$$

Recall the fact that $v_\tau(\theta_t\omega)\in\mathcal{K}(\theta_t\omega)$, this implies that $\|v_\tau(\theta_t\omega)\|^2+\|\varphi_\tau(\theta_t\omega)\|^2\leq R(\theta_{-t}\omega)$ is tempered. Thus there exist $T_4(\varepsilon,\omega)>\tau$ and $N_1(\varepsilon,\omega)$ such that, for $t>T_4(\varepsilon,\omega)$ and $M>N_2(\varepsilon,\omega)$, we have

$$\frac{5\eta_0}{M}\int_{\tau}^{t} e^{\gamma(h-t)}\|v(h,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))\|^2dh<\varepsilon. \tag{3.42}$$

Finally, for the last two terms in the right side of (3.36), by (3.38), there exist $N_3(\varepsilon,\omega)>0$ and $T_5(\varepsilon,\omega)$, it yields that

$$\begin{aligned}
&\frac{4\eta_0}{M}\int_{0}^{\tau-t} e^{\gamma h}\|y_1(\theta_h\omega)\|^2dh\leq\frac{4r(\omega)\eta_0}{M}\int_{\tau-t}^{0}e^{\frac{\gamma h}{2}}dh\leq\frac{8\eta_0 r(\omega)}{M\gamma}<\varepsilon,\\
&\frac{L_f e^{\tau}}{2}e^{\gamma(\tau-t)}\max_{h\in[0,\tau]}\sum_{i\in\mathbf{Z}}\eta(\frac{|i|}{M})|y_{1i}(\theta_h\omega)|^2.
\end{aligned}$$

Set

$$T(\varepsilon,\omega)=\max\{T_0(\varepsilon,\omega),T_1(\varepsilon,\omega),T_2(\varepsilon,\omega)+\tau,T_3(\varepsilon,\omega)+\tau,T_4(\varepsilon,\omega),T_5(\varepsilon,\omega)\},$$

and

$$N=\max\{N_1(\varepsilon,\omega),N_2(\varepsilon,\omega),N_3(\varepsilon,\omega)\}.$$

Then by combining (3.37), (3.40), (3.41) and (3.42), it follows that for $t>T>\tau$ and $M>N$, we have

$$\sum_{|i|>2M}(v_i^2(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))+v_i^2(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega)))<C\varepsilon.$$

Thus we have

$$
\begin{aligned}
&\sum_{|i|>2M}(|u_i(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))|^2+|v_i(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))|^2)\\
\le\ & 2\sum_{|i|>2M}(|v_i(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))|^2+|\varphi_i(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))|^2)\\
&+2\sum_{|i|>2M}(|y_{1i}(\theta_{-t}\omega)|^2+|y_{2i}(\theta_{-t}\omega)|^2)\\
<\ & C\varepsilon.
\end{aligned}
$$

Therefore, for $t > T(\varepsilon,\omega)+\tau$, we have $t+s > T(\varepsilon,\omega)$ for any $s\in[-\tau,0]$, and

$$\sum_{|i|>2M}(|u_{it}(s,\theta_{-t-s}\omega,v_\tau(\theta_{-t-s}\omega))|^2+|\varphi_i(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))|^2)<\varepsilon.$$

By replacing ω in above estimate with $\theta_s\omega$, we obtain

$$\sum_{|i|>2M}(|u_{it}(s,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))|^2+|\varphi_i(t,\theta_{-t}\omega,v_\tau(\theta_{-t}\omega))|^2)<\varepsilon.$$

This completes the proof. ∎

By Theorem 3.1, Lemmas 3.1 and 3.2, we have the following result.

Theorem 3.4. *Assumptions (H1)-(H2) hold. Then for P-a.e. $\omega\in\Omega$, RDS ϕ_t associated with equation (3.1) possesses a $\mathcal{D}$-random attractor.*

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