TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 3, pp. 1057-1077, June 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

CONVERGENCE THEOREMS FOR VARIATIONAL INEQUALITIES EQUILIBRIUM PROBLEMS AND NONEXPANSIVE MAPPINGS BY HYBRID METHOD

Shahram Saeidi

Abstract. In this paper, we introduce iterative schemes for finding a common element of the set of common fixed points for a left amenable semigroup of nonexpansive mappings, the set of solutions of the variational inequalities for a family of α-inverse-strongly monotone mappings and the set of solutions of a system of equilibrium problems in a Hilbert space. We establish weak and strong convergence theorems for the sequences generated by our proposed schemes. Moreover, we present various applications to the additive semigroup of nonnegative real numbers and families of strictly pseudocontractive mappings.

1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H. A mapping T of C into C is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $Fix(T)$ the set of fixed points of T.

T is *strictly pseudocontractive* if there exists κ with $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C.$$

If $k = 0$, then T is nonexpansive.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e., the set

$$EP(F) := \{x \in C : F(x, y) \geq 0, \ \forall y \in C\}.$$

Received February 12, 2010, accepted May 20, 2011.
Communicated by Wataru Takahashi.
2010 *Mathematics Subject Classification*: 47H09, 47H10, 47J20, 43A07, 47H20, 74G15.
Key words and phrases: Amenable semigroup, Equilibrium problem, Inverse-strongly monotone mapping, Iteration, Nonexpansive mapping, Projection.

Let $\mathcal{G} = \{F_i\}_{i\in I}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$. The system of equilibrium problems for $\mathcal{G} = \{F_i\}_{i\in I}$ is to determine common equilibrium points for $\mathcal{G} = \{F_i\}_{i\in I}$, i.e., the set

$$EP(\mathcal{G}) := \{x \in C : F_i(x, y) \geq 0,\ \forall y \in C,\ \forall i \in I\}. \tag{1.1}$$

Many problems in applied sciences, such as monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, as well as certain fixed point problems reduce into finding some element of $EP(F)$; see [3, 10, 11]. The formulation (1.1), extends this formalism to systems of such problems, covering in particular various forms of feasibility problems [2, 9].

Recall that a mapping $A : C \to H$ is called α-inverse-strongly monotone [4], if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2,\ \ \forall x, y \in C.$$

It is easy to see that if $A : C \to H$ is α-inverse-strongly monotone, then it is a $\frac{1}{\alpha}$-Lipschitzian mapping.

Let $A : C \to H$ be a mapping. The classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0,\ \ \forall v \in C. \tag{1.2}$$

The set of solutions of variational inequality (1.2) is denoted by $VI(C, A)$. Put $A = I - T$, where $T : C \to H$ is a strictly pseudocontractive mapping with κ. It is known that A is $\frac{1-\kappa}{2}$-inverse-strongly monotone and $A^{-1}(0) = Fix(T) = \{x \in C : Tx = x\}$.

Recently, weak and strong convergence theorems for finding a common element of $EP(F)$, $VI(C, A)$ and $Fix(T)$, have been studied by many authors (see e.g., [6, 7, 8, 12, 13, 15, 16, 18, 20, 21, 22, 25] and references therein).

In this paper, motivated by [18, 20], we introduce iterative algorithms for finding a common element of the set of common fixed points for a left amenable semigroup of nonexpansive mappings on C, the set of solutions of a system of equilibrium problems $EP(\mathcal{G})$ for a family $\mathcal{G} = \{F_i : i = 1, \ldots, M\}$ of bifunctions from $C\times C$ into $\mathbb{R}$ and the set of solutions of variational inequalities $VI(C, A_j)$ for a family $\{A_j : j = 1 \ldots N\}$ of α-inverse-strongly monotone mappings from C into H. We establish some weak and strong convergence theorems for the sequences generated by our proposed algorithms. We obtain our strong convergence results via the hybrid method; see [16]. Various applications to the additive semigroup of nonnegative real numbers and the families of strictly pseudocontractive mappings are also presented.

2. Preliminaries

Let C be a nonempty closed and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction. Given any $r > 0$ and $x \in H$, the operator $J_r^F : H \to C$ defined by

$$J_r^F(x) := \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}$$

is called the resolvent of F.

Lemma 2.1. ([10]). *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ satisfy*

$(A1)$ $F(x,x) = 0$ *for all* $x \in C$*;*

$(A2)$ F *is monotone, i.e.* $F(x,y) + F(y,x) \leq 0$ *for all* $x, y \in C$*;*

$(A3)$ *for all* $x, y, z \in C$,

$$\liminf_{t \to 0} F(tz + (1-t)x, y) \leq F(x,y);$$

$(A4)$ *for all* $x \in C$*,* $y \longmapsto F(x,y)$ *is convex and lower semicontinuous.*

Then:

(1) J_r^F *is single-valued;*

(2) J_r^F *is firmly nonexpansive, i.e.*

$$\|J_r^F x - J_r^F y\|^2 \leq \langle J_r^F x - J_r^F y, x - y\rangle,\ \forall x, y \in H;$$

(3) $Fix(J_r^F) = EP(F)$*;*

(4) $EP(F)$ *is closed and convex.*

Recall the metric (nearest point) projection P_C from a Hilbert space H onto a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is known that P_C is a nonexpansive mapping and satisfies:

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle,\ \ \forall x, y \in H. \tag{2.1}$$

P_C is characterized as follows:

$$y = P_C x \iff \langle x - y, y - z\rangle \geq 0,\ \ \forall z \in C.$$

In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \iff u = P_C(u - \lambda A u),\ \ \forall \lambda > 0. \tag{2.2}$$

A set-valued mapping $T : H \to 2^H$ is said to be monotone, if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply that $\langle f - g, x - y\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is said to be maximal, if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping is maximal, if and only if for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$, $\forall (y, g) \in G(T)$ imply that $f \in Tx$. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\},$$

and define

$$Tv = \begin{cases} Av + N_C v, & v \in C; \\ \varnothing, v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [13, 17]). It is easy to show that for given $\lambda \in [0, 2\alpha]$, the mapping $(I - \lambda A) : C \to H$ is nonexpansive.

The following lemma is well known; see, for instance, [24].

Lemma 2.2. *Let C be a closed convex subset of H and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \varnothing$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.3. ([22]). *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $y \in C$,*

$$\|x_{n+1} - y\| \leq \|x_n - y\|,$$

for every $n \in \mathbb{N}$. Then, $\{P_C(x_n)\}$ converges strongly to some $z \in C$.

Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real valued functions on S with supremum norm. For each $s \in S$, we define l_s and r_s on $l^\infty(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty(S)$. Let X be a subspace of $l^\infty(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e. $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $l^\infty(S)$ is amenable when S is a commutative semigroup or a solvable group. A net $\{\mu_\alpha\}$ of means on X is said to be left regular if $\lim_\alpha \|l_s^* \mu_\alpha - \mu_\alpha\| = 0$ for each $s \in S$, where l_s^* is the adjoint operator of l_s; see, for instance, [14].

Let C be a nonempty closed and convex subset of H. A family $\mathcal{S} = \{T(s) : s \in S\}$ is called a nonexpansive semigroup on C if for each $s \in S$ the mapping $T(s) : C \to C$ is nonexpansive and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $Fix(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$. For a nonexpansive mapping $T : C \to C$, we denote by $F_\varepsilon(T)$ the ε-approximate fixed points of T; i.e., $F_\varepsilon(T) = \{x \in C : \|x - Tx\| \leq \varepsilon\}$.

If C is bounded, then $F_\varepsilon(T) \neq \varnothing$, for each $\varepsilon > 0$ (see [24]). For $D \subset C$, we denote $F_\varepsilon(T; D) = F_\varepsilon(T) \cap D$.

The open ball of radius r centered at 0 is denoted by B_r. For a subset A of H, we denote by $\overline{co}A$ the closed convex hull of A.

The following lemmas can be found in [23, 14, 19].

Lemma 2.4. *Let f be a function of semigroup S into a Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let X be a subspace of $l^\infty(S)$ containing 1 and all the functions $t \to \langle f(t), x^*\rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$\langle f_\mu, x^*\rangle = \mu_t\langle f(t),\ x^*\rangle$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X then

$$\int f(t)d\mu(t) \in \overline{co}\{f(t) : t \in S\}.$$

We can write f_μ by $\int f(t)d\mu(t)$.

Lemma 2.5. *Let C be a closed convex subset of a Hilbert space H, $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup from C into C such that $Fix(\mathcal{S}) \neq \varnothing$ and let X be a subspace of $l^\infty(S)$ such that $1 \in X$ and let the mapping $t \mapsto \langle T(t)x, y\rangle$ be an element of X for each $x \in C$ and $y \in H$, and let μ be a mean on X.*

If we write $T(\mu)x$ instead of $\int T_t x d\mu(t)$, then the following hold.

(*i*) *$T(\mu)$ is a nonexpansive mapping from C into C.*

(*ii*) *$T(\mu)x = x$ for each $x \in Fix(\mathcal{S})$.*

(*iii*) *$T(\mu)x \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$.*

(*iv*) *If μ is left invariant, then $T(\mu)$ is a nonexpansive retraction from C onto $Fix(\mathcal{S})$.*

3. Strong Convergence

The following is our main strong convergence result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H, let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C, let $\mathcal{G} = \{F_j : j = 1, \ldots, M\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$ which satisfy $(A1)$-$(A4)$, let $\mathcal{A} = \{A_k : k = 1 \ldots N\}$ be a finite family of α-inverse-strongly monotone mappings from C into H, and let $\mathcal{F} := \cap_{k=1}^N VI(C, A_k) \cap Fix(\mathcal{S}) \cap EP(\mathcal{G}) \neq \varnothing$.*

Let X be a left invariant subspace of $l^\infty(S)$ such that $1 \in X$, and the function $t \longmapsto \langle T(t)x, y\rangle$ is an element of X for each $x \in C$ and $y \in H$; and let $\{\mu_n\}$ be a left regular sequence of means on X.

Let $\{\alpha_n\}$ *be a sequence in* $[a, 1]$ *for some* $a \in (0, 1)$*, let* $\{\lambda_{k,n}\}_{k=1}^N$ *be sequences in* $[c, d] \subset (0, 2\alpha)$ *and let* $\{r_{j,n}\}_{j=1}^M$ *be sequences in* $(0, \infty)$ *such that* $\liminf_n r_{j,n} > 0$ *for every* $j \in \{1, \ldots, M\}$*.*

If $\{x_n\}$ *is the sequence generated by* $x_1 = x \in H$*,*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T(\mu_n)v_n, \\ v_n = P_C(I - \lambda_{N,n}A_N)\ldots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)u_n, \\ u_n = J_{r_{M,n}}^{F_M}\ldots J_{r_{2,n}}^{F_2}J_{r_{1,n}}^{F_1}x_n, \\ C_n = \{z \in H : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x), \ \forall n \ge 1, \end{cases}$$

then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $P_{\mathcal{F}}(x)$*.*

Proof. Take

$$\mathcal{J}_n^k := J_{r_{k,n}}^{F_k}\ldots J_{r_{2,n}}^{F_2}J_{r_{1,n}}^{F_1}, \ \forall k \in \{1, \ldots, M\},$$

$$\mathcal{J}_n^0 := I,$$

and

$$\mathcal{P}_n^k := P_C(I - \lambda_{k,n}A_k)\ldots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1), \ \forall k \in \{1, \ldots, N\},$$

$$\mathcal{P}_n^0 := I.$$

So, we can write

$$y_n = (1-\alpha_n)x_n + \alpha_n T(\mu_n)\mathcal{P}_n^N\mathcal{J}_n^M x_n.$$

We shall divide the proof into several steps.

Step 1. The sequence $\{x_n\}$ is well defined.

Proof of Step 1. The sets C_n and Q_n are closed and convex subsets of H for every $n \in \mathbb{N}$; see [20]. So, $C_n \cap Q_n$ is a closed convex subset of H for any $n \in \mathbb{N}$. Let $p \in \mathcal{F}$. Since, for each $k \in \{1, \ldots, M\}$, $J_{r_{k,n}}^{F_k}$ is nonexpansive, and from Lemma 2.1 we have

$$\|u_n - p\| = \|\mathcal{J}_n^M x_n - p\| = \|\mathcal{J}_n^M x_n - \mathcal{J}_n^M p\| \le \|x_n - p\|. \tag{3.1}$$

On the other hand, since $A_k : C \to H$ is α-inverse-strongly monotone and $\lambda_{k,n} \in [c, d] \subset [0, 2\alpha]$, $P_C(I - \lambda_{k,n}A_k)$ is nonexpansive. Thus $\mathcal{P}_n^N$ is nonexpansive. From (2.2), we have $\mathcal{P}_n^N p = p$. It follows that

$$\|v_n - p\| = \|\mathcal{P}_n^N u_n - \mathcal{P}_n^N p\| \le \|u_n - p\| \le \|x_n - p\|. \tag{3.2}$$

So, we have

$$(3.3)\qquad \begin{aligned}\|y_n - p\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T(\mu_n)v_n - p)\|\\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|T(\mu_n)v_n - p\|\\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|v_n - p\|\\ &\le \|x_n - v\|.\end{aligned}$$

It follows that $p \in C_n$; thus, $\mathcal{F} \subset C_n$, for every $n \in \mathbb{N}$. Next, we show by induction that

$$\mathcal{F} \subset C_n \cap Q_n$$

for each $n \in \mathbb{N}$. Since $\mathcal{F} \subset C_1$ and $Q_1 = H$, we get $\mathcal{F} \subset C_1 \cap Q_1$. Suppose that $\mathcal{F} \subset C_k \cap Q_k$ for $k \in \mathbb{N}$. Then, there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x)$. Therefore, for each $z \in C_k \cap Q_k$, we have

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0.$$

So, we get

$$\mathcal{F} \subset C_k \cap Q_k \subset Q_{k+1}.$$

From this and $\mathcal{F} \subset C_n$ $(\forall n)$, we have

$$\mathcal{F} \subset C_{k+1} \cap Q_{k+1}.$$

This means that the sequence $\{x_n\}$ is well defined.

Step 2. The sequences $\{x_n\}$, $\{y_n\}$, $\{\mathcal{J}_n^k x_n\}_{k=1}^M$ and $\{\mathcal{P}_n^k u_n\}_{k=1}^N$ are bounded and

$$(3.4)\qquad \lim_{n\to\infty} \|x_n - x\| = c, \quad \text{for some } c \in \mathbb{R}.$$

Proof of Step 2. From $x_{n+1} = P_{C_n \cap Q_n}(x)$, we have

$$\|x_{n+1} - x\| \le \|z - x\|,\ \forall z \in C_n \cap Q_n.$$

Since $P_{\mathcal{F}}(x) \in \mathcal{F} \subset C_n \cap Q_n$, we have

$$(3.5)\qquad \|x_{n+1} - x\| \le \|P_{\mathcal{F}}(x) - x\|,$$

for every $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded. So, from (3.1), (3.2) and (3.3), the sequences $\{\mathcal{J}_n^k x_n\}_{k=1}^M$, $\{\mathcal{P}_n^k u_n\}_{k=1}^N$ and $\{y_n\}$ are also bounded.

It is easy to show that $x_n = P_{Q_n}(x)$. From this and $x_{n+1} \in Q_n$, we have

$$\|x - x_n\| \le \|x - x_{n+1}\|,$$

for every $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, there exists $c \in \mathbb{R}$ such that (3.4) holds.

Step 3. $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Proof of Step 3. Since $x_n = P_{Q_n}(x)$, $x_{n+1} \in Q_n$ and $(x_n + x_{n+1})/2 \in Q_n$, we have

$$\begin{aligned}\|x - x_n\|^2 &\le \|x - \frac{x_n + x_{n+1}}{2}\|^2 \\ &= \|\frac{1}{2}(x - x_n) + \frac{1}{2}(x - x_{n+1}\|^2 \\ &= \frac{1}{2}\|x - x_n\|^2 + \frac{1}{2}\|x - x_{n+1}\|^2 - \frac{1}{4}\|x_n - x_{n+1}\|^2.\end{aligned}$$

So, we get

$$\frac{1}{4}\|x_n - x_{n+1}\|^2 \le \frac{1}{2}\|x - x_{n+1}\|^2 - \frac{1}{2}\|x - x_n\|^2.$$

From (3.4), we obtain $\lim_{n\to\infty} \|x_n - x_{n+1}\|^2 = 0$.

Step 4. $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof of Step 4. From $x_{n+1} \in C_n$, we have

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \le 2\|x_n - x_{n+1}\|.$$

Now, apply Step 3.

Step 5. $\lim_{n\to\infty} \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\| = 0$, $\forall k \in \{0, 1, \ldots, M-1\}$.

Proof of Step 5. Let $p \in \mathcal{F}$ and $k \in \{0, 1, \ldots, M-1\}$. Since $J_{r_{k+1,n}}^{F_{k+1}}$ is firmly nonexpansive, we obtain

$$\begin{aligned}&\|p - \mathcal{J}_n^{k+1} x_n\|^2 \\ &= \|J_{r_{k+1,n}}^{F_{k+1}} p - J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n\|^2 \\ &\le \langle J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n - p, \mathcal{J}_n^k x_n - p\rangle \\ &= \frac{1}{2}(\|J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n - p\|^2 + \|\mathcal{J}_n^k x_n - p\|^2 - \|\mathcal{J}_n^k x_n - J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n\|^2).\end{aligned}$$

It follows that

$$\|\mathcal{J}_n^{k+1} x_n - p\|^2 \le \|x_n - p\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2.$$

Therefore, by the convexity of $\|.\|^2$, we have

$$\begin{aligned}&\|y_n - p\|^2 \le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|T(\mu_n)v_n - p\|^2 \\ &\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|\mathcal{J}_n^{k+1} x_n - p\|^2 \\ &\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n(\|x_n - p\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2) \\ &= \|x_n - p\|^2 - \alpha_n\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2.\end{aligned}$$

Since $\{\alpha_n\} \subset [a,1]$, we get

$$a\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2 \le \alpha_n \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2$$
$$\le \|x_n - p\|^2 - \|y_n - p\|^2 \le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).$$

From this and Step 4, we get the desired result.

Step 6. $\lim_{n\to\infty} \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\| = 0$, $\forall k \in \{0, 1, \ldots, N-1\}$.

Proof of Step 6. Since $\{A_k : k = 1 \ldots N\}$ are α-inverse-strongly monotone, by the the choice of $\{\lambda_{k,n}\}$ for given $p \in \mathcal{F}$ and $k \in \{0, 1, \ldots, N-1\}$ we have

$$\begin{aligned}
&\|\mathcal{P}_n^{k+1} u_n - p\|^2 \\
&= \|P_C(I - \lambda_{k+1,n} A_{k+1})\mathcal{P}_n^k u_n - P_C(I - \lambda_{k+1,n} A_{k+1})p\|^2 \\
&\le \|(I - \lambda_{k+1,n} A_{k+1})\mathcal{P}_n^k u_n - (I - \lambda_{k+1,n} A_{k+1})p\|^2 \\
&\le \|\mathcal{P}_n^k u_n - p\|^2 + \lambda_{k+1,n}(\lambda_{k+1,n} - 2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2 \\
&\le \|x_n - p\|^2 + c(d - 2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2.
\end{aligned}$$

From this, we have

$$\begin{aligned}
\|y_n - p\|^2 &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|T(\mu_n)\mathcal{P}_n^N u_n - p\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|\mathcal{P}_n^{k+1} u_n - p\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n(\|x_n - p\|^2 + c(d-2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2) \\
&= \|x_n - p\|^2 + c(d - 2\alpha)\alpha_n \|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2.
\end{aligned}$$

So,

$$c(2\alpha - d)\alpha_n \|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2 \le \|x_n - p\|^2 - \|y_n - p\|^2$$
$$\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).$$

Since $\alpha_n \subset [a, 1]$, we obtain

$$\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\| \to 0 \ \ (n \to \infty). \tag{3.6}$$

From (2.1) and the fact that $I - \lambda_{k+1,n} A_{k+1}$ is nonexpansive, we have

$$\begin{aligned}
&\|\mathcal{P}_n^{k+1} u_n - p\|^2 \\
&= \|P_C(I - \lambda_{k+1,n} A_{k+1})\mathcal{P}_n^k u_n - P_C(I - \lambda_{k+1,n} A_{k+1})p\|^2 \\
&\le \langle (\mathcal{P}_n^k u_n - \lambda_{k+1,n} A_{k+1}\mathcal{P}_n^k u_n) - (p - \lambda_{k+1,n} A_{k+1}p), \mathcal{P}_n^{k+1} u_n - p\rangle \\
&= \frac{1}{2}\{\|(\mathcal{P}_n^k u_n - \lambda_{k+1,n} A_{k+1}\mathcal{P}_n^k u_n) - (p - \lambda_{k+1,n} A_{k+1}p)\|^2 + \|\mathcal{P}_n^{k+1} u_n - p\|^2 \\
&\quad - \|(\mathcal{P}_n^k u_n - \lambda_{k+1,n} A_{k+1}\mathcal{P}_n^k u_n) - (p - \lambda_{k+1,n} A_{k+1}p) - (\mathcal{P}_n^{k+1} u_n - p)\|^2\}
\end{aligned}$$

$$\le \frac{1}{2}\{\|\mathcal{P}_n^k u_n - p\|^2 + \|\mathcal{P}_n^{k+1} u_n - p\|^2$$
$$-\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n - \lambda_{k+1,n}(A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p)\|^2\}$$
$$= \frac{1}{2}\{\|\mathcal{P}_n^k u_n - p\|^2 + \|\mathcal{P}_n^{k+1} u_n - p\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2$$
$$+2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\rangle$$
$$-\lambda_{k+1,n}^2\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2\}.$$

This implies that

$$\begin{aligned}\|\mathcal{P}_n^{k+1} u_n - p\|^2 &\le \|\mathcal{P}_n^k u_n - p\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 \\ &\quad +2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\rangle \\ &\quad -\lambda_{k+1,n}^2\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2 \\ &\le \|x_n - p\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 \\ &\quad +2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\rangle.\end{aligned}$$

From this, we have

$$\begin{aligned}\|y_n - p\|^2 &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|\mathcal{P}_n^{k+1} u_n - p\|^2 \\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\{\|x_n - p\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 \\ &\quad +2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\rangle\} \\ &\le \|x_n - p\|^2 - \alpha_n\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 \\ &\quad +2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|,\end{aligned}$$

which implies that

$$\begin{aligned}a\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 \le \alpha_n\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\quad +2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|.\end{aligned}$$

Hence it follows from Step 4 and (3.6) that $\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\| \to 0$.

Step 7. $\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0$, $\forall t \in S$.

Proof of Step 7. Let $p \in \mathcal{F}$ and put

$$M_0 = \max\{\|x_n - p\| : n \in \mathbb{N}\}.$$

Set $D = \{y \in C : \|y - p\| \le M_0\}$. We remark D is a bounded closed convex set, $\{x_n\} \subseteq D$ and it is invariant under the mappings $\{\mathcal{J}_n^k : k = 1, \ldots, M$ and $n \in \mathbb{N}\}$, $\{\mathcal{P}_n^k : k = 1, \ldots, N$ and $n \in \mathbb{N}\}$ and $\mathcal{S}$. We will show that

$$\limsup_{n\to\infty}\sup_{y\in D}\|T(\mu_n)y - T(t)T(\mu_n)y\| = 0, \ \ \forall t\in S. \tag{3.7}$$

Our proof of (3.7) follows the lines of a proof in [1]. Let $\varepsilon > 0$. By [5, Theorem 1.2], there exists $\delta > 0$ such that

$$\{\overline{co}F_\delta(T(t);D) + B_\delta\}\cap C \subseteq F_\varepsilon(T(t);C),\ \forall t\in S. \tag{3.8}$$

By [5, Corollary 1.1], there also exists a natural number N such that

$$\|\frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y - T(t)(\frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y)\| \le \delta, \tag{3.9}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is left regular, there exists $n_0 \in \mathbb{N}$ such that $\|\mu_n - l^*_{t^i}\mu_n\| \le \delta/(M_0 + \|p\|)$ for $n \ge n_0$ and $i = 1,\ldots,N$. Then we have

$$\begin{aligned}
&\sup_{y\in D}\|T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y d\mu_n(s)\|\\
&= \sup_{y\in D}\sup_{\|z\|=1}|(\mu_n)_s\langle T(s)y, z\rangle - (\mu_n)_s\langle\frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y, z\rangle|\\
&\le \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}|(\mu_n)_s\langle T(s)y,z\rangle - (l^*_{t^i}\mu_n)_s\langle T(s)y, z\rangle|\\
&\le \max_{i=1,\ldots,N}\|\mu_n - l^*_{t^i}\mu_n\|(M_0 + \|p\|) \le \delta,\ \forall n \ge n_0.
\end{aligned}\tag{3.10}$$

On the other hand, noting Lemma 2.4, we have

$$\int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y d\mu_n(s) \in \overline{co}\{\frac{1}{N+1}\sum_{i=0}^{N} T(t)^i(T(s)y) : s\in S\}.$$

From (3.8), (3.9), (3.10) and the above, we have

$$\begin{aligned}
T(\mu_n)y &= \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y d\mu_n(s) + (T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y d\mu_n(s))\\
&\in \{\overline{co}\{\frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y : s\in S\} + B_\delta\}\cap C\\
&\subseteq \{\overline{co}F_\delta(T(t);D) + B_\delta\}\cap C \subseteq F_\varepsilon(T(t);C),
\end{aligned}$$

for all $y \in D$ and $n \ge n_0$. Therefore

$$\limsup_{n}\sup_{y\in D}\|T(t)T(\mu_n)y - T(\mu_n)y\| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get (3.7).

Now, observe that

$$a\|x_n - T(\mu_n)v_n\| \leq \alpha_n\|x_n - T(\mu_n)v_n\|$$
$$= \|x_n - y_n\| \to 0, \text{ as } n \to \infty.$$

So,

$$\|x_n - T(\mu_n)v_n\| \to 0, \text{ as } n \to \infty. \quad (3.11)$$

Let $t \in S$ and $\varepsilon > 0$. Then there exists $\delta > 0$ which satisfies (3.8). From (3.7) and (3.11), there exists $k_0 \in \mathbb{N}$ such that $T(\mu_n)y \in F_\delta(T(t); D)$, $\forall y \in D$, and $\|x_n - T(\mu_n)v_n\| < \delta$ for all $n > k_0$. Therefore,

$$\begin{aligned} x_n &= T(\mu_n)v_n + (x_n - T(\mu_n)v_n) \\ &\in \{F_\delta(T(t); D) + B_\delta\} \cap C \subseteq F_\varepsilon(T(t); C), \end{aligned}$$

for all $n > k_0$. This shows that $\limsup_n \|x_n - T(t)x_n\| \leq \varepsilon$, and since $\varepsilon > 0$ is arbitrary, we get $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$.

Step 8. The weak ω-limit set of $\{x_n\}$, $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

Proof of Step 8. Let $z_0 \in \omega_w(x_n)$ and let $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z_0. From Steps 5 and 6, we obtain also that

$$\mathcal{J}^k_{n_m} x_{n_m} \rightharpoonup z_0,$$

for all $k \in \{1, \ldots, M\}$, and

$$\mathcal{P}^k_{n_m} u_{n_m} \rightharpoonup z_0,$$

for all $k \in \{1, \ldots, N\}$. In particular, $u_{n_m} \rightharpoonup z_0$ and $v_{n_m} \rightharpoonup z_0$. We need to show that $z_0 \in \mathcal{F}$. First, we note from $x_{n_m} \rightharpoonup z_0$ and Lemma 2.2, $z_0 \in Fix(\mathcal{S})$.

Now, we prove $z_0 \in \cap_{i=1}^N VI(C, A_i)$. For this purpose, let $k \in \{1, \ldots, N\}$ and T_k be the maximal monotone mapping defined by

$$T_k x = \begin{cases} A_k z + N_C z, & z \in C; \\ \varnothing, & z \notin C. \end{cases}$$

For any given $(z, u) \in G(T_k)$, hence $u - A_k z \in N_C z$. Since $\mathcal{P}^k_n u_n \in C$, by the definition of N_C, we have

$$\langle z - \mathcal{P}^k_n u_n, u - A_k z\rangle \geq 0. \tag{3.12}$$

On the other hand, since $\mathcal{P}^k_n u_n = P_C(\mathcal{P}^{k-1}_n u_n - \lambda_{k,n} A_k \mathcal{P}^{k-1}_n u_n)$, we have

$$\langle z - \mathcal{P}^k_n u_n, \mathcal{P}^k_n u_n - (\mathcal{P}^{k-1}_n u_n - \lambda_{k,n} A_k \mathcal{P}^{k-1}_n u_n)\rangle \geq 0.$$

So

$$\langle z - \mathcal{P}_n^k u_n, \frac{\mathcal{P}_n^k u_n - \mathcal{P}_n^{k-1} u_n}{\lambda_{k,n}} + A_k \mathcal{P}_n^{k-1} u_n \rangle \geq 0.$$

By (3.12) and the α-inverse monotonicity, we have

$$\begin{aligned}
&\langle z - \mathcal{P}_{n_m}^k u_{n_m}, u\rangle \geq \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k z\rangle \\
\geq\ & \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k z\rangle \\
&-\langle z - \mathcal{P}_{n_m}^k u_{n_m}, \frac{\mathcal{P}_{n_m}^k u_{n_m} - \mathcal{P}_{n_m}^{k-1} u_{n_m}}{\lambda_{k,n_m}} + A_k \mathcal{P}_{n_m}^{k-1} u_{n_m}\rangle \\
=\ & \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k z - A_k \mathcal{P}_{n_m}^k u_{n_m}\rangle \\
&+\langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k \mathcal{P}_{n_m}^k u_{n_m} - A_k \mathcal{P}_{n_m}^{k-1} u_{n_m}\rangle \\
&-\langle z - \mathcal{P}_{n_m}^k u_{n_m}, \frac{\mathcal{P}_{n_m}^k u_{n_m} - \mathcal{P}_{n_m}^{k-1} u_{n_m}}{\lambda_{k,n_m}}\rangle \\
\geq\ & \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k \mathcal{P}_{n_m}^k u_{n_m} - A_k \mathcal{P}_{n_m}^{k-1} u_{n_m}\rangle \\
&-\langle z - \mathcal{P}_{n_m}^k u_{n_m}, \frac{\mathcal{P}_{n_m}^k u_{n_m} - \mathcal{P}_{n_m}^{k-1} u_{n_m}}{\lambda_{k,n_m}}\rangle.
\end{aligned}$$

Since $\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n\| \to 0$, $\mathcal{P}_{n_m}^k u_{n_m} \rightharpoonup z_0$ and $\{A_k : k = 1, \ldots, N\}$ are Lipschitz continuous, we have

$$\lim_{m\to\infty} \langle z - \mathcal{P}_{n_m}^k u_{n_m}, u\rangle = \langle z - z_0, u\rangle \geq 0.$$

Again since T_k is maximal monotone, hence $0 \in T_k z_0$. This shows that $z_0 \in VI(C, A_k)$. From this, it follows that

$$z_0 \in \cap_{i=1}^N VI(C, A_i).$$

Now, we note that by (A2) for given $y \in C$ and $k \in \{0, 1, \ldots, M-1\}$, we have

$$\frac{1}{r_{k+1,n}} \langle y - \mathcal{J}_n^{k+1} x_n, \mathcal{J}_n^{k+1} x_n - \mathcal{J}_n^k x_n\rangle \geq F_{k+1}(y, \mathcal{J}_n^{k+1} x_n).$$

Thus

$$\langle y - \mathcal{J}_{n_m}^{k+1} x_{n_m}, \frac{\mathcal{J}_{n_m}^{k+1} x_{n_m} - \mathcal{J}_{n_m}^k x_{n_m}}{r_{k+1,n_m}}\rangle \geq F_{k+1}(y, \mathcal{J}_{n_m}^{k+1} x_{n_m}). \tag{3.13}$$

By condition (A4), $F_i(y, .)$, $\forall i$, is lower semicontinuous and convex, and thus weakly semicontinuous. Step 5 and condition $\liminf_n r_{j,n} > 0$ imply that

$$\frac{\mathcal{J}_{n_m}^{k+1} x_{n_m} - \mathcal{J}_{n_m}^k x_{n_m}}{r_{k+1,n_m}} \to 0,$$

in norm. Therefore, letting $m \to \infty$ in (3.13) yields

$$F_{k+1}(y, z_0) \leq \lim_m F_{k+1}(y, \mathcal{J}_{n_m}^{k+1} x_{n_m}) \leq 0,$$

for all $y \in C$ and $k \in \{0, 1, \dots, M-1\}$. Replacing y with $y_t := ty + (1-t)z_0$ with $t \in (0,1)$ and using (A1) and (A4), we obtain

$$0 = F_{k+1}(y_t, y_t) \leq tF_{k+1}(y_t, y) + (1-t)F_{k+1}(y_t, z_0) \leq tF_{k+1}(y_t, y).$$

Hence $F_{k+1}(ty + (1-t)z_0, y) \geq 0$, for all $t \in (0,1)$ and $y \in C$. Letting $t \to 0^+$ and using (A3), we conclude $F_{k+1}(z_0, y) \geq 0$, for all $y \in C$ and $k \in \{0, \dots, M-1\}$. Therefore

$$z_0 \in \bigcap_{k=1}^{M} EP(F_k) = EP(\mathcal{G}).$$

Step 9. The sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{\mathcal{F}}(x)$.

Proof of Step 9. Let $z_0 \in \omega_w(x_n)$ and let $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z_0. From (3.5) and Step 8, we have

$$\begin{aligned} \|x - P_{\mathcal{F}}(x)\| &\leq \|x - z_0\| \leq \liminf_{m\to\infty} \|x - x_{n_m}\| \\ &\leq \limsup_{m\to\infty} \|x - x_{n_m}\| \leq \|x - P_{\mathcal{F}}(x)\|. \end{aligned}$$

Hence

$$\lim_{m\to\infty} \|x - x_{n_m}\| = \|x - z_0\| = \|x - P_{\mathcal{F}}(x)\|.$$

Since $z_0 \in \mathcal{F}$ and H is a Hilbert space, we obtain

$$x_{n_m} \longrightarrow z_0 = P_{\mathcal{F}}(x).$$

Since $z_0 \in \omega_w(x_n)$ was arbitrary, we get $x_n \longrightarrow P_{\mathcal{F}}(x)$. ■

4. Weak Convergence

The following is a weak convergence theorem.

Theorem 4.1. *Let C, $\mathcal{S}$, $\mathcal{G}$, $\mathcal{A}$, $\mathcal{F}$, X, $\{\mu_n\}$, $\{\lambda_{k,n}\}_{k=1}^N$ and $\{r_{n,j}\}_{j=1}^M$ be as in Theorem 3.1. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a, b \in (0,1)$.*

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \geq 1$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(\mu_n)v_n, \\ v_n = P_C(I - \lambda_{N,n}A_N)\dots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)u_n, \\ u_n = J_{r_{M,n}}^{F_M} \dots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \end{cases}$$

then the sequence $\{x_n\}$ converges weakly to $z_0 \in \mathcal{F}$, where $z_0 = \lim_{n\to\infty} P_{\mathcal{F}}(x_n)$.

Proof. We will apply the notations used in proof of Theorem 3.1.

Step 1. $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in \mathcal{F}$.

Proof of Step 1. Let $p \in \mathcal{F}$. Then

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T(\mu_n)v_n - p)\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|T(\mu_n)v_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|v_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{4.1}$$

From this, we obtain that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty}\|x_n - p\| \text{ exists.} \tag{4.2}$$

Step 2. $\lim_{n\to\infty}\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1}x_n\| = 0$, $\forall k \in \{0, 1, \ldots, M-1\}$.

Proof of Step 2. For $p \in \mathcal{F}$, as in Step 5 of Theorem 3.1, we get

$$\|\mathcal{J}_n^{k+1}x_n - p\|^2 \leq \|x_n - p\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1}x_n\|^2,$$

for all $k \in \{0, 1, \ldots, M-1\}$. Therefore, by (4.1), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|v_n - p\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|\mathcal{J}_n^{k+1}x_n - p\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\{\|x_n - p\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1}x_n\|^2\} \\ &\leq \|x_n - p\|^2 - a\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1}x_n\|^2. \end{aligned}$$

Applying (4.2), we have

$$\begin{aligned} &a\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1}x_n\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0. \end{aligned}$$

So, we get the desired result.

Step 3. $\lim_{n\to\infty}\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\| = 0$, $\forall k \in \{0, 1, \ldots, N-1\}$.

Proof of Step 3. For $p \in \mathcal{F}$ and $k \in \{0, 1, \ldots, N-1\}$, as in Step 6 of Theorem 3.1, we get

$$\|\mathcal{P}_n^{k+1}u_n - p\|^2 \leq \|x_n - p\|^2 + c(d-2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2.$$

From this and (4.1), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|v_n-p\|^2\\ &\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|\mathcal{P}_n^{k+1}u_n-p\|^2\\ &\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\{\|x_n-p\|^2+c(d-2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|^2\}\\ &= \|x_n-p\|^2+c(d-2\alpha)\alpha_n\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|^2.\end{aligned}$$

So,

$$\begin{aligned}&c(2\alpha-d)\alpha_n\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|^2\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2\to 0.\end{aligned}$$

Since $0<a\le\alpha_n$, we obtain

$$\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|\to 0 \ \ (n\to\infty). \tag{4.3}$$

Again, like that in Step 6 of Theorem 3.1, we have

$$\begin{aligned}\|\mathcal{P}_n^{k+1}u_n-p\|^2 \le\ &\|x_n-p\|^2-\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|^2\\ &+2\lambda_{k+1,n}\langle\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\rangle.\end{aligned}$$

Then, from this and (4.1), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|\mathcal{P}_n^{k+1}u_n-p\|^2\\ &\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\{\|x_n-p\|^2-\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|^2\\ &\quad+2\lambda_{k+1,n}\langle\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\rangle\}\\ &\le \|x_n-p\|^2-a\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|^2\\ &\quad+2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|,\end{aligned}$$

which implies that

$$\begin{aligned}a\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|^2 \le\ &\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &+2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|.\end{aligned}$$

Hence it follows from Step 1 and (4.3) that $\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|\to 0$.

Step 4. $\lim_{n\to\infty}\|x_n-T(\mu_n)v_n\|=0$.

Proof of Step 4. Note that for $p\in\mathcal{F}$ we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)(x_n-p)+\alpha_n(T(\mu_n)v_n-p)\|^2\\ &= (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|T(\mu_n)v_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-T(\mu_n)v_n\|^2\\ &\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|v_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-T(\mu_n)v_n\|^2\\ &\le \|x_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-T(\mu_n)v_n\|^2.\end{aligned}$$

So, from (4.2), we have

$$a(1-b)\|x_n - T(\mu_n)v_n\|^2 \le \alpha_n(1-\alpha_n)\|x_n - T(\mu_n)v_n\|^2$$
$$\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0, \text{ as } n \to \infty.$$

Now, from $a, b \in (0,1)$ we obtain

$$\|x_n - T(\mu_n)v_n\| \to 0, \text{ as } n \to \infty.$$

Step 5. $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$, for all $t \in S$.

Proof of Step 5. The proof is the same as Step 7 of Theorem 3.1; the only difference is that the assertion (3.11) in that proof follows here from Step 4.

Step 6. $\{x_n\}$ converges weakly to $z_0 \in \mathcal{F}$, where $z_0 = \lim_{n\to\infty} P_{\mathcal{F}}(x_n)$.

Proof of Step 6. Applying Steps 2, 3 and 5, by a proof similar to Step 8 of Theorem 3.1, we can show that the weak ω-limit set of $\{x_n\}$, $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

Now, (4.2) and the Opial's property of Hilbert space imply that $\omega_w(x_n)$ is singleton. Therefore, $x_n \rightharpoonup z_0$ for some $z_0 \in \mathcal{F}$.

Let $z_n = P_{\mathcal{F}}(x_n)$. Since $z_0 \in \mathcal{F}$, we have

$$\langle x_n - z_n, z_n - z_0 \rangle \ge 0.$$

Using $\|x_{n+1} - z_0\| \le \|x_n - z_0\|$ $(\forall n \in \mathbb{N})$ and Lemma 2.3, we have that $\{z_n\}$ converges strongly to some $y_0 \in \mathcal{F}$. Since $x_n \rightharpoonup z_0$, we have

$$\langle z_0 - y_0, y_0 - z_0 \rangle \ge 0.$$

Therefore, we obtain $z_0 = y_0 = \lim_{n\to\infty} P_{\mathcal{F}}(x_n)$. ∎

5. Applications

In this section, we deduce algorithms for a finite family of (non-self) strictly pseudocontractive mappings, as an application of the proposed algorithms. Moreover, we present various applications to the additive semigroup of nonnegative real numbers.

Corollary 5.1. *Let C, $\mathcal{S}$, $\mathcal{G}$, X, $\{\mu_n\}$, $\{\alpha_n\}$ and $\{r_{n,j}\}_{j=1}^M$ be as in Theorem 3.1. Let $\psi = \{T_j : j = 1 \ldots N\}$ be a finite family of strictly pseudocontractive mappings with $0 \le \kappa < 1$ from C into C such that $\mathcal{F} := Fix(\mathcal{S}) \cap Fix(\psi) \cap EP(\mathcal{G}) \neq \varnothing$ and $\{\lambda_{k,n}\}_{k=1}^N$ be sequences in $[c,d] \subset (0, 1-\kappa)$.*

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \ge 1$,

$$\begin{cases} y_n &= (1-\alpha_n)x_n + \alpha_n T(\mu_n)v_n, \\ v_n &= ((1-\lambda_{N,n})I+\lambda_{N,n}T_N)\dots((1-\lambda_{2,n})I+\lambda_{2,n}T_2)((1-\lambda_{1,n})I+\lambda_{1,n}T_1)u_n, \\ u_n &= J^{F_M}_{r_{M,n}}\dots J^{F_2}_{r_{2,n}}J^{F_1}_{r_{1,n}}x_n, \\ C_n &= \{z\in H : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z\in H : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n\cap Q_n}(x), \end{cases}$$

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{\mathcal{F}}(x)$.

Proof. Put $A_j = I - T_j$ for every $j \in \{1,\dots,N\}$. Then A_j is $\frac{1-k}{2}$-inverse-strongly monotone. We have that $Fix(T_j)$ is the solution set of $VI(C, A_j)$; i.e., $Fix(T_j) = VI(C, A_j)$. Therefore, $Fix(\psi) = \cap_{k=1}^N VI(C, A_k)$ and it suffices to apply Theorem 3.1. ■

Corollary 5.2. *Let C, $\mathcal{S}$, $\mathcal{G}$, X, $\{\mu_n\}$, $\{\alpha_n\}$ and $\{r_{n,j}\}_{j=1}^M$ be as in Theorem 4.1. Let $\psi = \{T_j : j = 1\dots N\}$ be a finite family of strictly pseudocontractive mappings with $0 \le \kappa < 1$ from C into C such that $\mathcal{F} := Fix(\mathcal{S}) \cap Fix(\psi) \cap EP(\mathcal{G}) \ne \varnothing$ and $\{\lambda_{k,n}\}_{k=1}^N$ be sequences in $[c,d] \subset (0, 1-\kappa)$.*

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \ge 1$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(\mu_n)v_n, \\ v_n = ((1-\lambda_{N,n})I+\lambda_{N,n}T_N)\dots((1-\lambda_{2,n})I+\lambda_{2,n}T_2)((1-\lambda_{1,n})I+\lambda_{1,n}T_1)u_n, \\ u_n = J^{F_M}_{r_{M,n}}\dots J^{F_2}_{r_{2,n}}J^{F_1}_{r_{1,n}}x_n, \end{cases}$$

then the sequence $\{x_n\}$ converges weakly to $z_0 \in \mathcal{F}$, where $z_0 = \lim_{n\to\infty} P_{\mathcal{F}}(x_n)$.

Remark 5.3. We may put

$$v_n = P_C(I - \lambda_{N,n}(I - T_N))\dots P_C(I - \lambda_{2,n}(I - T_2))P_C(I - \lambda_{1,n}(I - T_1))u_n,$$

in the schemes of Corollaries 4.1 and 4.2, and obtain schemes for families of non-self strictly pseudocontractive mappings.

Corollary 5.4. *Let C, $\mathcal{G}$, $\mathcal{A}$, $\{\alpha_n\}$, $\{r_{n,j}\}_{j=1}^M$ and $\{\lambda_{k,n}\}_{k=1}^N$ be as in Theorem 3.1. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings from C into C and $\mathcal{F} := \cap_{k=1}^N VI(C, A_k) \cap Fix(\mathcal{S}) \cap EP(\mathcal{G}) \ne \varnothing$.*

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$,

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(s)v_n ds, \\ v_n = P_C(I - \lambda_{N,n}A_N)\dots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)u_n, \\ u_n = J^{F_M}_{r_{M,n}}\dots J^{F_2}_{r_{2,n}}J^{F_1}_{r_{1,n}}x_n, \\ C_n = \{z\in H : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z\in H : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x),\ \forall n \ge 1, \end{cases}$$

where $\{t_n\}$ is an increasing sequence in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$, then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{\mathcal{F}}(x)$.

Proof. For $n \in \mathbb{N}$, define $\mu_n(f) = \frac{1}{t_n}\int_0^{t_n} f(t)dt$ for each $f \in C(\mathbb{R}_+)$, where $C(\mathbb{R}_+)$ denotes the space of all real valued bounded continuous functions on $\mathbb{R}_+$ with supremum norm. Then, $\{\mu_n\}$ is a regular sequence of means [24]. Further, for each $x \in C$, we have $T(\mu_n)x = \frac{1}{t_n}\int_0^{t_n} T(s)xds$. Now, apply Theorem 3.1 to conclude the result. ∎

Corollary 5.5. *Let C, $\mathcal{G}$, $\mathcal{A}$, $\{\alpha_n\}$, $\{r_{n,j}\}_{j=1}^M$ and $\{\lambda_{k,n}\}_{k=1}^N$ be as in Theorem 4.1. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings from C into C and $\mathcal{F} := \cap_{k=1}^N VI(C, A_k) \cap Fix(\mathcal{S}) \cap EP(\mathcal{G}) \neq \varnothing$.*

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n a_n \displaystyle\int_0^\infty \exp(-a_n s)T(s)v_n ds, \\ v_n = P_C(I - \lambda_{N,n}A_N)\dots P_C(I-\lambda_{2,n}A_2)P_C(I-\lambda_{1,n}A_1)u_n, \\ u_n = J_{r_{M,n}}^{F_M}\dots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \ \forall n \geq 1, \end{cases}$$

where $\{a_n\}$ is a decreasing sequence in $(0, \infty)$ such that $\lim_{n\to\infty} a_n = 0$, then the sequence $\{x_n\}$ converges weakly to $z_0 \in \mathcal{F}$, where $z_0 = \lim_{n\to\infty} P_{\mathcal{F}}(x_n)$.

Proof. For each $n \in \mathbb{N}$, define $\mu_n(f) = a_n \int_0^\infty \exp(-a_n t)f(t)dt$ for each $f \in C(\mathbb{R}_+)$. Then, $\{\mu_n\}$ is a regular sequence of means [24]. Further, for each $x \in C$, we have $T(\mu_n)x = r_n \int_0^\infty \exp(-r_n t)T(t)xdt$. Now, apply Theorem 4.1 to conclude the result. ∎

Acknowledgments

The author would like to thank the referee for some useful comments. This research was in part supported by a grant from IPM (No. 89470019).

References

1. S. Atsushiba and W. Takahashi, Approximation common fixed points of nonexpansive semigroups by the Mann iteration process, *Ann. Univ. Mariae Curie-Sklodowska*, **51** (1997), 1-16.

2. H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.*, **38** (1996), 367-426.

3. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63** (1994), 123-145.

4. F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.*, **20** (1967), 197-228.

5. R. E. Bruck, On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, *Israel J. Math.*, **38** (1981), 304-314.

6. L. C. Ceng, S. Al-Homidan, Q. H. Ansari and J. C. Yao, An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings, *J. Comput. Appl. Math.*, **223** (2009), 967-974.

7. L. C. Ceng, H. K. Xu and J. C. Yao, A hybrid steepest-descent method for variational inequalities in Hilbert spaces, *Appl. Anal.*, **87** (2008), 575-589.

8. V. Colao, G. Marino and H. K. Xu, An iterative method for finding common solutions of equilibrium and fixed point problems, *J. Math. Anal. Appl.*, **344** (2008), 340-352.

9. P. L. Combettes, Quasi-Fejerian analysis of some optimization algorithms, in: *Inherently Parallel Algorithms for Feasibility and Optimization*, (D. Butnariu, Y. Censor, and S. Reich, eds.), Elsevier, New York, 2001, pp. 115-152.

10. P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, **6** (2005), 117-136.

11. A. Gopfert, H. Riahi, C. Tammer and C. Zalinescu, *Variational Methods in Partially Ordered Spaces*, Springer-Verlag, New York, 2003.

12. H. Iiduka and W. Takahashi, Strong convergence studied by a hybrid type method for monotone operators in a Banach space, *Nonlinear Anal.*, **68** (2008), 3679-3688.

13. H. Iiduka and W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, *Nonlinear Anal.*, **61** (2005), 341-350.

14. A. T. Lau, N. Shioji and W. Takahashi, Existences of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces, *J. Funct. Anal.*, **161** (1999), 62-75.

15. G. Marino and H. K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, *J. Math. Anal. Appl.*, **329** (2007), 336-346.

16. K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, **279** (2003), 372-379.

17. R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.*, **149** (1970), 75-88.

18. S. Saeidi, Iterative algorithms for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of families and semigroups of nonexpansive mappings, *Nonlinear Anal.*, **70** (2009), 4195-4208.

19. S. Saeidi, Existence of ergodic retractions for semigroups in Banach spaces, *Nonlinear Anal.*, **69** (2008), 3417-3422.

20. A. Tada and W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem, *J. Optim. Theory Appl.*, **133** (2007), 359-370.

21. S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.*, **331** (2007), 506-515.

22. W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.*, **113** (2003), 417-428.

23. W. Takahashi, A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space, *Proc. Amer. Math. Soc.*, **81** (1981), 253-256.

24. W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

25. H. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces, *Nonlinear Anal.*, **69** (2008), 456-462.

Shahram Saeidi
Department of Mathematics
University of Kurdistan
Sanandaj 416, Kurdistan
and
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P. O. Box 19395-5746
Tehran, Iran
E-mail: sh.saeidi@uok.ac.ir
shahram_saeidi@yahoo.com