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LIPSCHITZ CONTINUITY OF AN APPROXIMATE SOLUTION MAPPING TO EQUILIBRIUM PROBLEMS

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Abstract. In this paper, with respect to Hausdorff metric, we establish the Lipschitz continuity of an approximate solution mapping for a parametric scalar equilibrium problem. Simultaneously, we give some applications to parametric optimization problems and parametric variational inequalities. Our main results are new and strengthen some results in the recent literature.

1. INTRODUCTION

The equilibrium problems provide a unifying framework for investigating a large variety of problems of variational analysis such as variational inequalities, optimization problems and minimax problems. Especially, a great deal of research has been devoted to finding the existence of solutions to equilibrium problems in various versions; see, for example, [8, 10, 16, 25, 28] and the references therein. Among many desirable properties of equilibrium problems, the stability analysis of solutions is an essential topic in optimization theory and applications. Stability may be understood as lower or upper semicontinuity, continuity, and Lipschitz or Hölder continuity. The semicontinuities, especially the lower semicontinuity, of solution mappings for parametric equilibrium problems and parametric variational inequalities have been of increasing interest in the literature, such as [1, 3, 4, 11-15, 17, 21-23, 26, 31]. On the other hand, Hölder continuity of solutions to parametric vector equilibrium problems has also been discussed recently, see [2, 5, 6, 11, 12, 24, 29], although there may be less works in the literature devoted to this property than to semicontinuity.

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In [2, 5-7, 11, 12, 29, 30], by the assumptions of strong monotonicity and/or strong pseudomonotonicity, the authors obtained the Hölder continuity of solutions to parametric equilibrium problems under the case that the solution is unique in some neighborhood of a given point, namely the solution map is single-valued. However, for a parametric equilibrium problem, generally speaking, the solution mapping is set-valued. Naturally, it is a need to study the Hölder continuous properties of the set-valued solution mappings. Very recently, by using the different assumptions,which is weaker than pseudomonotonicity, Li et al. [20, 24] have studied Hölder continuity of the solution mappings for parametric generalized vector equilibrium problems when they are set-valued ones. Furthermore, Li and Li [27] defined another concept of strong monotonicity and obtained Hölder continuity of the solution mappings for a parametric weak vector equilibrium problems by a scalarization method.

On the other hand, exact solutions of the problems may not exist in many practical problems because the data of the problems are not sufficiently "regular". Moreover, these mathematical models are solved usually by numerical methods (iterative procedures or heuristic algorithms) which produce approximations to the exact solutions. So it is impossible to obtain an exact solution of many practical problems. Naturally, investigating approximate solutions of parametric equilibrium problems is of interest in both practical applications and computations. However, to the best of our knowledge, there was only a few results devoted to this direction in the literature. In [19], Kimura and Yao have established the existence results for two types of approximate generalized vector equilibrium problems, and further obtained the semicontinuity of approximate solution mappings. In [18], Khanh and Luu have discussed the semicontinuity of the approximate solution mappings of parametric multivalued quasivariational inequalities in topological vector spaces. In [4], Anh and Khanh have considered two kinds of approximate solution mappings to parametric generalized vector quasiequilibrium problems and established the sufficient conditions for Hausdorff semicontinuity (or Berge semicontinuity). In [27], Li and Li have investigated the Hausdorff continuity (or Berge continuity) of the approximate solution mapping for a parametric scalar equilibrium problem. By using a scalarization method, they obtained a sufficient condition of the lower semicontinuity of the approximate solution mapping for a parametric vector equilibrium problem.

Motivated by the work reported in [4, 19, 18, 27], this paper aims to establish the Lipschitz continuity of the approximate solution mappings for a parametric scalar equilibrium problem (PSEP). Our sufficient conditions for Lipschitz continuity of the approximate solution mappings are different from the corresponding ones in [2, 11, 12, 24, 29]. In this paper, the crucial assumptions are not strong monotonicity and/or strong pseudomonotonicity but concavity and Lipschitz continuity. Our main proof methods are also different from the corresponding ones in [2, 11, 12, 24, 29]. By using the monotonicity of the approximate solution mappings (with respect to the set-inclusion) for (PSEP), we established the Lipschitz continuity of the approximate solution mappings for (PSEP). Our consequences are new and strengthen the corresponding results in [4, 19, 27].

The rest of the paper is organized as follows. In Section 2, we discuss the Lipschitz continuity of the approximate solution mapping for (PSEP), and give some examples to illustrate our results. In Section 3, by the results of Section 2, we give some applications to optimization problems and variational inequalities.

2. Lipschitz Continuity of an Approximate Solution Mapping to (PSEP)

In this section, we consider the following parametric scalar equilibrium problem of finding $\bar{x} \in E$ such that

$$\text{(PSEP)} \qquad f(\bar{x}, y, \mu) + \epsilon \geq 0, \qquad \forall y \in E,$$

where $f : E \times E \times M \to R, E$ is a nonempty convex compact subset of X, $M \subset Y$ is a nonempty compact subset and ϵ is a nonnegative real number; X, Y are two normed spaces.

For any $\epsilon \geq 0$ and $\mu \in M$, by $S_\epsilon(\mu)$ denotes the approximate solution set of (PSEP), i.e.,

$$S_\epsilon(\mu) = \{\bar{x} \in E : f(\bar{x}, y, \mu) + \epsilon \geq 0, \quad \forall y \in E\}.$$

Throughout this section, we assume that $S_\epsilon(\mu) \neq \emptyset$ for any $\epsilon \geq 0$ and $\mu \in M$.

Lemma 2.1. *If for every $y \in E$ and $\mu \in M$, $f(\cdot, y, \mu)$ is upper semicontinuous on E, then for any $\epsilon \geq 0$, the approximate solution set $S_\epsilon(\mu)$ of (PSEP) is a compact set.*

Proof. Since E is a compact set, it suffices to show that $S_\epsilon(\mu)$ is a closed set of E for any given $\mu \in M$. Indeed, take any sequence $x_n \in S_\epsilon(\mu)$ with $x_n \to x_0$. Then, it follows from the definition of $S_\epsilon(\mu)$ that $x_n \in E$ and for any $y \in E$, $f(x_n, y, \mu) + \epsilon \geq 0$. Then, $x_n \to x_0 \in E$ and $f(x_0, y, \mu) + \epsilon \geq \sup_{n\to\infty} f(x_n, y, \mu) + \epsilon \geq 0$ since E is a compact subset and $f(\cdot, y, \mu)$ is upper semicontinuous. So, $x_0 \in S_\epsilon(\mu)$ and the proof is complete. ■

Before formulating the main results of this section, let us recall the definition of Hausdorff metric between two nonempty closed bounded subsets $A, B \subset X$:

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(a, B) := \inf_{b\in B} \|a - b\|$.

Lemma 2.2. *Let $\epsilon_0 > 0$. Suppose that the following conditions are satisfied:*

(i) *For each $y \in E$, $\mu \in M$, $f(\cdot, y, \mu)$ is upper semicontinuous on E;*

(*ii*) *For each* $y \in E$, $\mu \in M$, $f(\cdot, y, \mu)$ *is a concave function, i.e., for any* $x_1, x_2 \in E$ *and any* $\lambda \in [0,1]$, $f(\lambda x_1 + (1-\lambda)x_2, y, \mu) \geq \lambda f(x_1, y, \mu) + (1-\lambda) f(x_2, y, \mu)$*;*

(*iii*) *There exists a constant* $\gamma > 0$ *such that* $S_{\epsilon_0}(\mu) \subset \gamma B$ *for any* $\mu \in M$*, where* B *is a unit ball of the origin.*

Then, there exists a constant $a > 0$ *such that the approximate solution mapping* $S_{\cdot}(\cdot)$ *of* (*PSEP*) *satisfies the following condition:* $\forall \epsilon_1, \epsilon_2 \in [0, \epsilon_0] : \epsilon_1^2 + \epsilon_2^2 \neq 0$, $\forall \mu \in M$,

$$H(S_{\epsilon_1}(\mu), S_{\epsilon_2}(\mu)) \leq \frac{a}{\max\{\epsilon_1, \epsilon_2\}} |\epsilon_1 - \epsilon_2|. \tag{1}$$

Proof. (a) We claim that there exists a constant $a > 0$ such that

$$H(S_\epsilon(\mu), S_0(\mu)) \leq a, \quad \forall \epsilon \in (0, \epsilon_0], \forall \mu \in M. \tag{2}$$

For any $\epsilon \in (0, \epsilon_0]$ and $\mu \in M$, by the definitions of $S_{\cdot}(\mu)$, we get

$$S_0(\mu) \subset S_\epsilon(\mu) \subset S_{\epsilon_0}(\mu).$$

By Lemma 2.1, $S_0(\mu), S_\epsilon(\mu)$ and $S_{\epsilon_0}(\mu)$ are compact sets. Then, it follows from the definition of Hausdorff metric that

$$\begin{aligned} H(S_0(\mu), S_\epsilon(\mu)) &\leq H(S_0(\mu), S_{\epsilon_0}(\mu)) \\ &\leq \sup_{x \in S_{\epsilon_0}(\mu)} \sup_{y \in S_{\epsilon_0}(\mu)} \|x - y\|. \end{aligned} \tag{3}$$

Therefore, let $a = 2\gamma$, this together with (3) yield (2) by virtue of (iii).

(b) By virtue of Lemma 2.1, $S_\epsilon(\mu)$ is compact for any $\epsilon \geq 0$ and $\mu \in M$. Thus, we only need to verify that $S_{\cdot}(\mu)$ satisfies (1). Obviously, the conclusion is trivially if $\epsilon_1 = \epsilon_2$. Without loss of generality, we assume that $\epsilon_1 < \epsilon_2$. By the definition of $S_{\cdot}(\mu)$, we get $S_{\epsilon_1}(\mu) \subset S_{\epsilon_2}(\mu)$. Thus, we only need to show that for any $\bar{x}_2 \in S_{\epsilon_2}(\mu)$, there exists $\bar{x}_1 \in S_{\epsilon_1}(\mu)$ satisfying

$$\|\bar{x}_2 - \bar{x}_1\| \leq \frac{a}{\epsilon_2} |\epsilon_2 - \epsilon_1|. \tag{4}$$

Let $\bar{x}_2 \in S_{\epsilon_2}(\mu)$ and pick $\bar{x}_0 \in S_0(\mu)$ such that

$$\|\bar{x}_2 - \bar{x}_0\| \leq a.$$

Then, for any $y \in E$, we have

$$f(\bar{x}_2, y, \mu) + \epsilon_2 \geq 0 \tag{5}$$

and

(6) $$f(\bar{x}_0, y, \mu) \geq 0.$$

After multiplying (5) by $\frac{\epsilon_1}{\epsilon_2}$ and (6) by $\frac{\epsilon_2-\epsilon_1}{\epsilon_2}$ and summing, we get

(7) $$\frac{\epsilon_1}{\epsilon_2} f(\bar{x}_2, y, \mu) + \epsilon_1 + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2} f(\bar{x}_0, y, \mu) \geq 0.$$

Then, it follows from (ii) and (7) that

$$f(\frac{\epsilon_1}{\epsilon_2}\bar{x}_2 + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2}\bar{x}_0, y, \mu) + \epsilon_1 \geq 0,$$

which implies

$$\bar{x}_1 := \frac{\epsilon_1}{\epsilon_2}\bar{x}_2 + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2}\bar{x}_0 \in S_{\epsilon_1}(\mu).$$

Thus,

$$\begin{aligned} \|\bar{x}_2 - \bar{x}_1\| &= \frac{\epsilon_2 - \epsilon_1}{\epsilon_2}\|\bar{x}_2 - \bar{x}_0\| \\ &\leq \frac{a}{\epsilon_2}|\epsilon_2 - \epsilon_1|. \end{aligned}$$

So, (4) holds and the proof is complete. ■

Theorem 2.1. *Let* $0 < \widetilde{\epsilon} < \epsilon_0$. *Suppose that the following conditions are satisfied:*

(i) *For each* $y \in E$, $\mu \in M$, $f(\cdot, y, \mu)$ *is upper semicontinuous on* E*;*

(ii) *For each* $y \in E$, $\mu \in M$, $f(\cdot, y, \mu)$ *is a concave function;*

(iii) *There exists a constant* $\gamma > 0$ *such that* $S_{\epsilon_0}(\mu) \subset \gamma B$ *for any* $\mu \in M$*, where* B *is a unit ball of the origin.*

Then, there exists a constant $a > 0$ *such that the approximate solution mapping* $S_{\cdot}(\cdot)$ *of (PSEP) satisfies the following Lipschitz condition:* $\forall \epsilon_1, \epsilon_2 \in [\widetilde{\epsilon}, \epsilon_0]$, $\forall \mu \in M$,

(8) $$H(S_{\epsilon_1}(\mu), S_{\epsilon_2}(\mu)) \leq \frac{a}{\widetilde{\epsilon}}|\epsilon_1 - \epsilon_2|.$$

Proof. Since $\widetilde{\epsilon} \leq \max\{\epsilon_1, \epsilon_2\}$, it follows from (1) that (8) holds.

Remark 2.1. The assumption (iii) of Theorem 2.1 is an upper estimate of $S_{\epsilon_0}(M) := \bigcup_{\mu \in M} S_{\epsilon_0}(\mu)$ and seems to be very strict. But the assumption (iii) is filled under the following case: "For each $x, y \in E$, $f(x, y, \cdot)$ is continuous on M". Indeed, by Theorem 3.1 in [26], $S_{\epsilon_0}(\cdot)$ is upper semicontinuous on M. Then, by Proposition 11 in Section 1 of Chapter 3 [9], $S_{\epsilon_0}(M)$ is a compact subset of E since M is a compact subset.

Remark 2.2. When $f(x, y, \mu) = f(x, y)$, i.e., $f(x, y, \mu)$ does not depend on μ, the (PSEP) reduces to the model discussed in [19]. In [19], Kimura and Yao discussed the upper semicontinuity and lower semicontinuity of the approximate solution set $S_{(\cdot)}$ of (PSEP). However, in Theorem 2.1, we obtained the Lipschitz continuity of the approximate solution set $S_{\cdot}(\mu)$ for (PSEP) which strengthen the corresponding ones of [19]. Simultaneously, the assumptions and proof method of Theorem 2.1 are very different from the corresponding ones in [19].

Remark 2.3. In Theorem 2.1, we have obtained actually that for any $\widetilde{\epsilon}$ satisfying $0 < \widetilde{\epsilon} < \epsilon_0$ and $\mu \in M$, the approximate solution mapping $S_{\cdot}(\mu)$ of (PSEP) is Lipschitz continuous on $[\widetilde{\epsilon}, \epsilon_0]$. However, $\widetilde{\epsilon}$ can not be equal to zero, namely, $S_{\cdot}(\mu)$ may not be Lipschitz continuous on $[0, \epsilon_0]$. The following example explains the case.

Example 2.1. Let $\epsilon_0 > 0$ and small. Let $X = Z = R, E = [0, 1], M = [1, 2]$ and $f(x, y, \mu) = \mu(-x^2 - y^2 + 1)$. Obviously, all assumptions of Theorem 2.1 are satisfied. It is easy to show that the solution mapping S of (PSEP) is given by

$$S_\epsilon(\mu) = [0, \sqrt{\frac{\epsilon}{\mu}}], \quad \forall \mu \in M, \forall \epsilon \in [0, \epsilon_0].$$

However, for any $\mu \in M$, $S_{\cdot}(\mu)$ is not Lipschitz on $[0, \epsilon_0]$. Indeed, taking $\mu = 1$ and $x_1 = \sqrt{\epsilon_1} \in S_{\epsilon_1}(1)$ as a selection, there is no $L > 0$ such that

$$d(x_1, S_0(1)) \leq L\epsilon_1,$$

for every $0 < \epsilon_1 \leq \epsilon_0$.

Theorem 2.2. *Suppose that all conditions of Lemma 2.1 are satisfied, and (iii) is replaced by:* (iii') *For each* $x, y \in E$, $f(x, y, \cdot)$ *is* ℓ*-Lipschitz on* M. *Then for any* $\epsilon \in (0, \epsilon_0]$, *there exists a constant* $\kappa > 0$ *such that the approximate solution mapping* $S_\epsilon(\cdot)$ *of (PSEP) satisfying the following Lipschitz condition:*

$$H(S_\epsilon(\mu_1), S_\epsilon(\mu_2)) \leq \frac{\kappa\ell}{\epsilon}\|\mu_1 - \mu_2\|, \qquad \forall \mu_1, \mu_2 \in M. \tag{9}$$

Suppose furthermore that there exists a constant $\widetilde{\epsilon} > 0$ *such that* $\widetilde{\epsilon} < \epsilon_0$. *Then there exists a constant* $\widetilde{\kappa} \geq \kappa$ *such that the approximate solution mapping* $S_{\cdot}(\cdot)$ *of (PSEP) satisfying the following Lipschitz condition:*

$$H(S_\epsilon(\mu_1), S_\epsilon(\mu_2)) \leq \frac{\widetilde{\kappa}\ell}{\widetilde{\epsilon}}\|\mu_1 - \mu_2\|, \qquad \forall \epsilon \in [\widetilde{\epsilon}, \epsilon_0], \forall \mu_1, \mu_2 \in M. \tag{10}$$

Proof. By virtue of Lemma 2.1, for any $\epsilon \in (0, \epsilon_0]$ and $\mu \in M$, $S_\epsilon(\mu)$ is a compact subset of E. Thus, (9) and (10) are well-defined.

(a) First, we verify that (9) holds. For any $\mu_1, \mu_2 \in M$, there are two cases to be considered.

Case 1. $\|\mu_1-\mu_2\| \le \frac{\epsilon}{\ell}$. The conclusion (9) is trivial if $\mu_1=\mu_2$. Thus, we assume without loss of generality that $\mu_1 \neq \mu_2$. Let $r := \ell\|\mu_1-\mu_2\|$. Obviously, $0<r\le\epsilon$. Let $\bar{x}\in S_{\epsilon-r}(\mu_1)$ be arbitrarily given. Then, for any $y\in E$, we get

$$\bar{x}\in E \quad \text{and} \quad f(\bar{x},y,\mu_1)+\epsilon-r\ge 0.$$

Therefore,

$$f(\bar{x},y,\mu_2)+f(\bar{x},y,\mu_1)-f(\bar{x},y,\mu_2)\ge -\epsilon+r. \tag{11}$$

From the Lipschitz continuity of $f(x,y,\cdot)$ on M, we get

$$|f(x,y,\mu_1)-f(x,y,\mu_2)|\le \ell\|\mu_1-\mu_2\|=r. \tag{12}$$

Thus, (11) and (12) together yield that

$$f(\bar{x},y,\mu_2)\ge -\epsilon, \quad \forall y\in E,$$

that is $\bar{x}\in S_\epsilon(\mu_2)$. So for r defined above and μ_1 close enough to μ_2,

$$S_{\epsilon-r}(\mu_1)\subset S_\epsilon(\mu_2).$$

Therefore, by Remark 2.1 and (1)

$$\begin{aligned}\sup_{\bar{x}_2\in S_\epsilon(\mu_1)}\inf_{\bar{x}_1\in S_\epsilon(\mu_2)}\|\bar{x}_2-\bar{x}_1\| &\le \sup_{\bar{x}_2\in S_\epsilon(\mu_1)}\inf_{\bar{x}_1\in S_{\epsilon-r}(\mu_1)}\|\bar{x}_2-\bar{x}_1\|\\ &\le H(S_\epsilon(\mu_1),S_{\epsilon-r}(\mu_1))\\ &\le \frac{ar}{\max\{\epsilon,\epsilon-r\}}\\ &= \frac{a\ell}{\epsilon}\|\mu_1-\mu_2\|.\end{aligned} \tag{13}$$

Due to the symmetry between μ_1 and μ_2, the same estimate is also valid, i.e.,

$$\sup_{\bar{x}_1\in S_\epsilon(\mu_2)}\inf_{\bar{x}_2\in S_\epsilon(\mu_1)}\|\bar{x}_1-\bar{x}_2\|\le\frac{a\ell}{\epsilon}\|\mu_1-\mu_2\|. \tag{14}$$

Thus, by the definition of Hausdorff metric, (13) and (14) together implies

$$H(S_\epsilon(\mu_1),S_\epsilon(\mu_2))\le\frac{a\ell}{\epsilon}\|\mu_1-\mu_2\|. \tag{15}$$

Case 2. $\|\mu_1-\mu_2\|>\frac{\epsilon}{\ell}$. Noting that M is a compact subset, let us consider the finite open covering of the set of M by the open balls of the radius $\frac{\epsilon}{4\ell}$. Let μ_1,μ_2 be two centers of the open balls. The number of these balls will be denoted by N and their centers by $\{\nu_i : i = 1, ..., N\}$. Obviously, $\{\nu_i : i = 1, ..., N\}$ includes two points μ_1, μ_2. Then the sequence of points $\{\nu_{i_m} : m = 1, ..., L, L \leq N\}$ exists such that

$$\nu_{i_1} = \mu_1, \qquad \nu_{i_m} = \mu_2$$

and

$$\|\nu_{i_m} - \nu_{i_{m+1}}\| \leq \frac{\epsilon}{2\ell}.$$

Therefore, it follows from the triangle inequality of Hausdorff metric and (15) that

$$\begin{aligned} H(S_\epsilon(\mu_1), S_\epsilon(\mu_2)) &\leq \sum_{m=1}^{L} H(S_\epsilon(\nu_{i_m}), S_\epsilon(\nu_{i_{m+1}})) \\ &\leq \frac{a\ell}{\epsilon} \sum_{m=1}^{L} \|\nu_{i_m} - \nu_{i_{m+1}}\| \\ &\leq L\frac{a\ell}{\epsilon} \bullet \frac{\epsilon}{2\ell} \\ &\leq N\frac{a\ell}{\epsilon} \bullet \frac{\epsilon}{2\ell} \\ &\leq N\frac{a\ell}{2\epsilon}\|\mu_1 - \mu_2\|. \end{aligned} \tag{16}$$

Let $\kappa := \max\{a, Na/2\}$. Thus, from (15) and (16), we have that (9) holds.

(b) Now, we show that (10) is satisfied. For any $\mu_1, \mu_2 \in M$ and $\epsilon \in [\widetilde{\epsilon}, \epsilon_0]$, there are also two cases to be considered.

If $\|\mu_1 - \mu_2\| \leq \frac{\epsilon}{\ell}$, then it follows from (15) that

$$\begin{aligned} H(S_\epsilon(\mu_1), S_\epsilon(\mu_2)) &\leq \frac{a\ell}{\epsilon}\|\mu_1 - \mu_2\| \\ &\leq \frac{a\ell}{\widetilde{\epsilon}}\|\mu_1 - \mu_2\| \end{aligned} \tag{17}$$

since $\widetilde{\epsilon} \leq \epsilon$.

If $\|\mu_1 - \mu_2\| > \frac{\epsilon}{\ell}$, then we consider the finite open covering of the set of M by the open balls of the radius $\frac{\widetilde{\epsilon}}{4\ell}$ since M is a compact subset. The number of these balls will be denoted by $\widetilde{N}$. Then $\widetilde{N} \geq N$ (N is defined as above) since $\widetilde{\epsilon} \leq \epsilon$. Following the proof of Case 2 in (a), $\widetilde{\epsilon} < \epsilon$ and (16), one has

$$\begin{aligned} H(S_\epsilon(\mu_1), S_\epsilon(\mu_2)) &\leq \widetilde{N}\frac{a\ell}{\widetilde{\epsilon}} \bullet \frac{\widetilde{\epsilon}}{2\ell} \\ &\leq \widetilde{N}\frac{a\ell}{\widetilde{\epsilon}} \bullet \frac{\epsilon}{2\ell} \\ &\leq \widetilde{N}\frac{a\ell}{2\widetilde{\epsilon}}\|\mu_1 - \mu_2\|. \end{aligned} \tag{18}$$

Thus, (17) and (18) together yield that (10) holds by letting $\widetilde{\kappa} := \max\{a, \widetilde{N}a/2\} \geq \kappa$ and the proof is complete. ■

The following example is given to illustrate that the concavity of $f(\cdot, y, \mu)$ in Theorem 2.2 is essential.

Example 2.2. Let $\epsilon > 0$ and small. Let $X = Z = R, E = [1, 2], M = [0, 1]$ and $f(x, y, \mu) = \mu x(x - y) - \epsilon$. Then, all conditions of Theorem 2.2 except for (ii) are satisfied. Direct computation shows that $S_\epsilon(0) = [1, 2]$ and $S_\epsilon(\mu) = \{2\}, \forall \mu \in (0, 1]$. Clearly, we see that $S_\epsilon(\cdot)$ is even not l.s.c at $\mu = 0$. Hence the assumption (ii) in Theorem 2.2 is essential.

The following example illustrates that Theorem 2.2 is applicable.

Example 2.3. Let $X = Z = R, E = [0, 1], M = [0, \frac{1}{2}]$ and $f(x, y, \mu) = y - x + \mu$. Then, it is clear that all assumptions of Theorem 2.1 hold and Theorem 2.1 is applicable. Moreover, it follows from direct computation that for any $\epsilon \in (0, \epsilon_0]$, $S_\epsilon(\mu) = [0, \mu + \epsilon]$, $\forall \mu \in M$. Thus, the approximate solution mapping $S_\epsilon(\cdot)$ of (PSEP) satisfies (9).

Combing Theorems 2.1 and 2.2, we can obtain the following result.

Theorem 2.3. *Let* $0 < \widetilde{\epsilon} < \epsilon_0$. *Suppose that the following conditions are satisfied:*

(*i*) *For each* $y \in E$, $\mu \in M$, $f(\cdot, y, \mu)$ *is upper semicontinuous on* E*;*

(*ii*) *For each* $y \in E$, $\mu \in M$, $f(\cdot, y, \mu)$ *is a concave function;*

(*iii*) *For each* $x, y \in E$, $f(x, y, \cdot)$ *is* ℓ*-Lipschitz on* M.

Then there exist constants $a > 0$ *and* $\widetilde{\kappa} > 0$ *such that the approximate solution mapping* $S_\cdot(\cdot)$ *of (PSEP) satisfying the following Lipschitz condition:*

$$
\begin{aligned}
&H(S_{\epsilon_1}(\mu_1), S_{\epsilon_2}(\mu_2)) \\
&\leq \frac{a}{\widetilde{\epsilon}}|\epsilon_1 - \epsilon_2| + \frac{\widetilde{\kappa}\ell}{\widetilde{\epsilon}}\|\mu_1 - \mu_2\|, \ \forall \epsilon_1, \epsilon_2 \in [\widetilde{\epsilon}, \epsilon_0], \forall \mu_1, \mu_2 \in M.
\end{aligned} \tag{19}
$$

Proof. Since

$$H(S_{\epsilon_1}(\mu_1), S_{\epsilon_2}(\mu_2)) \leq H(S_{\epsilon_1}(\mu_1), S_{\epsilon_1}(\mu_2)) + H(S_{\epsilon_1}(\mu_2), S_{\epsilon_2}(\mu_2)),$$

it follows from (8) and (10) that (19) holds. ■

Remark 2.4. In [26], Li and Li have established the H-continuity and B-continuity of the approximate solution mapping $S_\cdot(\cdot)$ for (PSEP), respectively. Herein, under Hausdorff metric, the Lipschitz continuity of the solution mapping $S_\cdot(\cdot)$ is established in Theorem 2.3.

3. Applications

As pointed out in Introduction, equilibrium problems contain many problems as special cases, including optimization problems, variational inequalities, complementarity problems, etc., we can derive from Theorems 2.1, 2.2 and 2.3 consequences for such special cases. In this section, we only discuss classic optimization and variational inequality problems.

3.1. Optimization Problems

Let X and Y be normed spaces, $E \subset X$ be a nonempty convex compact subset, $M \subset Y$ be a nonempty compact subset and let ϵ be a nonnegative constant. Let f be defined as

$$f(x, y, \mu) = \phi(y, \mu) - \phi(x, \mu)$$

where $\phi : E \times M \to R$ is a function. Then (PSEP) reduces to the parametric optimization problem of finding $\bar{x} \in E$ such that

$$\text{(POP)} \qquad \min\{\phi(x, \mu) : x \in E\}$$

For any $\epsilon \geq 0$ and $\mu \in M$, by $\bar{S}_\epsilon(\mu)$ denotes the approximate solution set of (PSEP), i.e.,

$$\bar{S}_\epsilon(\mu) = \{\bar{x} \in E : \inf_{y \in E} \phi(y, \mu) + \epsilon \geq \phi(\bar{x}, \mu)\}.$$

We assume that $\bar{S}_\epsilon(\mu) \neq \emptyset$ for any $\epsilon \geq 0$ and $\mu \in M$.

Corollary 3.1. *Let $0 < \widetilde{\epsilon} < \epsilon_0$. Suppose that the following conditions are satisfied:*

- (i) *For each $y \in E$, $\mu \in M$, $\phi(\cdot, \mu)$ is lower continuous on E;*
- (ii) *For each $y \in E$, $\mu \in M$, $\phi(\cdot, \mu)$ is a convex function, i.e., for any $x_1, x_2 \in E$ and any $\lambda \in [0, 1]$, $\phi(\lambda x_1 + (1-\lambda)x_2, \mu) \leq \lambda\phi(x_1, \mu) + (1-\lambda)\phi(x_2, \mu)$;*
- (iii) *For each $x \in E$, $\phi(x, \cdot)$ is $\frac{\ell}{2}$-Lipschitz on M.*

Then

- (a) *for any $\epsilon \in (0, \epsilon_0]$, there exists a constant $\kappa > 0$ such that the approximate solution mapping $\bar{S}_\epsilon(\cdot)$ of (POP) satisfying the following Lipschitz condition:*

$$H(\bar{S}_\epsilon(\mu_1), \bar{S}_\epsilon(\mu_2)) \leq \frac{\kappa\ell}{\epsilon}\|\mu_1 - \mu_2\|, \qquad \forall \mu_1, \mu_2 \in M;$$

- (b) *there exist constants $a > 0$ and $\widetilde{\kappa} \geq \kappa > 0$ such that the approximate solution mapping $\bar{S}_{\cdot}(\cdot)$ of (POP) satisfying the following Lipschitz condition:*

$$H(\bar{S}_{\epsilon_1}(\mu_1), \bar{S}_{\epsilon_2}(\mu_2)) \leq \frac{a}{\widetilde{\epsilon}}|\epsilon_1 - \epsilon_2| + \frac{\widetilde{\kappa}\ell}{\widetilde{\epsilon}}\|\mu_1 - \mu_2\|, \ \forall \epsilon_1, \epsilon_2 \in [\widetilde{\epsilon}, \epsilon_0], \forall \mu_1, \mu_2 \in M.$$

3.2. Variational Inequalities

Let X be a normed space, X^* be the topological dual space of X, Y be normed spaces, $E \subset X$ be a nonempty convex compact subset, $M \subset Y$ be a nonempty compact subset and let ϵ be a nonnegative constant. Let f be defined as

$$f(x, y, \mu) = \langle g(x, \mu), y - x \rangle,$$

where $g : X \times \Lambda \to X^*$ is a single-valued map. Then (PSEP) collapses to the parametric variational inequality of finding $\bar{x} \in E$ such that

$$\text{(PVI)} \qquad \langle g(\bar{x}, \mu), y - \bar{x} \rangle + \epsilon \geq 0, \qquad \forall y \in E.$$

Denote the solution set map of (PVI) by $S^I_{\cdot}(\cdot)$. We also assume that $S^I_\epsilon(\mu) \neq \emptyset$ for any $\epsilon \geq 0$ and $\mu \in M$.

Corollary 3.2. *Let $0 < \widetilde{\epsilon} < \epsilon_0$. Suppose that the following conditions are satisfied:*

(i) *For any $y \in E$, $\mu \in M$, any sequence $\{x_n\} \subset E$ with $x_n \to x_0$,*

$$\limsup_n \langle g(x_n, \mu), y - x_n \rangle \leq \langle g(x_0, \mu), y - x_0 \rangle;$$

(ii) *For any $y \in E$, $\mu \in M$, $g(\cdot, \mu)$ is a affine function, i.e., for any $x_1, x_2 \in E$ and any $\lambda \in [0, 1]$, $g(\lambda x_1 + (1 - \lambda)x_2, \mu) = \lambda g(x_1, \mu) + (1 - \lambda)g(x_2, \mu)$;*

(iii) *For any $\mu \in M$, $g(\cdot, \mu)$ is monotone on E, i.e., for any $x_1, x_2 \in E$, $\langle g(x_1, \mu) - g(x_2, \mu), x_1 - x_2 \rangle \geq 0$;*

(iv) *For any $x \in E$, $g(x, \cdot)$ is ℓ-Lipschitz on M.*

Then

(a) *for any $\epsilon \in (0, \epsilon_0]$, there exists a constant $\kappa > 0$ such that the approximate solution mapping $S^I_\epsilon(\cdot)$ of (PVI) satisfying the following Lipschitz condition:*

$$H(S^I_\epsilon(\mu_1), S^I_\epsilon(\mu_2)) \leq \frac{\kappa \ell \rho}{\epsilon} \|\mu_1 - \mu_2\|, \qquad \forall \mu_1, \mu_2 \in M;$$

(b) *there exist constants $a > 0$ and $\widetilde{\kappa} \geq \kappa > 0$ such that the approximate solution mapping $S^I_{\cdot}(\cdot)$ of (PVI) satisfying the following Lipschitz condition:*

$$H(S^I_{\epsilon_1}(\mu_1), S^I_{\epsilon_2}(\mu_2)) \leq \frac{a}{\widetilde{\epsilon}}|\epsilon_1 - \epsilon_2| + \frac{\widetilde{\kappa} \ell \rho}{\widetilde{\epsilon}} \|\mu_1 - \mu_2\|, \ \forall \epsilon_1, \epsilon_2 \in [\widetilde{\epsilon}, \epsilon_0], \forall \mu_1, \mu_2 \in M,$$

where $\rho := \sup_{x,y \in E} \|x - y\| < \infty$.

Proof. To apply Theorems 2.2 and 2.3, we only need to verify that the concavity and Lipschitz continuity of f. First, $\forall \mu \in M, \forall x_1, x_2, y \in E, \forall \lambda \in [0,1]$,

$$\begin{aligned}
&f(\lambda x_1 + (1-\lambda)x_2, y, \mu) - (\lambda f(x_1, y, \mu) + (1-\lambda) f(x_2, y, \mu)) \\
=&\ \langle \lambda g(x_1, \mu) + (1-\lambda) g(x_2, \mu), y - (\lambda x_1 + (1-\lambda)x_2)\rangle \\
&-\lambda \langle g(x_1, \mu), y - x_1\rangle - (1-\lambda)\langle g(x_2, \mu), y - x_2\rangle \\
=&\ \lambda(1-\lambda)\langle g(x_1, \mu) - g(x_2, \mu), x_1 - x_2\rangle \\
\geq&\ 0,
\end{aligned}$$

where we used the assumptions (ii) and (iii). Since E is compact, $\rho := \sup_{x,y\in E} \|x - y\| < \infty$. Second, $\forall x, y, \in E, \forall \mu_1, \mu_2 \in M$,

$$\begin{aligned}
|f(x, y, \mu_1) - f(x, y, \mu_2)| &= |\langle g(x, \mu_1), y - x\rangle - \langle g(x, \mu_2), y - x\rangle| \\
&\leq \|g(x, \mu_1) - g(x, \mu_2)\| \|x - y\| \\
&\leq \ell\rho \|\mu_1 - \mu_2\|.
\end{aligned}$$

Remark 3.5. If for any $\mu \in M$, $g(\cdot, \mu)$ is continuous on E, then the assumption (i) of Corollary 3.2 holds.

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REFERENCES

1. L. Q. Anh and P. Q. Khanh, Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems, *J. Math. Anal. Appl.*, **294** (2004), 699-711.

2. L. Q. Anh and P. Q. Khanh, On the Hŏlder continuity of solutions to multivalued vector equilibrium problems, *J. Math. Anal. Appl.*, **321** (2006), 308-315.

3. L. Q. Anh and P. Q. Khanh, Various kinds of semicontinuity and solution sets of parametric multivalued symmetric vector quasiequilibrium problems, *J. Glob. Optim.*, **41** (2008), 539-558.

4. L. Q. Anh and P. Q. Khanh, Semicontinuity of the approximate solution sets of multivalued quasiequilibrium problems, *Numer. Funct. Anal. Optim.*, **29** (2008), 24-42.

5. L. Q. Anh and P. Q. Khanh, Uniqueness and Hŏlder continuity of the solution to multivalued equilibrium problems in metric spaces, *J. Glob. Optim.*, **37** (2007), 449-465.

6. L. Q. Anh and P. Q. Khanh, Sensitivity analysis for multivalued quasiequilibrium problems in metric spaces: Hölder continuity of solutions, *J. Glob. Optim.*, **42** (2008), 515-531.

7. L. Q. Anh and P. Q. Khanh, Hölder continuity of the unique solution to quasiequilibrium problems in metric spaces, *J. Optim. Theory Appl.*, **141** (2009), 37-54.

8. Q. H. Ansari and J. C. Yao, An existence result for the generalized vector equilibrium problem, *Appl. Math. Lett.*, **12** (1999), 53-56.

9. J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley, New York, 1984.

10. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63** (1994), 123-145.

11. M. Bianchi and R. Pini, A note on stability for parametric equilibrium problems, *Oper. Res. Lett.*, **31** (2003), 445-450.

12. M. Bianchi and R. Pini, Sensitivity for parametric vector equilibria, *Optimization*, **55** (2006), 221-230.

13. C. R. Chen, S. J. Li and K. L. Teo, Solution semicontinuity of parametric generalized vector equilibrium problems, *J. Glob. Optim.*, **45** (2009), 309-318.

14. C. R. Chen and S. J. Li, Semicontinuity of the solution set map to a set-valued weak vector variational inequality, *J. Ind. Manag. Optim.*, **3** (2007), 519-528.

15. Y. H. Cheng and D. L. Zhu, Global stability results for the weak vector variational inequlality, *J. Glob. Optim.*, **32** (2005), 543-550.

16. F. Giannessi (ed.), *Vector Variational Inequalities and Vector Equilibria: Mathematical Theories*, Kluwer Academic Publishers, Dordrecht, Holland, 2000.

17. N. J. Huang, J. Li and H. B. Thompson, Stability for parametric implicit vector equilibrium problems, *Math. Comput. Modelling.*, **43** (2006), 1267-1274.

18. P. Q. Khanh and L. M. Luu, Lower and upper semicontinuity of the solution sets and the approxiamte solution sets to parametric multivalued quasivariational inequalities, *J. Optim. Theory Appl.*, **133** (2007), 329-339.

19. K. Kimura and J. C. Yao, Semicontinuity of solutiong mappings of parametric generalized vector equilibrium problems, *J. Optim. Theory Appl.*, **138** (2008), 429-443.

20. S. J. Li, C. R. Chen, X. B. Li and K. L. Teo, Hölder continuity and upper estimates of solutions to vector quasiequilibrium problems, *European J. Oper. Res.*, **210** (2011), 148-157.

21. S. J. Li and C. R. Chen, Stability of weak vector variational inequality, *Nonlinear Anal.*, **70** (2009), 1528-1535.

22. S. J. Li, G. Y. Chen and K. L. Teo, On the stability of generalized vector quasivariational inequlity problems, *J. Optim. Theory Appl.*, **113** (2002), 283-295.

23. S. J. Li and Z. M. Fang, On the stability of a dual weak vector variational inequality problem, *J. Ind. Manag. Optim.*, **4** (2008), 155-165.

24. S. J. Li, X. B. Li, L. N. Wang and K. L. Teo, The Hölder continuity of solutions to generalized vector equilibrium problems, *European J. Oper. Res.*, **199** (2009), 334-338.

25. X. B. Li and S. J. Li, Existences of solutions for generalized vector quasi-equilibrium problems, *Optim. Lett.*, **4** (2010), 17-28.

26. X. B. Li and S. J. Li, Continuity of approximate solution mapping for parametric equilibrium problems, *J. Glob. Optim.*, **51** (2011), 541-548.

27. S. J. Li and X. B. Li, Hölder continuity of solutions to parametric weak vector equilibrium problems, *J. Optim. Theory Appl.*, **149** (2011), 540-553.

28. S. J. Li, X. Q. Yang and G. Y. Chen, Generalized vector quasi-equilibrum problems, *Math. Methods Oper. Res.*, **61** (2005), 385-397.

29. M. Ait Mansour and H. Riahi, Sensitivity analysis for abstract equilibrium problems, *J. Math. Anal. Appl.*, **306** (2005), 684-691.

30. M. Ait Mansour and L. Scrimali, Hölder continuity of solutions to elastic traffic networkmodels, *J. Glob. Optim.*, **40** (2008), 175-184.

31. M. M. Wong, Lower semicontinuity of the solution map to a parametric vector variational inequality, *J. Glob. Optim.*, **46** (2010), 435-446.

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