TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 3, pp. 1007-1016, June 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

ELLIPTIC NUMERICAL RANGES OF BORDERED MATRICES

Mao-Ting Chien* and Kuo-Chih Hung

Abstract. Let $A \in M_n(C)$. The numerical range of A is defined as $W(A) = \{x^*Ax : x \in C^n, |x| = 1\}$. We give a necessary and sufficient condition for $W(A)$ to be an elliptic disk. The criterion is applied to prove the elliptic numerical ranges of certain bordered matrices.

1. INTRODUCTION

Let $A \in M_n$. The numerical range of A is the set of complex numbers

$$W(A) = \{x^*Ax : x \in C^n, |x| = 1\}.$$

It is well known (see [5]) that $W(A)$ is convex, and $W(A)$ is an elliptic disk if $n = 2$. It is of great interest in determining matrices with elliptic numerical ranges. There have been a number of interesting papers on the subject, see [1, 2, 3, 4, 6]. Let $A \in M_n$ be of the form

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix},$$

where $B \in M_k$ and $E \in M_{n-k}$. The matrix A is called a bordered matrix of B. In this paper, we focus on the class of bordered matrices with $B = \alpha I_k$, a scalar matrix. In Section 2, a criterion for the numerical range to be an elliptic disk is given. In Section 3, this criterion is applied to bordered matrices to have elliptic numerical range.

2. AN ELLIPTIC CRITERION

Let $A \in M_n$. For any θ, consider the Hermitian part of $e^{i\theta}A$

Received February 10, 2010, accepted May 11, 2011.
Communicated by Ngai-Ching Wong.
2010 *Mathematics Subject Classification*: 15A60.
Key words and phrases: Bordered matrices, Elliptic disk, Numerical range.
The author would like to thank the Department of Aerospace and Mechanical Engineering, University of Southern California, for the warm hospitality during the visit when part of this work was carried out. Partial support of this visit from Taiwan National Science Council is gratefully acknowledged.
*Corresponding author.

$$H_\theta(A) = (e^{i\theta}A + e^{-i\theta}A^*)/2.$$

Marcus and Pesce [7] showed that $W(A)$ is a circular disk centered at the origin if and only if the maximal eigenvalue $\lambda_{\max}(H_\theta(A))$ of $H_\theta(A)$ is constant for all θ. The result is generalized in [3] that $W(A)$ is a regular elliptic disk centered at the origin having major axis on x-axis of length $2a$ and minor axis on y-axis of length $2b$ if and only if

$$\lambda_{\max}(H_\theta(A)) = (a^2 - c^2 \sin^2\theta)^{1/2},$$

for $0 \le \theta < 2\pi$, where $c = (a^2 - b^2)^{1/2}$. A criterion for general elliptic numerical range is modified as follows.

Theorem 1. *Let $A \in M_n(C)$, and $a, b \in R$. Then $W(A)$ is an elliptic disk centered at the point $p + qi$, major axis parallel to the vector $e^{i\alpha}$ of length $2a$ and minor axis of length $2b$ if and only if*

$$\lambda_{\max}\Big(H_\theta(A)\Big) = p\cos\theta - q\sin\theta + \Big(a^2 - c^2\sin^2(\theta+\alpha)\Big)^{1/2}$$

for $0 \le \theta < 2\pi$, where $c = (a^2 - b^2)^{1/2}$.

Proof. It is clear that $W(A)$ is an elliptic disk centered at the point $p+qi$, major axis parallel to the vector $e^{i\alpha}$ of length $2a$ and minor axis of length $2b$ if and only if $W(e^{-i\alpha}(A-(p+qi)I))$ is a regular elliptic disk centered at the origin, major axis of length $2a$ and minor axis of length $2b$. Then by [3], we have

$$\lambda_{\max}\Big(H_\theta(e^{-i\alpha}(A-(p+qi)I))\Big) = (a^2 - c^2\sin^2\theta)^{1/2}$$

for $0 \le \theta < 2\pi$, where $c = (a^2 - b^2)^{1/2}$. The conclusion then follows from the observation:

$$\begin{aligned} &\lambda_{\max}\Big(H_\theta(e^{-i\alpha}(A-(p+qi)I))\Big) \\ &= \lambda_{\max}\Big(H_{\theta-\alpha}(A-(p+qi)I)\Big) \\ &= \lambda_{\max}\Big(H_{\theta-\alpha}(A) - \Re(p+qi)e^{i(\theta-\alpha))}\Big). \end{aligned}$$

■

3. Bordered Matrices

Consider the following bordered matrix of a scalar matrix

$$(1) \qquad A = \begin{pmatrix} \alpha & 0 & \cdots & 0 & a_{1n} \\ 0 & \alpha & \ddots & \vdots & a_{2n} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \alpha & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & \beta \end{pmatrix}.$$

Linden [6] shows that $W(A)$ is an elliptic disk. We give a different proof using Theorem 1.

Theorem 2. *Let A be an $n \times n$ matrix defined as in (1). Then $W(A)$ is an elliptic disk centered at the point $(\alpha+\beta)/2$, major axis parallel to the vector $e^{i\frac{\phi}{2}}$ of length*

$$\left(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta-\alpha|^2}{2}+2\,|\sum_{j=1}^{n} a_{jn}a_{nj}+\frac{(\beta-\alpha)^2}{4}\,|\right)^{\frac{1}{2}},$$

and minor axis of length

$$\left(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta-\alpha|^2}{2}-2\,|\sum_{j=1}^{n-1} a_{jn}a_{nj}+\frac{(\beta-\alpha)^2}{4}\,|\right)^{\frac{1}{2}},$$

where

$$\sum_{j=1}^{n-1} a_{jn}a_{nj}+\frac{(\beta-\alpha)^2}{4}=|\sum_{j=1}^{n-1} a_{jn}a_{nj}+\frac{(\beta-\alpha)^2}{4}\,|e^{i\phi}.$$

Proof. We may assume $\alpha=0$. Then the characteristic polynomial of $H_\theta(A)$ becomes

$$t^n-(\frac{\beta e^{i\theta}+\overline{\beta}e^{-i\theta}}{2})t^{n-1}-\Big(\sum_{j=1}^{n-1}|\frac{a_{jn}e^{i\theta}+\overline{a_{nj}}e^{-i\theta}}{2}|^2\Big)t^{n-2}.$$

It follows that

$$\begin{aligned}&\lambda_{\max}(H_\theta(A))\\ =&\frac{1}{2}\left((\frac{\beta e^{i\theta}+\overline{\beta}e^{-i\theta}}{2})+\Big((\frac{\beta e^{i\theta}+\overline{\beta}e^{-i\theta}}{2})^2+4\sum_{j=1}^{n-1}|\frac{a_{jn}e^{i\theta}+\overline{a_{nj}}e^{-i\theta}}{2}|^2\Big)^{\frac{1}{2}}\right)\\ =&\frac{1}{2}\left(\Re\beta e^{i\theta}+\Big((\Re\beta e^{i\theta})^2+4\sum_{j=1}^{n-1}|\frac{a_{jn}e^{i\theta}+\overline{a_{nj}}e^{-i\theta}}{2}|^2\Big)^{\frac{1}{2}}\right).\end{aligned}$$

Suppose $\beta=p+qi$. Then $p\cos\theta-q\sin\theta=\Re\beta e^{i\theta}$. We compute that

$$\begin{aligned}&\lambda_{\max}(H_\theta(A))-(p\cos\theta-q\sin\theta)/2\\ =&\frac{1}{2}\left(\Re\beta e^{i\theta}+\Big((\Re\beta e^{i\theta})^2+4\sum_{j=1}^{n-1}|\frac{a_{jn}e^{i\theta}+\overline{a_{nj}}e^{-i\theta}}{2}|^2\Big)^{\frac{1}{2}}\right)-\frac{1}{2}\Re\beta e^{i\theta}\\ =&\frac{1}{2}\Big(\sum_{j=1}^{n-1}|a_{jn}e^{i\theta}+\overline{a_{nj}}e^{-i\theta}|^2+\frac{1}{4}|\beta e^{i\theta}+\overline{\beta}e^{-i\theta}|^2\Big)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta|^2}{2}+2\Re(\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4})e^{2i\theta}\Big)^{\frac{1}{2}}\\
&= \frac{1}{2}\Big(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta|^2}{2}+2\Re|\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}|e^{2i(\theta+\frac{\phi}{2})}\Big)^{\frac{1}{2}}\\
&= \frac{1}{2}\Bigg(\Big(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta|^2}{2}+2|\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}|\Big)\times\cos^2(\theta+\frac{\phi}{2})\\
&\quad+\Big(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta|^2}{2}-2|\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}|\Big)\times\sin^2(\theta+\frac{\phi}{2})\Bigg)^{\frac{1}{2}}\\
&= \frac{1}{2}\Bigg(\Big(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta|^2}{2}+2|\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}|\Big)\\
&\quad-\Big((\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta|^2}{2}+2|\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}|)\\
&\quad-\Big(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta|^2}{2}-2|\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}|\Big)\Big)\times\sin^2(\theta+\frac{\phi}{2})\Bigg)^{\frac{1}{2}},
\end{aligned}
$$

where

$$\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}=|\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}|e^{i\phi}.$$

By Theorem 1, $W(A)$ is an elliptic disk centered at the point $\beta/2$, major axis parallel to the vector $e^{i\frac{\phi}{2}}$ of length

$$\Big(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta|^2}{2}+2|\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}|\Big)^{\frac{1}{2}},$$

and minor axis of length

$$\Big(\sum_{j=1}^{n-1}(|a_{jn}|^2+|a_{nj}|^2)+\frac{|\beta|^2}{2}-2|\sum_{j=1}^{n-1}a_{jn}a_{nj}+\frac{\beta^2}{4}|\Big)^{\frac{1}{2}}.$$ ∎

As a consequence of Theorem 2, we obtain

Corollary 3. *The numerical range of an $n\times n$ of the form (1) is invariant under interchange of the (n,j) and (j,n) entries for any $j=1,2,\ldots,n-1$.*

The numerical ranges of special bordered matrices are examined in [1].

Theorem 4. ([1]). *Let A be a bordered matrix of αI_m of the form*

$$\begin{pmatrix} \alpha I_m & B \\ (kB)^* & \beta I_{n-m} \end{pmatrix}. \tag{2}$$

Then $W(A)$ is an elliptic disk with foci at

$$\lambda_{1,2} = \frac{\alpha + \beta \pm \sqrt{(\alpha-\beta)^2 + 4kr^2}}{2},$$

major axis of length

$$\sqrt{\frac{1}{2}|\ \alpha - \beta\ |^2 + r^2(|k|^2+1) + |\ \frac{1}{2}(\alpha-\beta)^2 + 2r^2k\ |},$$

and minor axis of length

$$\sqrt{\frac{1}{2}|\ \alpha - \beta\ |^2 + r^2(|k|^2+1) - |\ \frac{1}{2}(\alpha-\beta)^2 + 2r^2k\ |},$$

where $r = ||B||_2 = \max\{\sqrt{\lambda} : \lambda \in \sigma(B^*B)\}$.

In the following, we interchange the $(n-1)$-th and n-th rows of the bordered matrices (1) discussed in Theorem 2 which also produce elliptic numerical range. This step plays an important role in relaxing the interchange of two rows and two columns of the matrices (1) which is addressed in Theorem 6.

Theorem 5. *Let A be an $n \times n$ matrix of the form*

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & a_{1n} \\ \vdots & \ddots & \ddots & \vdots & a_{2n} \\ 0 & \cdots & \cdots & 0 & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & a_{nn} \end{pmatrix}.$$

If $a_{n-1,n} = 0$ and $a_{n-1,n-1} = a_{nn}$, then $W(A)$ is an elliptic disk with foci at 0 and a_{nn}, major axis of length $(r^2 + |a_{nn}|^2)^{\frac{1}{2}}$, and minor axis of length r, where

$$\begin{aligned} r^2 \ = \ & \frac{1}{2}\sum_{i=1}^{n-2}\Big(|a_{n-1,i}|^2 + |a_{in}|^2\Big) \\ & + \frac{1}{2}\sqrt{|\ \sum_{i=1}^{n-2}\Big(|a_{n-1,i}|^2 - |a_{in}|^2\Big)\ |^2 + 4\ |\ \sum_{i=1}^{n-2} a_{n-1,i}a_{in}\ |^2}. \end{aligned}$$

Proof. The hermitian part

$$H_\theta(A) = \frac{e^{i\theta}A+e^{-i\theta}A^*}{2}$$

$$= \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{2}\overline{a_{n-1,1}}e^{-i\theta} & \frac{1}{2}a_{1n}e^{i\theta} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2}\overline{a_{n-1,n-2}}e^{-i\theta} & \frac{1}{2}a_{n-2,n}e^{i\theta} \\ \frac{1}{2}a_{n-1,1}e^{i\theta} & \cdots & \frac{1}{2}a_{n-1,n-2}e^{i\theta} & \frac{1}{2}(a_{nn}e^{i\theta}+\overline{a_{nn}}e^{-i\theta}) & 0 \\ \frac{1}{2}\overline{a_{1n}}e^{-i\theta} & \cdots & \frac{1}{2}\overline{a_{n-2,n}}e^{-i\theta} & 0 & \frac{1}{2}(a_{nn}e^{i\theta}+\overline{a_{nn}}e^{-i\theta}) \end{pmatrix}.$$

Since $H_\theta(A)$ is a matrix of the form (2) in Theorem 4, it follows that $W\Big(H_\theta(A)\Big)$ is a line segment (a degenerated elliptic disk) with center at

$$\frac{1}{2}\Big(\frac{a_{nn}e^{i\theta}+\overline{a_{nn}}e^{-i\theta}}{2}\Big) = \frac{p\cos\theta - q\sin\theta}{2},$$

where $a_{nn} = p + qi$. Moreover,

$$\lambda_{\max}\Big(H_\theta(A)\Big) = \frac{p\cos\theta - q\sin\theta}{2} + \frac{1}{2}\sqrt{\frac{1}{4}|a_{nn}e^{i\theta}+\overline{a_{nn}}e^{-i\theta}|^2 + r^2}, \tag{4}$$

where

$$r = 2||U||_2, \text{ and } U = \begin{pmatrix} \frac{1}{2}\overline{a_{n-1,1}}e^{-i\theta} & \frac{1}{2}a_{1n}e^{i\theta} \\ \vdots & \vdots \\ \frac{1}{2}\overline{a_{n-1,n-2}}e^{-i\theta} & \frac{1}{2}a_{n-2,n}e^{i\theta} \end{pmatrix} \tag{5}$$

Direct computation shows that

$$\begin{aligned} r^2 &= 4||U||_2^2 = 4\max\{\lambda : \lambda \in \sigma(U^*U)\} \\ &= \frac{1}{2}\sum_{i=1}^{n-2}\Big(|a_{n-1,i}|^2 + |a_{in}|^2\Big) \\ &\quad + \frac{1}{2}\sqrt{|\sum_{i=1}^{n-2}\Big(|a_{n-1,i}|^2 - |a_{in}|^2\Big)|^2 + 4|\sum_{i=1}^{n-2}a_{n-1,i}a_{in}|^2}. \end{aligned}$$

From (4), we have that

$$\begin{aligned}
&\lambda_{\max}\Big(H_\theta(A)\Big) - \frac{p\cos\theta - q\sin\theta}{2} \\
= &\ \frac{1}{2}\Big(r^2 + \frac{1}{4}|a_{nn}e^{i\theta} + \overline{a_{nn}}e^{-i\theta}|^2\Big)^{\frac{1}{2}} \\
= &\ \frac{1}{2}\Big(r^2 + \frac{|a_{nn}|^2}{2} + 2\Re(\frac{{a_{nn}}^2}{4})e^{2i\theta}\Big)^{\frac{1}{2}} \\
= &\ \frac{1}{2}\Big(r^2 + \frac{|a_{nn}|^2}{2} + \frac{1}{2}\Re|{a_{nn}}^2|e^{2i(\theta+\frac{\phi}{2})}\Big)^{\frac{1}{2}} \\
= &\ \frac{1}{2}\Bigg(\Big(r^2 + \frac{|a_{nn}|^2}{2} + \frac{1}{2}|{a_{nn}}^2|\Big) \times \cos^2(\theta + \frac{\phi}{2}) \\
&\ + \Big(r^2 + \frac{|a_{nn}|^2}{2} - \frac{1}{2}|{a_{nn}}^2|\Big) \times \sin^2(\theta + \frac{\phi}{2})\Bigg)^{\frac{1}{2}} \\
= &\ \frac{1}{2}\Bigg((r^2 + |a_{nn}|^2) - \Big((r^2 + |a_{nn}|^2) - r^2\Big) \times \sin^2(\theta + \frac{\phi}{2})\Bigg)^{\frac{1}{2}},
\end{aligned}$$

where ${a_{nn}}^2 = |{a_{nn}}^2|e^{i\phi}$. The conclusion then follows from Theorem 1. ∎

The result of Theorem 5 can be extended to more general form.

Theorem 6. *Let A be an $n \times n$ matrix such that all entries of A are zero except that the k_1-th row, k_2-th column are possibly nonzero, and the diagonals $a_{ii} = \alpha$ for $i \neq k_1, k_2$, $a_{k_1k_1} = a_{k_2k_2} = \beta$ and $a_{k_1k_2} = 0$. Then $W(A)$ is an elliptic disk foci at α and β, major axis of length $\sqrt{r^2 + |\beta - \alpha|^2}$ and minor axis of length r, where*

$$\begin{aligned}
r^2 = &\ \frac{1}{2}\sum_{\substack{i=1\\ i\neq k_1,k_2}}^{n}\Big(|a_{k_1,i}|^2 + |a_{i,k_2}|^2\Big) \\
&+\frac{1}{2}\sqrt{|\sum_{\substack{i=1\\ i\neq k_1,k_2}}^{n}\Big(|a_{k_1,i}|^2 - |a_{i,k_2}|^2\Big)|^2 + 4|\sum_{\substack{i=1\\ i\neq k_1,k_2}}^{n} a_{k_1,i}\, a_{i,k_2}|^2}.
\end{aligned}$$

Proof. At first, consider the matrix $A - \alpha I$, and then choose a permutation matrix P, interchanging rows k_1 and $n-1$, and columns k_2 and n. Then $P^T(A - \alpha I)P$ is an $n \times n$ matrix of the form in Theorem 5. ∎

Example. Consider

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $W(A)$ is not an elliptic disk, indeed it has a flat portion on its boundary. And $W(B)$ is an ovular not an elliptic disk. These examples indicate that the conditions $a_{k_1k_2} = 0$ and $a_{k_1k_1} = a_{k_2k_2}$ are essential for the elliptic numerical range.

Finally, the idea of the proof in Theorem 5 is applied to guarantee certain bordered matrices having the same numerical range.

Theorem 7. *Let $A_1, A_2, A_3 \in M_n$ be bordered matrices of αI_k of the form*

$$A_j = \begin{pmatrix} \alpha I_k & C_j \\ D_j & \beta I_{n-k} \end{pmatrix},$$

$j = 1, 2, 3$, *where*

$$C_1 = \begin{pmatrix} a_{1,k+1} & a_{1,k+2} & \cdots & a_{1n} \\ a_{2,k+1} & a_{2,k+2} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,k+1} & a_{k,k+2} & \cdots & a_{kn} \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{2,k+1} & a_{2,k+2} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,k+1} & a_{k,k+2} & \cdots & a_{kn} \end{pmatrix}, D_2 = \begin{pmatrix} \overline{a_{1,k+1}} & 0 & \cdots & 0 \\ \overline{a_{1,k+2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & 0 & \cdots & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & a_{1,k+2} & \cdots & a_{1n} \\ 0 & a_{2,k+2} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{k,k+2} & \cdots & a_{kn} \end{pmatrix}, D_3 = \begin{pmatrix} \overline{a_{1,k+1}} & \overline{a_{2,k+1}} & \cdots & \overline{a_{k,k+1}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

Then $W(A_1) = W(A_2) = W(A_3)$ is an elliptic disk with foci at α and β, major axis of length $\sqrt{r^2 + |\beta - \alpha|^2}$ and minor axis of length r, where $r = ||U||_2$, and

$$U = \begin{pmatrix} a_{1,k+1} & a_{1,k+2} & \cdots & a_{1n} \\ a_{2,k+1} & a_{2,k+2} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,k+1} & a_{k,k+2} & \cdots & a_{kn} \end{pmatrix}.$$

Proof. We compute that

$$H_\theta(A_j) = \frac{e^{i\theta}A_j + e^{-i\theta}A_j^*}{2} = \begin{pmatrix} \Re(\alpha e^{i\theta}) I_k & U_j \\ \overline{U_j} & \Re(\beta e^{i\theta}) I_{n-k} \end{pmatrix},$$

$j = 1, 2, 3$, where

$$U_1 = \frac{1}{2}\begin{pmatrix} a_{1,k+1}e^{i\theta} & a_{1,k+2}e^{i\theta} & \cdots & a_{1n}e^{i\theta} \\ a_{2,k+1}e^{i\theta} & a_{2,k+2}e^{i\theta} & \cdots & a_{2n}e^{i\theta} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,k+1}e^{i\theta} & a_{2,k+2}e^{i\theta} & \cdots & a_{kn}e^{i\theta} \end{pmatrix},$$

$$U_2 = \frac{1}{2}\begin{pmatrix} a_{1,k+1}e^{-i\theta} & a_{1,k+2}e^{-i\theta} & \cdots & a_{1n}e^{-i\theta} \\ a_{2,k+1}e^{i\theta} & a_{2,k+2}e^{i\theta} & \cdots & a_{2n}e^{i\theta} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,k+1}e^{i\theta} & a_{2,k+2}e^{i\theta} & \cdots & a_{kn}e^{i\theta} \end{pmatrix},$$

$$U_3 = \frac{1}{2}\begin{pmatrix} a_{1,k+1}e^{-i\theta} & a_{1,k+2}e^{i\theta} & \cdots & a_{1n}e^{i\theta} \\ a_{2,k+1}e^{-i\theta} & a_{2,k+2}e^{i\theta} & \cdots & a_{2n}e^{i\theta} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,k+1}e^{-i\theta} & a_{2,k+2}e^{i\theta} & \cdots & a_{kn}e^{i\theta} \end{pmatrix}.$$

We see that $H_\theta(A_1)$, $H_\theta(A_2)$ and $H_\theta(A_3)$ are the form of the same type as that of (3) for each θ. Assume $\alpha+\beta = p+qi$. Then by a similar argument of the proof of Theorem 5, following (4) and (5), we obtain that

$$\lambda_{\max}\Big(H_\theta(A_j)\Big) = \frac{p\cos\theta - q\sin\theta}{2} + \frac{1}{2}\sqrt{|\Re(\alpha-\beta)e^{i\theta}|^2 + 4r_j^2},$$

where

$$r_j = ||U_j||_2 = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } U_j^*U_j\},$$

$j = 1,2,3$. Direct computations show that

$$U_1^*U_1 = U_2^*U_2 = V^*U_3^*U_3V,$$

where $V = \mathrm{diag}(e^{2i\theta}, 0, \ldots, 0)$. Thus $r_1 = r_2 = r_3$, denote the common value r. By Theorem 5, $W(A_1) = W(A_2) = W(A_3)$ is an elliptic disk with foci at α and β, major axis of length $\sqrt{r^2 + |\beta-\alpha|^2}$ and minor axis of length r. ■

REFERENCES

1. E. S. Brown and I. M. Spitkovsky, On matrices with elliptical numerical ranges, *Linear and Multilinear Algebra*, **52** (2004), 177-193.

2. W. Calbeck, Numerical ranges of 3×3 companion matrices, *Linear Algebra Appl.*, **428** (2008), 2715-2722.

3. M. T. Chien, L. Yeh, Y. T. Yeh and F. Z. Lin, On geometric properties of the numerical range, *Linear Algebra Appl.*, **274** (1998), 389-410.

4. H. L. Gau, Numerical ranges of 4×4 matrices, *Taiwanese J. Math.*, **10** (2006), 117-128.

5. R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

6. H. Linden, Containment regions for zeros of polynomials from numerical ranges of companion matrices, *Linear Algebra Appl.*, **350** (2002), 125-145.

7. M. Marcus and C. Pesce, Computer generated numerical ranges and some resulting theorems, *Linear and Multilinear Algebra*, **20** (1987), 121-157.

Mao-Ting Chien
Department of Mathematics
Soochow University
Taipei 11102, Taiwan
E-mail: mtchien@scu.edu.tw

Kuo-Chih Hung
Department of Mathematics
National Tsing Hua University
Hsinchu 300, Taiwan
E-mail: kchung@mx.nthu.edu.tw