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GENERALIZED WEIGHTED COMPOSITION OPERATORS FROM AREA NEVANLINNA SPACES TO BLOCH-TYPE SPACES

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Abstract. Let $H(\mathbb{D})$ denote the class of all analytic functions on the open unit disk $\mathbb{D}$ of $\mathbb{C}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The generalized weighted composition operator is defined by

$$D^n_{\varphi,u} f = u f^{(n)} \circ \varphi,\ \ f \in H(\mathbb{D}).$$

The boundedness and compactness of generalized weighted composition operators from area Nevanlinna spaces to Bloch-type spaces and little Bloch-type spaces are characterized in this paper.

1. Introduction

Let μ be a positive continuous function on $[0, 1)$. We say that μ is normal if there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that (see [14])

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1), \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a} = 0;$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta,1), \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b} = \infty.$$

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. We denote by $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$. Let $p \in [1, \infty)$, $\alpha > -1$. An $f \in H(\mathbb{D})$ is said to belong to the area Nevanlinna space $\mathcal{N}^p_\alpha = \mathcal{N}^p_\alpha(\mathbb{D})$, if

$$\|f\|^p_{\mathcal{N}^p_\alpha} = \int_{\mathbb{D}} \big[\log(1 + |f(z)|)\big]^p dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$. From [1], we see that the area Nevanlinna space $\mathcal{N}^p_\alpha$ is a linear topological vector space with respect to $F-$norm $\|\cdot\|_{\mathcal{N}^p_\alpha}$. Under $\|\cdot\|_{\mathcal{N}^p_\alpha}$,

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the topology of $\mathcal{N}_\alpha^p$ is stronger than that of local uniform convergence. This is a consequence of the following estimate (see, e.g., [1])

$$\log(1+|f(z)|) \le C\frac{\|f\|_{\mathcal{N}_\alpha^p}}{(1-|z|^2)^{(2+\alpha)/p}},\ f\in\mathcal{N}_\alpha^p, \tag{1}$$

where C is depend only on p and α.

In this paper, a subset A of $\mathcal{N}_\alpha^p$ is called bounded if there exists a positive number r such that $A\subset\{f\in\mathcal{N}_\alpha^p : \|f\|_{\mathcal{N}_\alpha^p} < r\}$. Given a Banach space X, we say that a linear map $T:\mathcal{N}_\alpha^p\to X$ is bounded if $T(A)\subset X$ is bounded for every bounded subset A of $\mathcal{N}_\alpha^p$. We say that T is compact if $T(A)\subset X$ is relatively compact for every bounded subset $A\subset\mathcal{N}_\alpha^p$.

Suppose μ is a normal function on $[0,1)$. The Bloch-type space $\mathcal{B}_\mu=\mathcal{B}_\mu(\mathbb{D})$ is the space of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_\mu}=|f(0)|+\sup_{z\in\mathbb{D}}\mu(|z|)\left|f'(z)\right|<\infty.$$

With the norm $\|\cdot\|_{\mathcal{B}_\mu}$, $\mathcal{B}_\mu$ is a Banach space. If $\mu(|z|)=1-|z|^2$, we denote $\mathcal{B}_\mu$ simply by $\mathcal{B}$, which is the well-known classical Bloch space. Let $\mathcal{B}_{\mu,0}$ denote the subspace of $\mathcal{B}_\mu$ consisting of those $f\in\mathcal{B}_\mu$ such that

$$\lim_{|z|\to1}\mu(|z|)\left|f'(z)\right|=0.$$

This function space is called the little Bloch-type space.

Let n be a nonnegative integer, φ be an analytic self-map of $\mathbb{D}$ and $u\in H(\mathbb{D})$. The generalized weighted composition operator $D_{\varphi,u}^n$ is defined by

$$D_{\varphi,u}^n f=uf^{(n)}\circ\varphi,\ f\in H(\mathbb{D}), \tag{2}$$

where $f^{(0)}=f$. The generalized weighted composition operator $D_{\varphi,u}^n$ can be regarded as a product of composition operator C_φ, multiplication operator M_u and the n-th differentiation operator D^n. The generalized weighted composition operator $D_{\varphi,u}^n$ was introduced in [25] by the second author of this paper, and studied in [25, 27, 29, 17].

It is interesting to provide a function theoretic characterization of φ and u when they induce a bounded or compact operator between spaces of analytic functions in the unit disk, the polydisk and the unit ball. The books [2, 24] contain plenty of information on this topic for $D_{\varphi,u}^n$ in the case of $n=0$ and $u(z)=1$, i.e. for the composition operator C_φ.

In the case of $n=0$, $D_{\varphi,u}^n$ is the weighted composition operator uC_φ. The second author of this paper studied the weighted composition operator from the area Nevanlinna space to the Bloch space in [28]. Weighted composition operators between other analytic function spaces are studied, for example in [3, 4, 7, 8, 9, 10, 11, 13, 15, 16, 18, 19, 21, 22, 26, 28] (see also related references therein). The case of $n = 1$ and $u(z) = \varphi'(z)$, that is $D^n_{\varphi,u} = DC_\varphi$, was studied in [5, 6, 23]. The case of $n = 1$ and $u(z) = 1$, that is $D^n_{\varphi,u} = C_\varphi D$, was studied in [5, 23].

In this paper we study the generalized weighted composition operator. We give some sufficient and necessary conditions for the boundedness and compactness of the generalized weighted composition operator from the area Nevanlinna space to the Bloch-type space and the little Bloch-type space.

Throughout this paper C denotes a positive constant which may be different at different occurrences.

2. Main Results and Proofs

In this section, we give some auxiliary results which will be used in proving the main results of this paper. They are incorporated in the lemmas which follow.

Lemma 1. *Let n be a nonnegative integer, $1 \le p < \infty$ and $\alpha > -1$. Then there exists some C such that for each $f \in \mathcal{N}^p_\alpha$ and $z \in \mathbb{D}$,*

$$|f^{(n)}(z)| \le \frac{1}{(1-|z|^2)^n} \exp\Big[\frac{C\|f\|_{\mathcal{N}^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}\Big]. \tag{3}$$

Proof. For $z \in \mathbb{D}$ and $\xi \in \partial\mathbb{D}$, we have

$$1 - \left|z + \frac{1-|z|}{2}\xi\right|^2 \ge 1 - \frac{(1+|z|)^2}{4} \ge \frac{1-|z|^2}{4}.$$

From this, and then using Cauchy integral formula and (1), we have

$$\begin{aligned}
|f^{(n)}(z)| &= \left|\frac{n!}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi\right| \\
&\le \frac{n!2^n}{2\pi(1-|z|)^n}\int_{\partial\mathbb{D}} |f(z + \frac{1-|z|}{2}\xi)||d\xi| \\
&\le \frac{1}{2\pi(1-|z|^2)^n}\int_{\partial\mathbb{D}} (1 + n!4^n|f(z + \frac{1-|z|}{2}\xi)|)|d\xi| \\
&\le \frac{1}{2\pi(1-|z|^2)^n}\int_{\partial\mathbb{D}} \exp\Big[n!4^n\log(1 + |f(z + \frac{1-|z|}{2}\xi)|)\Big]|d\xi| \\
&\le \frac{1}{2\pi(1-|z|^2)^n}\int_{\partial\mathbb{D}} \exp\Big[\frac{C\|f\|_{\mathcal{N}^p_\alpha}}{(1-|z+\frac{1-|z|}{2}\xi|^2)^{\frac{2+\alpha}{p}}}\Big]|d\xi|
\end{aligned}$$

$$\le \frac{1}{2\pi(1-|z|^2)^n}\int_{\partial\mathbb{D}} \exp\Big[\frac{C\|f\|_{\mathcal{N}_\alpha^p}}{(\frac{1-|z|^2}{4})^{\frac{2+\alpha}{p}}}\Big]|d\xi|$$
$$\le \frac{1}{(1-|z|^2)^n}\exp\Big[\frac{C\|f\|_{\mathcal{N}_\alpha^p}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}\Big],$$

from which the result follows.

Lemma 2. *A closed set K in $\mathcal{B}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(|z|)|f'(z)| = 0.$$

Proof. The proof is similar to the proof of Lemma 1 in [12], so we omit it here.

The following criterion for compactness follows from arguments similar, for example, to those outlined in Lemma 2.3 of [21].

Lemma 3. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, $1 \le p < \infty$ and $\alpha > -1$. The operator $D^n_{\varphi,u} : \mathcal{N}_\alpha^p \to \mathcal{B}_\mu$ is compact if and only if for each bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{N}_\alpha^p$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|D^n_{\varphi,u}f_k\|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$.*

The next Lemma 4 is the classic Faàdi Bruno's formula (see, e.g. [20]).

Lemma 4. *If $f(z)$ is an analytic function in complex plane and $\varphi(z) \in H(\mathbb{D})$, then for each positive integer n,*

$$(f\circ\varphi)^{(n)}(z) = \sum \frac{n!}{k_1!k_2!\cdots k_n!} f^{(k)}(\varphi(z))\prod_{j=1}^{n}(\frac{\varphi^{(j)}(z)}{j!})^{k_j},\ z\in\mathbb{D},$$

where the sum is over all different solutions in nonnegative integers $k_1, k_2, \cdots, k_n$ of $k = k_1 + k_2 + \cdots + k_n$ and $n = k_1 + 2k_2 + \cdots + nk_n$.

Now, we are ready to formulate and prove the main results of this paper.

Theorem 1. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, $1 \le p < \infty$, $\alpha > -1$ and μ is a normal function on $[0, 1)$. Then for each positive integer n, $D^n_{\varphi,u} : \mathcal{N}_\alpha^p \to \mathcal{B}_\mu$ is bounded if and only if for all $c > 0$,*

$$M_1(c) := \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] < \infty \tag{4}$$

and

$$M_2(c) := \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] < \infty. \tag{5}$$

Proof. Suppose that $D^n_{\varphi,u} : \mathcal{N}^p_\alpha \to \mathcal{B}_\mu$ is bounded. By utilizing test functions $\frac{z^n}{n!}$ and $\frac{z^{n+1}}{(n+1)!}$, we obtain $\sup_{z\in\mathbb{D}} \mu(|z|)|u'(z)| < \infty$ and $\sup_{z\in\mathbb{D}} \mu(|z|)|u'(z)\varphi(z) + u(z)\varphi'(z)| < \infty$. Hence $u \in \mathcal{B}_\mu$ and $\sup_{z\in\mathbb{D}} \mu(|z|)|u(z)\varphi'(z)| < \infty$.

For each $c > 0$ and $z \in \mathbb{D}$, take $f_z(w) = \exp(ch_1(w)) - 1$, where

$$h_1(w) = \Big[\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}w)^2}\Big]^{\frac{2+\alpha}{p}},\ w \in \mathbb{D}.$$

Using Lemma 4.2.2 in [24] and the inequality $|e^t - 1| \le e^{|t|} - 1$, $t \in \mathbb{C}$, we have

$$\int_{\mathbb{D}} \big[\log(1+|f_z(w)|)\big]^p dA_\alpha(w) \le \int_{\mathbb{D}} c^p |h_1(w)|^p dA_\alpha(w) < \infty.$$

Then $f_z \in \mathcal{N}^p_\alpha$ for all $z \in \mathbb{D}$. By Lemma 4, for each positive integer n,

$$\begin{aligned} &f_z^{(n)}(w) = [\exp(ch_1(w))]^{(n)} \\ &= \sum \frac{n!\exp^{(k)}(ch_1(w))}{k_1!k_2!\cdots k_n!}\prod_{j=1}^n\Big(\frac{ch_1^{(j)}(w)}{j!}\Big)^{k_j} \\ &= \sum \frac{n!\exp(ch_1(w))}{k_1!k_2!\cdots k_n!}\prod_{j=1}^n\Big[\frac{c2\tau(2\tau+1)\cdots(2\tau+j-1)\overline{\varphi(z)}^j(1-|\varphi(z)|^2)^\tau}{j!(1-\overline{\varphi(z)}w)^{2\tau+j}}\Big]^{k_j}, \end{aligned}$$

where and thereafter $\tau = \frac{2+\alpha}{p}$. Then

$$\begin{aligned} &f_z^{(n)}(\varphi(z)) \\ &= \sum \frac{n!\exp(ch_1(\varphi(z)))}{k_1!k_2!\cdots k_n!}\prod_{j=1}^n\Big[\frac{c2\tau(2\tau+1)\cdots(2\tau+j-1)\overline{\varphi(z)}^j(1-|\varphi(z)|^2)^\tau}{j!(1-|\varphi(z)|^2)^{2\tau+j}}\Big]^{k_j} \\ &= \sum \frac{n!\exp(ch_1(\varphi(z)))}{k_1!k_2!\cdots k_n!}\prod_{j=1}^n\Big[\frac{c2\tau(2\tau+1)\cdots(2\tau+j-1)\overline{\varphi(z)}^j}{j!(1-|\varphi(z)|^2)^{\tau+j}}\Big]^{k_j} \\ &= \sum \frac{n!\overline{\varphi(z)}^n\exp(ch_1(\varphi(z)))}{k_1!k_2!\cdots k_n!(1-|\varphi(z)|^2)^{k\tau+n}}\prod_{j=1}^n\Big[\frac{c2\tau(2\tau+1)\cdots(2\tau+j-1)}{j!}\Big]^{k_j} \qquad (6) \\ &= \sum \frac{n!\overline{\varphi(z)}^n\exp(ch_1(\varphi(z)))}{k_1!k_2!\cdots k_n!(1-|\varphi(z)|^2)^{k\tau+n}}\prod_{j=1}^n\Big[\frac{c2\tau(2\tau+1)\cdots(2\tau+j-1)}{j!}\Big]^{k_j} \\ &= \frac{\overline{\varphi(z)}^n\exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{n\tau+n}}\sum\frac{n!\prod_{j=1}^n\Big[\frac{c2\tau(2\tau+1)\cdots(2\tau+j-1)}{j!}\Big]^{k_j}}{k_1!k_2!\cdots k_n!}(1-|\varphi(z)|^2)^{(n-k)\tau} \\ &= \frac{\overline{\varphi(z)}^n\exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{n\tau+n}}P_{n-1}[2\tau,(1-|\varphi(z)|^2)^\tau]. \end{aligned}$$

Here $P_{n-1}[\lambda, x]$ is the n-1-degree polynomial, i.e.

$$\begin{aligned} P_{n-1}[\lambda, x] &= \sum \frac{n! \prod_{j=1}^{n} \left[\frac{c\lambda(\lambda+1)\cdots(\lambda+j-1)}{j!}\right]^{k_j}}{k_1! k_2! \cdots k_n!} x^{n-k} \\ &= \sum_{k=1}^{n} n! x^{n-k} \sum_{(k_1, k_2, \cdots, k_n) \in S_k} \frac{\prod_{j=1}^{n} \left[\frac{c\lambda(\lambda+1)\cdots(\lambda+j-1)}{j!}\right]^{k_j}}{k_1! k_2! \cdots k_n!}, \end{aligned}$$

and S_k is the set of solutions in nonnegative integers $k_1, k_2, \cdots, k_n$ of $k = k_1 + k_2 + \cdots + k_n$ and $n = k_1 + 2k_2 + \cdots + nk_n$. It is easy to see that for a fixed parameter λ, polynomial $P_{n-1}[\lambda, (1-|\varphi(z)|^2)^\tau]$ is a bounded real-valued function for all $z \in \mathbb{D}$, and the constant term of $P_{n-1}[\lambda, x]$ is $(c\lambda)^n$. Then $P_{n-1}[\lambda, (1-|\varphi(z)|^2)^\tau] \geq (c\lambda)^n$. On the other hand, for a fixed $x \in (0,1)$, $P_{n-1}[\lambda, x]$ is a monotonously increasing and unbounded function for $\lambda \in (0, +\infty)$. From (6), we get

$$\begin{aligned} & \sup_{w \in \mathbb{D}} \mu(|w|) \left| \left[u(w) f_z^{(n)}(\varphi(w)) \right]' \right| \\ =& \sup_{w \in \mathbb{D}} \mu(|w|) \left| u'(w) f_z^{(n)}(\varphi(w)) + u(w) f_z^{(n+1)}(\varphi(w)) \varphi'(w) \right| \\ \geq& \mu(|z|) \left| u'(z) f_z^{(n)}(\varphi(z)) + u(z) f_z^{(n+1)}(\varphi(z)) \varphi'(z) \right| \\ =& \Big| \frac{\mu(|z|) u'(z) \overline{\varphi(z)}^n \exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{n(\tau+1)}} P_{n-1}[2\tau, (1-|\varphi(z)|^2)^\tau] + \\ & \frac{\mu(|z|) u(z) \varphi'(z) \overline{\varphi(z)}^{n+1} \exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{(n+1)(\tau+1)}} P_n[2\tau, (1-|\varphi(z)|^2)^\tau] \Big| \\ =& |T_1 + T_2|. \end{aligned} \tag{7}$$

Here

$$T_1 = \frac{\mu(|z|) u'(z) \overline{\varphi(z)}^n \exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{n(\tau+1)}} P_{n-1}[2\tau, (1-|\varphi(z)|^2)^\tau]$$

and

$$T_2 = \frac{\mu(|z|) u(z) \varphi'(z) \overline{\varphi(z)}^{n+1} \exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{(n+1)(\tau+1)}} P_n[2\tau, (1-|\varphi(z)|^2)^\tau].$$

Hence

$$|T_1 + T_2| \leq C \| D_{\varphi,u}^n f_z \|_{\mathcal{B}_\mu}, \tag{8}$$

which implies that

$$|T_1| \leq |T_2| + C \| D_{\varphi,u}^n \|_{\mathcal{N}_\alpha^p \to \mathcal{B}_\mu}. \tag{9}$$

Now take $g_z(w) = R(z)\exp(ch_2(w)) - \exp(ch_1(w))$, where

$$h_2(w) = \Big[\frac{(1-|\varphi(z)|^2)^\kappa}{(1-\overline{\varphi(z)}w)^{\kappa+1}}\Big]^{\frac{2+\alpha}{p}},\ w \in \mathbb{D},$$

and

$$R(z) = \frac{P_{n-1}[2\tau,(1-|\varphi(z)|^2)^\tau]}{P_{n-1}[\kappa\tau,(1-|\varphi(z)|^2)^\tau]}.$$

By the monotonicity of the function $P_n[\lambda, x]$, there exists $\delta_0 >$ and $\kappa \in \mathbb{N}$ such that

(10)
$$\begin{aligned}&\Big|R(z)P_n[\kappa\tau,(1-|\varphi(z)|^2)^\tau] - P_n[2\tau,(1-|\varphi(z)|^2)^\tau]\Big|\\ =\ &R(z)P_n[\kappa\tau,(1-|\varphi(z)|^2)^\tau] - P_n[2\tau,(1-|\varphi(z)|^2)^\tau] \geq \delta_0.\end{aligned}$$

With the arguments similar to that on f_z, we obtain $g_z \in \mathcal{N}_\alpha^p$ for all $z \in \mathbb{D}$. Moreover

$$\begin{aligned}g_z^{(n)}(\varphi(z)) =\ &R(z)\frac{\overline{\varphi(z)}^n\exp(ch_2(\varphi(z)))}{(1-|\varphi(z)|^2)^{n\tau+n}}P_{n-1}[\kappa\tau,(1-|\varphi(z)|^2)^\tau] -\\ &\frac{\overline{\varphi(z)}^n\exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{n\tau+n}}P_{n-1}[2\tau,(1-|\varphi(z)|^2)^\tau] = 0,\end{aligned}$$

and

$$\begin{aligned}g_z^{(n+1)}(\varphi(z)) =\ &R(z)\frac{\overline{\varphi(z)}^{\,n+1}\exp(ch_2(\varphi(z)))}{(1-|\varphi(z)|^2)^{(n+1)(\tau+1)}}P_n[\kappa\tau,(1-|\varphi(z)|^2)^\tau]\\ &-\frac{\overline{\varphi(z)}^{n+1}\exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{(n+1)(\tau+1)}}P_n[2\tau,(1-|\varphi(z)|^2)^\tau]\\ =\ &\frac{\overline{\varphi(z)}^{n+1}\exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{(n+1)(\tau+1)}}\\ &\Big\{R(z)P_n[\kappa\tau,(1-|\varphi(z)|^2)^\tau] - P_n[2\tau,(1-|\varphi(z)|^2)^\tau]\Big\}.\end{aligned}$$

Therefore,

(11)
$$\begin{aligned}&C\|D_{\varphi,u}^n\|_{\mathcal{N}_\alpha^p\to\mathcal{B}_\mu} \geq \|D_{\varphi,u}^n\|_{\mathcal{N}_\alpha^p\to\mathcal{B}_\mu}\|g_z\|_{\mathcal{N}_\alpha^p} \geq \|D_{\varphi,u}^n g_z\|_{\mathcal{B}_\mu}\\ \geq\ &\sup_{w\in\mathbb{D}}\mu(|w|)\Big|\Big[u(w)g_z^{(n)}(\varphi(w))\Big]'\Big|\\ =\ &\sup_{w\in\mathbb{D}}\mu(|w|)\Big|u'(w)g_z^{(n)}(\varphi(w)) + u(w)g_z^{(n+1)}(\varphi(w))\varphi'(w)\Big|\\ \geq\ &\mu(|z|)\Big|u'(z)g_z^{(n)}(\varphi(z)) + u(z)g_z^{(n+1)}(\varphi(z))\varphi'(z)\Big|\\ =\ &\frac{\mu(|z|)\Big|u(z)\varphi'(z)\overline{\varphi(z)}^{n+1}\Big|\exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{(n+1)(\tau+1)}}\times\\ &\Big\{R(z)P_n[\kappa\tau,(1-|\varphi(z)|^2)^\tau] - P_n[2\tau,(1-|\varphi(z)|^2)^\tau]\Big\}.\end{aligned}$$

This implies that T_2 is bounded and hence T_1 is bounded by (9), i.e., for each $z \in \mathbb{D}$,

$$\frac{\mu(|z|)|u'(z)|\exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^n} \leq \frac{C(1-|\varphi(z)|^2)^{n\tau}}{|\varphi(z)|^n P_{n-1}[2\tau,(1-|\varphi(z)|^2)^\tau]} \tag{12}$$

and

$$\frac{\mu(|z|)|u(z)\varphi'(z)|\exp(ch_1(\varphi(z)))}{(1-|\varphi(z)|^2)^{n+1}} \leq \frac{C(1-|\varphi(z)|^2)^{(n+1)\tau}}{|\varphi(z)|^{n+1} P_n[2\tau,(1-|\varphi(z)|^2)^\tau]}, \tag{13}$$

which imply that (4) and (5) hold for all $c > 0$.

Conversely, suppose (4) and (5) hold for all $c > 0$. Let S be a bounded subset in $\mathcal{N}_\alpha^p$. Then there exists a positive number K such that $\|f\|_{\mathcal{N}_\alpha^p} \leq K$ for all $f \in S$. Then, by Lemma 1 we have

$$\begin{aligned}
&\|D_{\varphi,u}^n f\|_{\mathcal{B}_\mu} = \sup_{z\in\mathbb{D}} \mu(|z|)|(D_{\varphi,u}^n f)'(z)| + |u(0)f^{(n)}(\varphi(0))| \\
&= \sup_{z\in\mathbb{D}} \mu(|z|)|u'(z)f^{(n)}(\varphi(z)) \\
&\quad + u(z)f^{(n+1)}(\varphi(z))\varphi'(z)| + |u(0)f^{(n)}(\varphi(0))| \\
&\leq \sup_{z\in\mathbb{D}} \mu(|z|)|u'(z)f^{(n)}(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}} \mu(|z|)|u(z)f^{(n+1)}(\varphi(z))\varphi'(z)| + |u(0)f^{(n)}(\varphi(0))| \\
&\leq \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{C\|f\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] + \\
&\quad \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp\Big[\frac{C\|f\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] + |u(0)f^{(n)}(\varphi(0))| \\
&\leq M_1(CK) + M_2(CK) + |u(0)f^{(n)}(\varphi(0))| < \infty,
\end{aligned} \tag{14}$$

for all $f \in S$. This implies that $D_{\varphi,u}^n(S)$ is a bounded subset of $\mathcal{B}_\mu$, and then $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{B}_\mu$ is a bounded operator. The proof is completed.

Corollary 1. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, $1 \leq p < \infty$, $\alpha > -1$ and μ is a normal function on $[0,1)$. Then for each positive integer n, $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{B}_\mu$ is bounded if and only if the following conditions satisfied:*

(i) $u(z) \in \mathcal{B}_\mu$ and $\sup_{z\in\mathbb{D}} \mu(|z|)|u(z)\varphi'(z)| < \infty$;

(ii) for all $c > 0$,

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] = 0 \tag{15}$$

and

(16) $$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]=0.$$

Proof. Suppose $D^n_{\varphi,u}:\mathcal{N}^p_\alpha\to\mathcal{B}_\mu$ is bounded. From the proof part of Theorem 1, we get (i) directly. Moreover, (12) and (13) implies (15) and (16) hold for all $c>0$. On the other hand, employing (14), conditions (i) and (ii) lead that $D^n_{\varphi,u}:\mathcal{N}^p_\alpha\to\mathcal{B}_\mu$ is bounded. The proof is completed.

Theorem 2. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u\in H(\mathbb{D})$, $1\le p<\infty$, $\alpha>-1$ and μ is a normal function on $[0,1)$. Then for each positive integer n, $D^n_{\varphi,u}:\mathcal{N}^p_\alpha\to\mathcal{B}_\mu$ is compact if and only if $D^n_{\varphi,u}:\mathcal{N}^p_\alpha\to\mathcal{B}_\mu$ is bounded and for all $c>0$,*

(17) $$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{n}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]=0$$

and

(18) $$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]=0.$$

Proof. Let $\{z_k\}$ be a sequence such that $|\varphi(z_k)|\to 1$ as $k\to\infty$(if such a sequence does not exist then (17) and (18) automatically hold). For each $c>0$, take

$$f_k(w)=\exp\Big\{c\Big[\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}w)^2}\Big]\Big\}^{\frac{2+\alpha}{p}}-1.$$

From the proof of Theorem 1, we can conclude that $\{f_k\}$ is a bounded sequence in $\mathcal{N}^p_\alpha$ and $f_k\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. From the compactness of $D^n_{\varphi,u}$, we have $\lim_{k\to\infty}\|D^n_{\varphi,u}f_k\|_{\mathcal{B}_\mu}=0$. From (8), we get

$$\Big|\frac{\mu(|z_k|)u'(z_k)\overline{\varphi(z_k)}^n\exp(ch_1(\varphi(z_k)))}{(1-|\varphi(z_k)|^2)^{n(\tau+1)}}P_{n-1}[2\tau,(1-|\varphi(z_k)|^2)^\tau]\Big|\le\|D^n_{\varphi,u}f_k\|_{\mathcal{B}_\mu}$$
$$+\Big|\frac{\mu(|z_k|)u(z_k)\varphi'(z_k)\overline{\varphi(z_k)}^{n+1}\exp(ch_1(\varphi(z_k)))}{(1-|\varphi(z_k)|^2)^{(n+1)(\tau+1)}}P_n[2\tau,(1-|\varphi(z_k)|^2)^\tau]\Big|.$$

Thus

(19) $$\frac{\mu(|z_k|)|u'(z_k)|\exp(ch_1(\varphi(z_k)))}{(1-|\varphi(z_k)|^2)^n}\le\frac{(1-|\varphi(z_k)|^2)^{n\tau}\|D^n_{\varphi,u}f_k\|_{\mathcal{B}_\mu}}{|\varphi(z_k)|^nP_{n-1}[2\tau,(1-|\varphi(z_k)|^2)^\tau]}$$
$$+\frac{\mu(|z_k|)|u(z_k)\varphi'(z_k)\overline{\varphi(z_k)}|\exp(ch_1(\varphi(z_k)))P_n[2\tau,(1-|\varphi(z_k)|^2)^\tau]}{(1-|\varphi(z_k)|^2)^{(n+\tau+1)}P_{n-1}[2\tau,(1-|\varphi(z_k)|^2)^\tau]}.$$

Next set

$$g_k(w) = R(z_k)\exp(c\Big[\frac{(1-|\varphi(z_k)|^2)^{\kappa}}{(1-\overline{\varphi(z_k)}w)^{\kappa+1}}\Big]^{\frac{2+\alpha}{p}}) \\ -R(z_k)+1-\exp(c\Big[\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}w)^2}\Big]^{\frac{2+\alpha}{p}}).$$

Similarly, $\{g_k\}$ is a bounded sequence in $\mathcal{N}_\alpha^p$ and $g_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, then $\lim_{k\to\infty}\|D^n_{\varphi,u}g_k\|_{\mathcal{B}_\mu} = 0$. From (11), we have

$$\|D^n_{\varphi,u}g_k\|_{\mathcal{B}_\mu} \geq \frac{\mu(|z_k|)\Big|u(z_k)\varphi'(z_k)\overline{\varphi(z_k)}^{n+1}\Big|\exp(ch_1(\varphi(z_k)))}{(1-|\varphi(z_k)|^2)^{(n+1)(\tau+1)}}\times \\ \Big\{R(z_k)P_n[\kappa\tau,(1-|\varphi(z_k)|^2)^\tau]-P_n[2\tau,(1-|\varphi(z_k)|^2)^\tau]\Big\},$$

which together with (19) imply

$$\frac{(1-|\varphi(z_k)|^2)^{(n+1)\tau}\|D^n_{\varphi,u}g_k\|_{\mathcal{B}_\mu}}{|\varphi(z_k)|^{n+1}\Big\{R(z_k)P_n[\kappa\tau,(1-|\varphi(z_k)|^2)^\tau]-P_n[2\tau,(1-|\varphi(z_k)|^2)^\tau]\Big\}} \\ \geq \frac{\mu(|z_k|)\Big|u(z_k)\varphi'(z_k)\Big|\exp(ch_1(\varphi(z_k)))}{(1-|\varphi(z_k)|^2)^{n+1}} \tag{20}$$

and

$$\frac{(1-|\varphi(z_k)|^2)^{n\tau}\|D^n_{\varphi,u}g_k\|_{\mathcal{B}_\mu}}{|\varphi(z_k)|^{n}\Big\{R(z_k)P_n[\kappa\tau,(1-|\varphi(z_k)|^2)^\tau]-P_n[2\tau,(1-|\varphi(z_k)|^2)^\tau]\Big\}} \\ \geq \frac{\mu(|z_k|)|u(z_k)\varphi'(z_k)\varphi(z_k)|\exp(ch_1(\varphi(z_k)))}{(1-|\varphi(z_k)|^2)^{n+\tau+1}}. \tag{21}$$

From the last two inequalities we obtain the desired result.

Conversely, suppose that $D^n_{\varphi,u}:\mathcal{N}_\alpha^p \to \mathcal{B}_\mu$ is bounded and (17) and (18) hold for all $c > 0$. Let $\{f_k\}$ be a sequence in $\mathcal{N}_\alpha^p$ such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ and $\|f_k\|_{\mathcal{N}_\alpha^p} \leq K$. Then it is obvious that $f_k^{(n)}$ and $f_k^{(n+1)} \to 0$ uniformly on compact subsets of $\mathbb{D}$. Moreover, by (17) and (18) we have that for every $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$\frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] < \frac{\varepsilon}{2} \tag{22}$$

and

$$\frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] < \frac{\varepsilon}{2} \tag{23}$$

whenever $\delta < |\varphi(z)| < 1$.

By Lemma 1, (22) and (23) we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu(|z|)|(D^n_{\varphi,u}f_k)'(z)| \\
&= \sup_{z\in\mathbb{D}} \mu(|z|)|u'(z)f_k^{(n)}(\varphi(z)) + u(z)f_k^{(n+1)}(\varphi(z))\varphi'(z)| \\
&\le \sup_{z\in\mathbb{D}} \mu(|z|)|u'(z)f_k^{(n)}(\varphi(z))| + \sup_{z\in\mathbb{D}} |u(z)f_k^{(n+1)}(\varphi(z))\varphi'(z)| \\
&\le \sup_{|\varphi(z)|\le\delta} \mu(|z|)|u'(z)||f_k^{(n)}(\varphi(z))| + \sup_{|\varphi(z)|\le\delta} \mu(|z|)|u(z)\varphi'(z)||f_k^{(n+1)}(\varphi(z))| \\
&\quad + \sup_{\delta<|\varphi(z)|<1} \mu(|z|)|u'(z)||f_k^{(n)}(\varphi(z))| \\
&\quad + \sup_{\delta<|\varphi(z)|<1} \mu(|z|)|u(z)\varphi'(z)||f_k^{(n+1)}(\varphi(z))| \\
&\le C \sup_{|\varphi(z)|\le\delta} |f_k^{(n)}(\varphi(z))| + C \sup_{|\varphi(z)|\le\delta} |f_k^{(n+1)}(\varphi(z))| \\
&\quad + \sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{C\|f_k\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] \\
&\quad + \sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp\Big[\frac{C\|f_k\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] \\
&\le C \sup_{|w|\le\delta} |f_k^{(n)}(w)| + C \sup_{|w|\le\delta} |f_k^{(n+1)}(w)| + \varepsilon,
\end{aligned}$$

i.e.

$$\begin{aligned}
\|D^n_{\varphi,u}f_k\|_{\mathcal{B}_\mu} \le{}& C \sup_{|w|\le\delta} |f_k^{(n)}(w)| + C \sup_{|w|\le\delta} |f_k^{(n+1)}(w)| \\
&+\varepsilon + |u(0)f_k^{(n)}(\varphi(0))|.
\end{aligned}$$

This yields $\lim_{k\to\infty} \|D^n_{\varphi,u}f_k\|_{\mathcal{B}_\mu} = 0$. By Lemma 3, we see that the operator $D^n_{\varphi,u} : \mathcal{N}^p_\alpha \to \mathcal{B}_\mu$ is compact. The proof is completed.

From Theorem 1, Corollary 1 and Theorem 2 we can obtain the following corollary.

Corollary 2. *Suppose that* φ *is an analytic self-map of* $\mathbb{D}$, $u \in H(\mathbb{D})$, $1 \le p < \infty$, $\alpha > -1$ *and* μ *is a normal function on* $[0, 1)$. *Then for each positive integer* n, *the following statements are equivalent.*

(i) $D^n_{\varphi,u} : \mathcal{N}^p_\alpha \to \mathcal{B}_\mu$ *is bounded;*

(ii) $D^n_{\varphi,u} : \mathcal{N}^p_\alpha \to \mathcal{B}_\mu$ *is compact;*

(iii) $u \in \mathcal{B}_\mu$, $\sup_{z\in\mathbb{D}} \mu(|z|)|u(z)\varphi'(z)| < \infty$, *and both* (15) *and* (16) *hold for all* $c > 0$.

Theorem 3. *Suppose that* φ *is an analytic self-map of* $\mathbb{D}$, $u \in H(\mathbb{D})$, $1 \le p < \infty$, $\alpha > -1$ *and* μ *is a normal function on* $[0, 1)$. *Then for each positive integer* n, *the following statements are equivalent.*

(i) $D^n_{\varphi,u}:\mathcal{N}^p_\alpha\to\mathcal{B}_{\mu,0}$ is bounded;

(ii) $D^n_{\varphi,u}:\mathcal{N}^p_\alpha\to\mathcal{B}_{\mu,0}$ is compact;

(iii) $u\in\mathcal{B}_{\mu,0}$, $\lim_{|z|\to 1}\mu(|z|)|u(z)\varphi'(z)|=0$ and for all $c>0$,

$$\lim_{|z|\to 1}\frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]=0 \tag{24}$$

and

$$\lim_{|z|\to 1}\frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]=0. \tag{25}$$

Proof. $(ii)\Rightarrow(i)$. This implication is obvious.

$(i)\Rightarrow(iii)$. Suppose that $D^n_{\varphi,u}:\mathcal{N}^p_\alpha\to\mathcal{B}_{\mu,0}$ is bounded. By utilizing functions $\frac{z^n}{n!}$ and $\frac{z^{n+1}}{(n+1)!}$, we obtain

$$\lim_{|z|\to 1}\mu(|z|)|u'(z)|=0 \text{ and } \lim_{|z|\to 1}\mu(|z|)|u'(z)\varphi(z)+u(z)\varphi'(z)|=0.$$

Then

$$u\in\mathcal{B}_{\mu,0} \text{ and } \lim_{|z|\to 1}\mu(|z|)|u(z)\varphi'(z)|=0.$$

Since $D^n_{\varphi,u}:\mathcal{N}^p_\alpha\to\mathcal{B}_\mu$ is bounded, by Corollary 1, we conclude that the condition (15) and (16) hold for all $c>0$. Thus, for each $c,\varepsilon>0$, there exists a $t\in(0,1)$ such that

$$\frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]<\varepsilon, \tag{26}$$

whenever $t<|\varphi(z)|<1$. Moreover, from $u(z)\in\mathcal{B}_{\mu,0}$, we infer that there exists an $r\in(0,1)$ such that for $r<|z|<1$,

$$\mu(|z|)|u'(z)|<\varepsilon(1-t^2)^n\exp\Big[\frac{-c}{(1-t^2)^{\frac{2+\alpha}{p}}}\Big],$$

from which, if $r<|z|<1$ and $|\varphi(z)|\le t$, then we have

$$\frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]<\varepsilon. \tag{27}$$

From (26) and (27), we see that whenever $r<|z|<1$,

$$\frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]<\varepsilon,$$

which implies that (24) holds for all $c > 0$. Employing (16) and $\lim_{|z|\to 1} \mu(|z|)|u(z)\varphi'(z)| = 0$, with similar argument, we obtain (25) holds for all $c > 0$.

$(iii) \Rightarrow (ii)$. Suppose $u(z) \in \mathcal{B}_{\mu,0}$, $\lim_{|z|\to 1} \mu(|z|)|u(z)\varphi'(z)| = 0$ and (24) and (25) hold for all $c > 0$. From Lemma 2, $D^n_{\varphi,u} : \mathcal{N}^p_\alpha \to \mathcal{B}_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{f \in B_{\mathcal{N}^p_\alpha}} \mu(|z|)|(D^n_{\varphi,u} f)'(z)| = 0, \tag{28}$$

where $B_{\mathcal{N}^p_\alpha} = \{g \in \mathcal{N}^p_\alpha : \|g\|_{\mathcal{N}^p_\alpha} \le 1\}$ is the unit ball in the space $\mathcal{N}^p_\alpha$.

On the other hand, by Lemma 1, we have

$$\begin{aligned} \mu(|z|)|(D^n_{\varphi,u} f)'(z)| \le\ & \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{C\|f\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] + \\ & \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp\Big[\frac{C\|f\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]. \end{aligned} \tag{29}$$

Taking the supremum in (29) over the unit ball $B_{\mathcal{N}^p_\alpha}$, and letting $|z| \to 1$, from (24) and (25) we see that (28) holds and hence $D^n_{\varphi,u} : \mathcal{N}^p_\alpha \to \mathcal{B}_{\mu,0}$ is compact. The proof is completed.

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