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NONLINEAR CONDITIONS FOR COINCIDENCE POINT AND FIXED POINT THEOREMS

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Abstract. In this paper, we first establish some new types of fixed point theorem which generalize and improve Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem and many results in [W.-S. Du, Some new results and generalizations in metric fixed point theory, Nonlinear Anal. 73 (2010), 1439-1446] and references therein. Applying those new results, we also present some existence theorems of coincidence point and others.

1. Introduction

The celebrated Banach contraction principle (see, e.g., [1]) plays an important role in various fields of applied mathematical analysis. It is known that Banach contraction principle has been used to solve the existence of solutions for nonlinear integral equations and nonlinear differential equations in Banach spaces. It has also been applied to study the convergence of algorithms in computational mathematics. Since then a number of generalizations in various different directions of the Banach contraction principle have been investigated by several authors in the past; see [1, 3-6] and references therein.

Theorem BCP. (Banach)

Let (X, d) be a complete metric space and $T : X \to X$ be a selfmap. Assume that there exists a nonnegative number $\gamma < 1$ such that

$$d(T(x), T(y)) \leq \gamma d(x, y) \text{ for all } x, y \in X.$$

Then T has a unique fixed point in X. Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges to the fixed point.

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In 1969, Nadler [2] first gave a famous generalization of the Banach contraction principle for multivalued maps.

Theorem NA. (Nadler)

Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a k-contraction; that is, there exists a nonnegative number $k < 1$ such that

$$\mathcal{H}(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X,$$

where $\mathcal{CB}(X)$ is the class of all nonempty closed bounded subsets of X. Then there exists $v \in X$ such that $v \in Tv$.

In 1989, Mizoguchi and Takahashi [3] proved the following fixed point theorem which is a generalization of Nadler's fixed point theorem and gave a partial answer of Problem 9 in Reich [7]. It is worth to mention that the primitive proof of Mizoguchi-Takahashi's fixed point theorem is different. Recently, Suzuki [8] gave a very simple proof of Mizoguchi-Takahashi's fixed point theorem.

Theorem MT. (Mizoguchi and Takahashi)

Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued map. Assume that

$$\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

for all $x, y \in X$, where α is a function from $[0, \infty)$ into $[0, 1)$ satisfying $\limsup_{s \to t+0} \alpha(s) < 1$ for all $t \in [0, \infty)$. Then there exists $v \in X$ such that $v \in Tv$.

In 2007, M. Berinde and V. Berinde [4] proved the following interesting fixed point theorem which generalized Mizoguchi-Takahashi's fixed point theorem.

Theorem BB. (M. Berinde and V. Berinde)

Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map and $L \geq 0$. Assume that

$$\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$, where α is a function from $[0, \infty)$ into $[0, 1)$ satisfying $\limsup_{s \to t+0} \alpha(s) < 1$ for all $t \in [0, \infty)$. Then there exists $v \in X$ such that $v \in Tv$.

Very recently, the first author [6] established some new fixed point theorems for nonlinear multivalued contractive maps by using τ^0-metric (see Def. 2.1 below) and $\mathcal{MT}$-function (see Def. 2.3 below). Applying those results, the first author gave the generalizations of Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem and Banach contraction principle, Kannan's fixed point theorems and Chatterjea's fixed point theorems for nonlinear multivalued contractive maps in complete metric spaces; see [6] for more detail.

In this paper, we first establish a new type of fixed point theorem which generalizes and improves many results in [3-6, 10]. Some applications to the existence theorems of coincidence point and others are also given.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively. Let (X, d) be a metric space. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y \in A} d(x, y)$. Denote by $\mathcal{N}(X)$ the class of all nonempty subsets of X, $\mathcal{C}(X)$ the family of all nonempty closed subsets of X and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ defined by

$$\mathcal{H}(A, B) = \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X.

A point v in X is a fixed point of a map T if $v = Tv$ (when $T : X \to X$ is a single-valued map) or $v \in Tv$ (when $T : X \to \mathcal{N}(X)$ is a multivalued map). The set of fixed points of T is denoted by $\mathcal{F}(T)$.

Let $g : X \to X$ be a self-map and $T : X \to \mathcal{N}(X)$ be a multivalued map. A point x in X is said to be a *coincidence point* (see, for instance, [9, 10]) of g and T if $gx \in Tx$. The set of coincidence points of g and T is denoted by $\mathcal{COP}(g, T)$.

Recall that a function $p : X \times X \to [0, \infty)$ is called a *w-distance* [1, 11-15], if the following are satisfied:

$(w1)$ $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;

$(w2)$ for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is l.s.c.;

$(w3)$ for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

A function $p : X \times X \to [0, \infty)$ is said to be a *τ-function* [12, 14, 15], introduced and studied by Lin and Du, if the following conditions hold:

$(\tau 1)$ $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

$(\tau 2)$ if $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x, y_n) \leq M$ for some $M = M(x) > 0$, then $p(x, y) \leq M$;

$(\tau 3)$ for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;

$(\tau 4)$ for x, y, $z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

It is well known that the metric d is a w-distance and any w-distance is a τ-function, but the converse is not true; see [6, 12].

The following results are crucial in this paper.

Lemma 2.1. [15, Lemma 2.1]. *Let (X, d) be a metric space and $p : X \times X \to [0, \infty)$ be a function. Assume that p satisfies the condition $(\tau 3)$. If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

Lemma 2.2. [5, 6]. *Let A be a closed subset of a metric space (X, d) and $p : X \times X \to [0, \infty)$ be any function. Suppose that p satisfies $(\tau 3)$ and there exists $u \in X$ such that $p(u, u) = 0$. Then $p(u, A) = 0$ if and only if $u \in A$.*

Very recently, Du [5, 6] first introduced the concepts of τ^0-functions and τ^0-metrics as follows.

Definition 2.1. [5, 6]. Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a τ^0-function (resp. w^0-distance) if it is a τ-function (resp. w-distance) on X with $p(x, x) = 0$ for all $x \in X$.

Remark 2.1.

(a) It is obvious that any w^0-distance is a τ^0-function;

(b) If p is a τ^0-function, then, from $(\tau 4)$, $p(x, y) = 0$ if and only if $x = y$.

Example. [5, 6] Let $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$ and $0 < a < b$. Define the function $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \max\{a(y - x), b(x - y)\}.$$

Then p is nonsymmetric and hence p is not a metric. It is easy to see that p is a τ^0-function.

Definition 2.2. [5, 6]. Let (X, d) be a metric space and p be a τ^0-function (resp. w^0-distance). For any A, $B \in \mathcal{CB}(X)$, define a function $\mathcal{D}_p : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ by

$$\mathcal{D}_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\},$$

where $\delta_p(A, B) = \sup_{x\in A} p(x, B)$, then $\mathcal{D}_p$ is said to be the τ^0-*metric* (resp. w^0-*metric*) on $\mathcal{CB}(X)$ induced by p.

Clearly, any Hausdorff metric is a τ^0-metric, but the reverse is not true.

Theorem 2.1. [5, 6]. *Let (X, d) be a metric space and $\mathcal{D}_p$ a τ^0-metric defined as in Def. 2.3 on $\mathcal{CB}(X)$ induced by a τ^0-function p. Then for A, B, $C \in \mathcal{CB}(X)$, the following hold:*

(*i*) $\delta_p(A, B) = 0 \iff A \subseteq B$;

(*ii*) $\delta_p(A, B) \le \delta_p(A, C) + \delta_p(C, B)$;

(*iii*) *every τ^0-metric $\mathcal{D}_p$ is a metric on $\mathcal{CB}(X)$.*

Definition 2.3. [6, 16]. A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-$function$ if it satisfies Mizoguchi-Takahashi's condition (i.e. $\limsup_{s \to t+0} \varphi(s) < 1$ for all $t \in [0, \infty)$).

Obviously, if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function, then φ is an $\mathcal{MT}$-function. It is known that $\varphi : [0, \infty) \to [0, 1)$ is an $\mathcal{MT}$-function $if\ and\ only\ if$ for each $t \in [0, \infty)$, there exist $r_t \in [0, 1)$ and $\varepsilon_t > 0$ such that $\varphi(s) \le r_t$ for all $s \in [t, t + \varepsilon_t)$; see [16, Remark 2.5] for more details.

Definition 2.4. [17]. We say that $\varphi : [0, \infty) \to [0, 1)$ is a *function of contractive factor* if for any strictly decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have

$$0 \le \sup_{n \in \mathbb{N}} \varphi(x_n) < 1.$$

The following result tell us the relationship between $\mathcal{MT}$-functions and functions of contractive factor. It is essentially proved in [17], but we give the proof for the sake of completeness.

Lemma 2.3. [17]. *Any $\mathcal{MT}$-function is a function of contractive factor.*

Proof. Let $\varphi : [0, \infty) \to [0, 1)$ be an $\mathcal{MT}$-function and let $\{x_n\}_{n \in \mathbb{N}}$ be a strictly decreasing sequence in $[0, \infty)$. Then $t_0 := \lim_{n \to \infty} x_n = \inf_{n \in \mathbb{N}} x_n \ge 0$ exists. Since φ is an $\mathcal{MT}$-function, there exist $r_{t_0} \in [0, 1)$ and $\varepsilon_{t_0} > 0$ such that $\varphi(s) \le r_{t_0}$ for all $s \in [t_0, t_0 + \varepsilon_{t_0})$. On the other hand, there exists $\ell \in \mathbb{N}$, such that

$$t_0 \le x_n < t_0 + \varepsilon_{t_0}$$

for all $n \in \mathbb{N}$ with $n \ge \ell$. Hence $\varphi(x_n) \le r_{t_0}$ for all $n \ge \ell$. Let

$$\eta := \max\{\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_{\ell-1}), r_{t_0}\} < 1.$$

Then $\varphi(x_n) \le \eta$ for all $n \in \mathbb{N}$ and hence $0 \le \sup_{n \in \mathbb{N}} \varphi(x_n) \le \eta < 1$. Therefore φ is a function of contractive factor. ∎

3. New Types of Fixed Point Theorems

In this section, we first establish the following new type of fixed point theorem which is one of the main results of this paper. It improves and extends Nadler's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Berinde-Berinde's fixed point theorem and some results in [6].

Lemma 3.1. *Let (X, d) be a complete metric space, p be a τ^0-function and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that*

($\mathcal{P}1$) *there exists a function of contractive factor $\tau : [0, \infty) \to [0, 1)$ such that for each $x \in X$, if $y \in Tx$ with $y \ne x$ then there exists $z \in Ty$ such that*

$$p(y, z) \le \tau(p(x, y))p(x, y);$$

$(\mathcal{P}2)$ T *further satisfies one of the following conditions:*

- (*H1*) T *is closed;*
- (*H2*) *the map* $f : X \to [0, \infty)$ *defined by* $f(x) = p(x, Tx)$ *is l.s.c.;*
- (*H3*) *the map* $g : X \to [0, \infty)$ *defined by* $g(x) = d(x, Tx)$ *is l.s.c.;*
- (*H4*) *for any sequence* $\{x_n\}$ *in* X *with* $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} x_n = v$, *we have* $\lim_{n\to\infty} p(x_n, Tv) = 0$*;*
- (*H5*) $\inf\{p(x, z) + p(x, Tx) : x \in X\} > 0$ *for every* $z \notin \mathcal{F}(T)$.

Then $\mathcal{F}(T) \neq \emptyset$.

Proof. Let $x_1 \in X$ and $x_2 \in Tx_1$. If $x_1 = x_2$, then $x_1 \in \mathcal{F}(T)$. Otherwise, if $x_2 \neq x_1$, since p is a τ^0-function, by Remark 2.1 (b), we have $p(x_1, x_2) > 0$. It follows from $(\mathcal{P}1)$ that there exists $x_3 \in Tx_2$ such that

$$p(x_2, x_3) \leq \tau(p(x_1, x_2))p(x_1, x_2). \tag{3.1}$$

Let $\kappa : [0, \infty) \to [0, 1)$ by $\kappa(t) = \frac{1+\tau(t)}{2}$. Then $0 \leq \tau(t) < \kappa(t) < 1$ for all $t \in [0, \infty)$. By (3.1),

$$p(x_2, x_3) < \kappa(p(x_1, x_2))p(x_1, x_2). \tag{3.2}$$

If $p(x_2, x_3) = 0$, then $x_2 = x_3 \in \mathcal{F}(T)$. If $p(x_2, x_3) > 0$, then $x_3 \neq x_2$ and there exists $x_4 \in Tx_3$ such that

$$p(x_3, x_4) < \kappa(p(x_2, x_3))p(x_2, x_3).$$

By induction, we can obtain a sequence $\{x_n\}$ in X satisfying the following. For each $n \in \mathbb{N}$,

$$x_{n+1} \in Tx_n; \tag{3.3}$$

$$p(x_n, x_{n+1}) > 0; \tag{3.4}$$

$$p(x_{n+1}, x_{n+2}) < \kappa(p(x_n, x_{n+1}))p(x_n, x_{n+1}). \tag{3.5}$$

Since $\kappa(t) < 1$ for all $t \in [0, \infty)$, the sequence $\{p(x_n, x_{n+1})\}$ is strictly decreasing in $[0, \infty)$. Since τ is a function of contractive factor,

$$0 \leq \sup_{n\in\mathbb{N}} \tau(p(x_n, x_{n+1})) < 1.$$

Hence it follows that

$$0 < \sup_{n\in\mathbb{N}} \kappa(p(x_n, x_{n+1})) = \frac{1}{2}\left[1 + \sup_{n\in\mathbb{N}} \tau(p(x_n, x_{n+1}))\right] < 1.$$

Let $\lambda := \sup_{n\in\mathbb{N}} \kappa(p(x_n, x_{n+1}))$. So $\lambda \in (0, 1)$. We want to show that $\{x_n\}$ is a Cauchy sequence in X. Indeed, by (3.5), we have

$$p(x_{n+1}, x_{n+2}) < \kappa(p(x_n, x_{n+1}))p(x_n, x_{n+1}) \le \lambda p(x_n, x_{n+1}). \tag{3.6}$$

Hence, by (3.6), it implies that

$$p(x_{n+1}, x_{n+2}) < \lambda p(x_n, x_{n+1}) \ < \cdots < \lambda^n p(x_1, x_2) \quad \text{for each } n \in \mathbb{N}.$$

We claim that $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$. Let $\alpha_n = \frac{\lambda^n}{1-\lambda} p(x_1, x_2)$, $n \in \mathbb{N}$. For m, $n \in \mathbb{N}$ with $m > n$, we have

$$p(x_n, x_m) \le \sum_{j=n}^{m-1} p(x_j, x_{j+1}) < \alpha_n. \tag{3.7}$$

Since $\lambda \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and, by (3.7), we get

$$\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0. \tag{3.8}$$

Applying (c) of Lemma 2.1, $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. From $(\tau 2)$ and (3.7), we have

$$p(x_n, v) \le \alpha_n \quad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

Now, we verify that $v \in \mathcal{F}(T)$. If (H1) holds, since T is closed, $x_n \in Tx_{n-1}$ and $x_n \to v$ as $n \to \infty$, we have $v \in Tv$.

If (H2) holds, by the lower semicontinuity of f, $x_n \to v$ as $n \to \infty$ and (3.8), we obtain

$$\begin{aligned} p(v, Tv) &= f(v) \\ &\le \liminf_{m\to\infty} f(x_n) \\ &= \liminf_{m\to\infty} p(x_n, Tx_n) \\ &\le \lim_{n\to\infty} p(x_n, x_{n+1}) = 0, \end{aligned}$$

which implies $p(v, Tv) = 0$. By Lemma 2.2, we get $v \in \mathcal{F}(T)$.

Suppose that (H3) holds. Since $\{x_n\}$ is convergent in X, $d(x_n, x_{n+1}) = 0$. Since

$$\begin{aligned} d(v, Tv) &= g(v) \\ &\le \liminf_{m\to\infty} d(x_n, Tx_n) \\ &\le \lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \end{aligned}$$

we have $d(v, Tv) = 0$ and hence $v \in \mathcal{F}(T)$.

If (H4) holds, by (3.8), there exists $\{a_n\} \subset \{x_n\}$ with $\lim_{n\to\infty} \sup\{p(a_n, a_m) : m > n\} = 0$ and $\{b_n\} \subset Tv$ such that $\lim_{n\to\infty} p(a_n, b_n) = 0$. By $(\tau 3)$, $\lim_{n\to\infty} d(a_n, b_n) = 0$. Since $d(b_n, v) \leq d(b_n, a_n) + d(a_n, v)$, it implies $b_n \to v$ as $n \to \infty$. By the closedness of Tv, we have $v \in Tv$ or $v \in \mathcal{F}(T)$.

Finally, assume (H5) holds. On the contrary, suppose that $v \notin Tv$. Then, by (3.7) and (3.9), we have

$$\begin{aligned} 0 &< \inf_{x\in X}\{p(x, v) + p(x, Tx)\} \\ &\leq \inf_{n\in\mathbb{N}}\{p(x_n, v) + p(x_n, Tx_n)\} \\ &\leq \inf_{n\in\mathbb{N}}\{p(x_n, v) + p(x_n, x_{n+1})\} \\ &\leq \lim_{n\to\infty} 2\alpha_n \\ &= 0, \end{aligned}$$

a contradiction. Therefore $v \in \mathcal{F}(T)$. The proof is completed. ∎

Theorem 3.1. *Let (X, d) be a complete metric space, p be a τ^0-function and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that the condition $(\mathcal{P}2)$ as in Lemma 3.1 holds and further assume that*

$(\mathcal{P}3)$ *there exists a function of contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that for each $x \in X$, $p(y, Ty) \leq \varphi(p(x, y))p(x, y)$ for all $y \in Tx$.*

Then $\mathcal{F}(T) \neq \emptyset$.

Proof. We first prove that the condition $(\mathcal{P}1)$ in Lemma 3.1 holds. Define $\tau : [0, \infty) \to [0, 1)$ by $\tau(t) = \frac{1+\varphi(t)}{2}$. We claim that τ is also a function of contractive factor. Clearly, $0 \leq \varphi(t) < \tau(t) < 1$ for all $t \in [0, \infty)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence in $[0, \infty)$. Since φ is a function of contractive factor, we have

$$0 < \sup_{n\in\mathbb{N}} \tau(x_n) = \frac{1}{2}\left[1 + \sup_{n\in\mathbb{N}} \varphi(x_n)\right] < 1.$$

which means that τ is a function of contractive factor.

For each $x \in X$, let $y \in Tx$ with $y \neq x$. Then $p(x, y) > 0$. By $(\mathcal{P}3)$, we have

$$p(y, Ty) < \tau(p(x, y))p(x, y)$$

and hence there exists $z \in Ty$ such that

$$p(y, z) < \tau(p(x, y))p(x, y),$$

which show that $(\mathcal{P}1)$ holds. So all the conditions of Lamma 3.1 are satisfied. Therefore the conclusion follows from Lamma 3.1. ∎

Remark 3.1.

(a) [6, Theorem 2.1] is a special case of Lemma 3.1 and Theorem 3.1.

(b) $(\mathcal{P}1)$ and $(\mathcal{P}3)$ are equivalent. Indeed, in the proof of Theorem 3.1, we have shown that $(\mathcal{P}3)$ implies $(\mathcal{P}1)$. If $(\mathcal{P}1)$ holds, then it is easy to verify that $(\mathcal{P}3)$ also holds. Hence $(\mathcal{P}1)$ implies $(\mathcal{P}3)$ and we get the desired result. Therefore Lamma 3.1 can also be proved by using Theorem 3.1.

By using Theorem 3.1, we can establish the following existence theorem of coincidence point and fixed point.

Theorem 3.2. *Let (X, d) be a complete metric space, p be a τ^0-function, $T : X \to \mathcal{C}(X)$ be a multivalued map, $g : X \to X$ be a self-map, $\varphi : [0, \infty) \to [0, 1)$ be a function of contractive factor and $L \geq 0$. Suppose that the condition $(\mathcal{P}2)$ as in Lemma 3.1 holds and further assume*

$(\mathcal{P}4)$ *Tx is g-invariant (i.e. $g(Tx) \subseteq Tx$) for each $x \in X$;*

$(\mathcal{P}5)$ $p(y, Ty) \leq \varphi(p(x, y))p(x, y) + Lp(gy, Tx)$ *for all $x, y \in X$.*

Then $\mathcal{COP}(g, T) \cap \mathcal{F}(T) \neq \emptyset$.

Proof. We first prove that the condition $(\mathcal{P}3)$ in Theorem 3.1 holds. For each $x \in X$, if $y \in Tx$, from $(\mathcal{P}4)$, we have $gy \in Tx$ and hence $p(gy, Tx) = 0$ by using Lemma 2.2. So for each $x \in X$, it follows from $(\mathcal{P}5)$ that

$$p(y, Ty) \leq \varphi(p(x, y))p(x, y) \quad \text{for all } y \in Tx,$$

which say that $(\mathcal{P}3)$ holds.

Applying Theorem 3.1, $\mathcal{F}(T) \neq \emptyset$. So there exists $v \in X$ such that $v \in Tv$. By $(\mathcal{P}4)$, $gv \in Tv$. Therefore, $v \in \mathcal{COP}(g, T) \cap \mathcal{F}(T)$ and the proof is complete. ■

Using the definition of $\mathcal{D}_p$ and the same argument as in the proof of Theorem 3.2, we can obtain the following intersection theorem.

Theorem 3.3. *Let (X, d) be a complete metric space, p be a τ^0-function, $\mathcal{D}_p$ be a τ^0-metric on $\mathcal{CB}(X)$, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $g : X \to X$ be a self-map, $\varphi : [0, \infty) \to [0, 1)$ be a function of contractive factor and $L \geq 0$. Suppose that the conditions $(\mathcal{P}2)$ and $(\mathcal{P}4)$ hold and further assume*

$(\mathcal{P}6)$ $\mathcal{D}_p(Tx, Ty) \leq \varphi(p(x, y))p(x, y) + Lp(gy, Tx)$ *for all $x, y \in X$.*

Then $\mathcal{COP}(g, T) \cap \mathcal{F}(T) \neq \emptyset$.

The following result is immediate from Theorem 3.2 and Lemma 2.3.

Corollary 3.1. *Let (X, d) be a complete metric space, p be a τ^0-function, $T : X \to \mathcal{C}(X)$ be a multivalued map, $g : X \to X$ be a self-map, $\varphi : [0, \infty) \to [0, 1)$ be an $\mathcal{MT}$-function and $L \geq 0$. Suppose that the conditions $(\mathcal{P}2)$, $(\mathcal{P}4)$ and $(\mathcal{P}5)$ hold. Then $\mathcal{COP}(g, T) \cap \mathcal{F}(T) \neq \emptyset$.*

Corollary 3.2. *Let (X, d) be a complete metric space, p be a τ^0-function, $\mathcal{D}_p$ be a τ^0-metric on $\mathcal{CB}(X)$, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $g : X \to X$ be a self-map, $\varphi : [0, \infty) \to [0, 1)$ be an $\mathcal{MT}$-function and $L \geq 0$. Suppose that the conditions $(\mathcal{P}2)$, $(\mathcal{P}4)$ and $(\mathcal{P}6)$ hold. Then $\mathcal{COP}(g, T) \cap \mathcal{F}(T) \neq \emptyset$.*

Notice that we don't assume that the condition $(\mathcal{P}2)$ holds in the hypotheses of the following result, but g is assumed to be continuous.

Theorem 3.4. *Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $g : X \to X$ be a continuous self-map, $\varphi : [0, \infty) \to [0, 1)$ be a function of contractive factor and $L \geq 0$. Suppose that the condition $(\mathcal{P}4)$ holds and further assume*

$(\mathcal{P}7)$ $\mathcal{H}(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(gy, Tx)$ *for all* $x, y \in X$.

Then $\mathcal{COP}(g, T) \cap \mathcal{F}(T) \neq \emptyset$.

Proof. Let $p \equiv d$. Then $\mathcal{D}_p \equiv \mathcal{H}$. So $(\mathcal{P}6)$ and $(\mathcal{P}7)$ are identical. We claim that the condition $(\mathcal{P}2)$ as in Lemma 3.1 holds. Indeed, it suffices to show that (H4) in Lemma 3.1 holds. Let $\{x_n\}$ in X with $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = v$. Then, by $(\mathcal{P}4)$, $gx_{n+1} \in Tx_n, n \in \mathbb{N}$. Since $g : X \to X$ is continuous, $\lim_{n\to\infty} gx_n = gv$. It follows from $(\mathcal{P}7)$ that

$$\begin{aligned}\lim_{n\to\infty} d(x_{n+1}, Tv) &\leq \lim_{n\to\infty} \mathcal{H}(Tx_n, Tv)\\ &\leq \lim_{n\to\infty} \{\varphi(d(x_n, v))d(x_n, v) + Ld(gv, gx_{n+1})\} = 0.\end{aligned}$$

So $\lim_{n\to\infty} d(x_{n+1}, Tv) = 0$ and hence the condition (H4) holds. Therefore the conclusion follows from Theorem 3.3. ∎

As an application of Theorem 3.2, we have the following fixed point theorem.

Theorem 3.5. *Let (X, d) be a complete metric space, p be a τ^0-function, $T : X \to \mathcal{C}(X)$ be a multivalued map, $\varphi : [0, \infty) \to [0, 1)$ be a function of contractive factor and $L \geq 0$. Suppose that the condition $(\mathcal{P}2)$ as in Lamma 3.1 holds and further assume*

$(\mathcal{P}8)$ $p(y, Ty) \leq \varphi(p(x, y))p(x, y) + Lp(y, Tx)$ *for all* $x, y \in X$.

Then $\mathcal{F}(T) \neq \emptyset$.

Proof. Let $g \equiv id$ be the identity map on X. Hence the conclusion follows from Theorem 3.2. ∎

The following result is immediate from Theorem 3.3.

Theorem 3.6. *Let (X, d) be a complete metric space, p be a τ^0-function, $\mathcal{D}_p$ be a τ^0-metric on $\mathcal{CB}(X)$, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $\varphi : [0, \infty) \to [0, 1)$ be a function of contractive factor and $L \geq 0$. Suppose that the condition $(\mathcal{P}2)$ as in Lemma 3.1 holds and further assume*

$(\mathcal{P}9)$ $\mathcal{D}_p(Tx, Ty) \leq \varphi(p(x, y))p(x, y) + Lp(y, Tx)$ *for all* $x, y \in X$.

Then $\mathcal{F}(T) \neq \emptyset$.

Remark 3.2.

(a) [6, Theorem 2.2] and [6, Theorem 2.3] are special cases of Theorems 3.5 and Theorems 3.6, respectively.

(b) Obviously, Lemma 3.1 and Theorems 3.1-3.6 all improve and generalize Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem and Banach contraction principle.

REFERENCES

1. W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

2. S. B. Nadler, Jr., Multi-valued contraction mappings, *Pacific J. Math.*, **30** (1969), 475-488.

3. N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, *J. Math. Anal. Appl.*, **141** (1989), 177-188.

4. M. Berinde and V. Berinde, On a general class of multi-valued weakly Picard mappings, *J. Math. Anal. Appl.*, **326** (2007), 772-782.

5. W.-S. Du, Fixed point theorems for generalized Hausdorff metrics, *Int. Math. Forum*, **3** (2008), 1011-1022.

6. W.-S. Du, Some new results and generalizations in metric fixed point theory, *Nonlinear Anal.*, **73** (2010), 1439-1446.

7. S. Reich, Some problems and results in fixed point theory, *Contemp. Math.*, **21** (1983), 179-187.

8. T. Suzuki, Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's, *J. Math. Anal. Appl.*, **340** (2008), 752-755.

9. N. Hussain, A. Amini-Harandi and Y. J. Cho, *Approximate endpoints for set-valued contractions in metric spaces*, Fixed Point Theory and Applications, Vol. 2010, (2010), Article ID 614867, 13 pages, doi:10.1155/2010/614867.

10. W.-S. Du, *On coincidence point and fixed point theorems for nonlinear multivalued maps*, Topology and its Applications, (2011), doi: 10.1016/j.topol.2011.07.021.

11. O. Kada, T. Suzuki and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japon.*, **44** (1996), 381-391.

12. L.-J. Lin and W.-S. Du, Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces, *J. Math. Anal. Appl.*, **323** (2006), 360-370.

13. L.-J. Lin and W.-S. Du, Some equivalent formulations of the generalized Ekeland's variational principle and their applications, *Nonlinear Anal.*, **67** (2007), 187-199.

14. L.-J. Lin and W.-S. Du, On maximal element theorems, variants of Ekeland's variational principle and their applications, *Nonlinear Anal.*, **68** (2008), 1246-1262.

15. W.-S. Du, *Critical point theorems for nonlinear dynamical systems and their applications*, Fixed Point Theory and Applications, Vol. 2010, (2010), Article ID 246382, 16 pages, doi:10.1155/2010/246382.

16. W.-S. Du, *Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi-Takahashi's condition in quasiordered metric spaces*, Fixed Point Theory and Applications, Vol. 2010, (2010), Article ID 876372, 9 pages, doi:10.1155/2010/876372.

17. W.-S. Du, *Nonlinear contractive conditions for coupled cone fixed point theorems*, Fixed Point Theory and Applications, Volume 2010 (2010), Article ID 190606, 16 pages, doi:10.1155/2010/190606.

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