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EXISTENCE OF SOLUTIONS FOR A CLASS OF p-LAPLACIAN SYSTEMS WITH IMPULSIVE EFFECTS

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Abstract. By using the least action principle and the saddle point theorem, some existence theorems are obtained for second-order p-Laplacian systems with or without impulsive effects under weak sublinear growth conditions, we improve some existing results in the literature.

1. INTRODUCTION

Consider the second-order p-Laplacian systems with impulsive effects

$$(1.1)\qquad \begin{cases} \dfrac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0)-u(T) = \dot{u}(0)-\dot{u}(T) = 0, \\ \Delta \dot{u}^i(t_j) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) \\ \qquad = I_{ij}(u^i(t_j)), i = 1,2,...,N; j = 1,2,...,m, \end{cases}$$

where $p > 1, T > 0, t_0 = 0 < t_1 < t_2 < ... < t_m < t_{m+1} = T, u(t) = (u^1(t), u^2(t), ..., u^N(t)), I_{ij} : \mathbb{R} \to \mathbb{R}(i = 1,2,...,N; j = 1,2,...,m)$ are continuous and and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) *$F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0,T], \mathbb{R}^+)$ such that*

$$|F(t,x)| \le a(|x|)b(t), \quad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

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For the sake of convenience, in the sequel, we define $A = \{1, 2, ..., N\}, B = \{1, 2, ..., m\}$.

When $I_{ij} \equiv 0, p = 2$, (1.1) reduces to the second order Hamiltonian system, it has been proved that problem (1.1) has at least one solution by the least action principle and the minimax methods (see [2, 7-9, 11, 12, 15-18, 20-22, 25, 26]). Many solvability conditions are given, such as the coercive condition (see [2]), the periodicity condition (see [20]); the convexity condition (see [7]); the subadditive condition (see [15]); the bounded condition (see [8]).

When the nonlinearity $\nabla F(t, x)$ is bounded sublinearly, that is, there exist $f, g \in L^1([0, T], \mathbb{R}^+)$ and $\alpha \in [0, 1)$ such that

$$|\nabla F(t, x)| \leq f(t)|x|^{\alpha} + g(t) \tag{1.2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Tang [17] also proved the existence of solutions for problem (1.1) when $I_{ij} \equiv 0, p = 2$ under the condition

$$\lim_{|x| \to +\infty} |x|^{-2\alpha} \int_0^T F(t, x)dt \to +\infty, \tag{1.3}$$

or

$$\lim_{|x| \to +\infty} |x|^{-2\alpha} \int_0^T F(t, x)dt \to -\infty, \tag{1.4}$$

which generalizes Mawhin-Willem's results under bounded condition (see [8]).

However, there exists F neither satisfies (1.3) nor (1.4).

Let

$$F(t, x) = \sin\left(\frac{2\pi t}{T}\right)|x|^{7/4} + (0.6T - t)|x|^{3/2}.$$

It is easy to see that

$$\begin{aligned}|\nabla F(t, x)| &\leq \frac{7}{4}\left|\sin\left(\frac{2\pi t}{T}\right)\right||x|^{3/4} + \frac{3}{2}|0.6T - t||x|^{1/2} \\ &\leq \frac{7}{4}\left(\left|\sin\left(\frac{2\pi t}{T}\right)\right| + \varepsilon\right)|x|^{3/4} + \frac{T^3}{\varepsilon^2}\end{aligned}$$

for all $x \in \mathbb{R}^N$ and $t \in [0, T]$, where $\varepsilon > 0$. The above shows (1.2) holds with $\alpha = 3/4$ and

$$f(t) = \frac{7}{4}\left(\left|\sin\left(\frac{2\pi t}{T}\right)\right| + \varepsilon\right), \quad g(t) = \frac{T^3}{\varepsilon^2}.$$

However, $F(t, x)$ neither satisfies (1.3) nor (1.4). In fact,

$$\begin{aligned}&|x|^{-2\alpha}\int_0^T F(t,x)dt\\ &=|x|^{-3/2}\int_0^T\left[\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4}+(0.6T-t)|x|^{3/2}\right]dt=0.1T^2.\end{aligned}$$

The above example shows that it is valuable to further improve (1.3) and (1.4).

Impulsive differential equations arising from the real world describe the dynamics of processes in which sudden, discontinuous jumps occur. For the background, theory and applications of impulsive differential equations, we refer the readers to the monographs and some recent contributions as [1, 3, 4, 13, 20].

Some classical tools such as fixed point theorems in cones [1, 5, 19], the method of lower and upper solutions [3, 23] have been widely used to study impulsive differential equations.

Recently, the Dirichlet and periodic boundary conditions problems with impulses in the derivative are studied by variational method. For some general and recent works on the theory of critical point theory and variational methods we refer the readers to [10, 14, 19, 27, 28]. It is a novel approach to apply variational methods to the impulsive boundary value problem (IBVP for short). For $I_{ij} \not\equiv 0, i \in A, j \in B$, some special cases of are studied (1.1) when the gradient of the nonlinearity grow sublinearly by variational method, (see [28]).

Inspired by the above results [6, 15, 21, 22, 25, 26, 28], we devote to study the existence of solutions for problem (1.1) under condition (1.2). Our results generalize the previous work, which seems not to have been considered in the literature.

Throughout this paper, we let $q \in (1, +\infty)$ such that $1/p + 1/q = 1$.

2. Preliminaries and the Variational Setting

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we construct a variational structure. With this variational structure, we can reduce the problem of finding solutions of (1.1) to that of seeking the critical points of a corresponding functional.

Let $W_T^{1,p}$ be the Sobolev space

$$\begin{aligned}W_T^{1,p}&=\{u:[0,T]\to\mathbb{R}^N \mid u \text{ is absolutely continuous},\\ u(0)&=u(T),\ \dot{u}\in L^p([0,T],\mathbb{R}^N)\},\end{aligned}$$

it is a reflexive Banach space with the norm defined by

$$\|u\|=\|u\|_{W_T^{1,p}}=\left(\int_0^T[|\dot{u}(t)|^p+|u(t)|^p]\,dt\right)^{\frac{1}{p}}$$

for $u \in W_T^{1,p}$.

Let us recall that

$$\|u\|_p = \left(\int_0^T |u(t)|^p dt\right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

We have the following fact.

Take $v \in W_T^{1,p}$ and multiply the two sides of the equality

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) = \nabla F(t, u(t)), \tag{2.1}$$

by v and integrate from 0 to T:

$$\int_0^T ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t))dt = \int_0^T (\nabla F(t, u(t)), v(t))dt.$$

The first term is now

$$\int_0^T ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t))dt = \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t))dt$$

and

$$\begin{aligned}
&\int_{t_j}^{t_{j+1}} ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t))dt \\
=\ & (|\dot{u}(t_{j+1}^-)|^{p-2}\dot{u}(t_{j+1}^-), v(t_{j+1}^-)) - (|\dot{u}(t_j^+)|^{p-2}\dot{u}(t_j^+), v(t_j^+)) \\
& - \int_{t_j}^{t_{j+1}} (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))dt \\
=\ & \sum_{i=1}^{N} \Big(|\dot{u}^i(t_{j+1}^-)|^{p-2}\dot{u}^i(t_{j+1}^-)v^i(t_{j+1}^-) - |\dot{u}^i(t_j^+)|^{p-2}\dot{u}^i(t_j^+)v^i(t_j^+)\Big) \\
& - \int_{t_j}^{t_{j+1}} (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^T ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t))dt \\
&= \sum_{j=1}^{m}\sum_{i=1}^{N} \Delta\dot{u}^i(t_j)v^i(t_j) + |\dot{u}(T)|^{p-2}\dot{u}(T)v(T) \\
&\quad -|\dot{u}(0)|^{p-2}\dot{u}(0)v(0) - \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))dt \\
&= -\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j) - \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))dt.
\end{aligned}$$

Combining with (2.1), we get

$$\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j) + \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))dt + \int_0^T (\nabla F(t, u(t), v(t)))dt = 0.$$

Considering the above, we introduce the following concept for the solution for problem (1.1).

Definition 2.1. We say that a function $u \in W_T^{1,p}$ is a weak solution of problem (1.1) if the identity

$$\int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))dt + \sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j)) = -\int_0^T (\nabla F(t, u(t)), v(t))dt$$

holds for any $v \in W_T^{1,p}$.

The corresponding functional φ on $W_T^{1,p}$ given by

$$\begin{aligned} \varphi(u) &= \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt + \int_0^T F(t, u(t))dt + \sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt \\ &= \psi(u) + \phi(u), \end{aligned} \tag{2.2}$$

where

$$\psi(u) = \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt + \int_0^T F(t, u(t))dt$$

and

$$\phi(u) = \sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt.$$

It follows from assumption (A) that $\psi \in C^1(W_T^{1,p}, \mathbb{R})$. By the continuity of $I_{ij}, i \in A, j \in B$, one has that $\phi \in C^1(W_T^{1,p}, \mathbb{R})$. Thus, $\varphi \in C^1(W_T^{1,p}, \mathbb{R})$. For any $v \in W_T^{1,p}$, we have

$$\begin{aligned} < \varphi'(u), v > &= \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))dt + \sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u^i(t_j)v^i(t_j)) \\ &\quad + \int_0^T (\nabla F(t, u(t)), v(t))dt. \end{aligned} \tag{2.3}$$

By Definition 2.1, the weak solutions of problem (1.1) correspond to the critical points of φ.

To prove our main results, we need the following useful lemma, see [29].

Lemma 2.1. *Let* $u \in W_T^{1,p}$ *and* $\int_0^T u(t)dt = 0$. *Then*

$$\|u\|_\infty \le \left(\frac{T}{q+1}\right)^{1/q} \left(\int_0^T |\dot{u}(s)|^p ds\right)^{1/p}, \tag{2.4}$$

and

$$\int_0^T |u(s)|^p ds \le \frac{T^p \Theta(p,q)}{(q+1)^{p/q}} \int_0^T |\dot{u}(s)|^p ds, \tag{2.5}$$

where

$$\Theta(p,q) = \int_0^1 [s^{q+1} + (1-s)^{q+1}]^{p/q} ds. \tag{2.6}$$

Proof. Fix $t \in [0,T]$, for every $\tau \in [0,T]$, one has

$$u(t) = u(\tau) + \int_\tau^t \dot{u}(s)ds. \tag{2.7}$$

Set

$$\phi(s) = \begin{cases} s, & 0 \le s \le t, \\ T-s, & t \le s \le T. \end{cases} \tag{2.8}$$

Integrating (2.7) over $[0,T]$ and using the Hölder inequality, we obtain

$$\begin{aligned} Tu(t) &= \left|\int_0^T u(\tau)d\tau + \int_0^T \int_\tau^t \dot{u}(s)dsd\tau\right| \\ &\le \int_0^t \int_\tau^t |\dot{u}(s)|dsd\tau + \int_t^T \int_t^\tau |\dot{u}(s)|dsd\tau \\ &= \int_0^t s|\dot{u}(s)|ds + \int_t^T (T-s)|\dot{u}(s)|ds \\ &= \int_0^T \phi(s)|\dot{u}(s)|ds \\ &\le \left(\int_0^T [\phi(s)]^q ds\right)^{1/p} \left(\int_0^T |\dot{u}(s)|^p ds\right)^{1/p} \\ &= \frac{1}{(q+1)^{1/q}} [t^{q+1} + (T-t)^{q+1}]^{1/q} \left(\int_0^T |\dot{u}(s)|^p\right)^{1/p}. \end{aligned} \tag{2.9}$$

Since $t^{q+1} + (T-t)^{q+1} \le T^{q+1}$ for $t \in [0,T]$, it follows from (2.9) that (2.4) holds.

On the other hand, from (2.9), we have

$$\begin{aligned} T^p \int_0^T |u(t)|^p dt &\le \frac{1}{(q+1)p/q} \left(\int_0^T |\dot{u}(s)|^p ds \right) \int_0^T [t^{q+1} + (T-t)^{q+1}]^{p/q} dt \\ &\le \frac{T^{1+p(q+1)/q}}{(q+1)^{p/q}} \left(\int_0^T |\dot{u}(s)|^p ds \right) \int_0^1 [s^{q+1} + (1-s)^{q+1}]^{p/q} ds \\ &= \frac{T^{2p}\Theta(p,q)}{(q+1)^{p/q}} \int_0^T |\dot{u}(s)|^p ds. \end{aligned}$$

Then (2.5) holds. The proof is completed. ∎

3. Main Results and Their Proofs

Theorem 1.1. *Suppose that (A) holds and F, I_{ij} satisfy the following conditions:*

(I1) *For any $i \in A, j \in B$,*

$$I_{ij}(t) \ge 0, \quad \forall\ t \in \mathbb{R}; \tag{3.1}$$

(F1) *There exist $f, g \in L^1([0,T], \mathbb{R}^+)$ and $\alpha \in [0, p-1)$ such that*

$$|\nabla F(t,x)| \le f(t)|x|^\alpha + g(t) \tag{3.2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(F2)

$$\liminf_{|x|\to+\infty} |x|^{-q\alpha} \int_0^T F(t,x)dt > \frac{2^{q\alpha}T}{q(q+1)} \left(\int_0^T f(t)dt \right)^q. \tag{3.3}$$

Then problem (1.1) has at least one solution which minimizes the functional φ on $W_T^{1,p}$.

Theorem 1.2. *Suppose that (A) and (F1) hold, and the following conditions are satisfied:*

(I2) *There exist $a_{ij}, b_{ij} > 0$ and $\beta_{ij} \in (0,1), \gamma \in [0,\alpha)$ such that*

$$|I_{ij}(t)| \le a_{ij} + b_{ij}|t|^{\gamma\beta_{ij}}, \quad \text{for every } t \in \mathbb{R}, i \in A, j \in B; \tag{3.4}$$

(I3) *For any $i \in A, j \in B$,*

$$I_{ij}(t)t \le 0, \quad \forall\ t \in \mathbb{R}; \tag{3.5}$$

(F3)

$$\limsup_{|x|\to+\infty} |x|^{-q\alpha} \int_0^T F(t,x)dt < -\frac{(p+2q)2^{q\alpha}T}{pq(q+1)} \left(\int_0^T f(t)dt\right)^q . \tag{3.6}$$

Then problem (1.1) *has at least one solution in* $W_T^{1,p}$.

Remark 1.1. When $I_{ij} \equiv 0$, problem (1.1) degenerates to the corresponding ones for second order ordinary differential system, Theorem 1.1 and Theorem 1.2 still hold and we generalize the previous work [17].

Replaced by the condition

(F1') *There exist* $f, g \in L^1([0,T], \mathbb{R}^+)$ *such that*

$$|\nabla F(t,x)| \le f(t)|x| + g(t) \tag{3.7}$$

for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$.

Zhao and Wu [25, 26] proved the existence of solutions for problem (1.1) with $p = 2$ with no impulisive effects, that is, condition (F1) reduces to linearly bounded gradiant condition, based on this case, we generalize the results.

Theorem 1.3. *Suppose that* (A), $(I1)$ *hold, and the following conditions are satisfied:*

(f)

$$\int_0^T f(t)dt < \frac{2^{1-p}}{p} \left(\frac{T}{q+1}\right)^{-p/q}; \tag{3.8}$$

(F4) *There exist* $f, g \in L^1([0,T], \mathbb{R}^+)$ *such that*

$$|\nabla F(t,x)| \le f(t)|x|^{p-1} + g(t) \tag{3.9}$$

for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$;

(F5)

$$\begin{aligned} &\liminf_{|x|\to+\infty} |x|^{-p} \int_0^T F(t,x)dt \\ &> \frac{2^p}{q} \left(\frac{T^{p/q}}{(q+1)^{p/q} - 2^{p-1}pT^{p/q}\int_0^T f(t)dt}\right)^{q/p} \left(\int_0^T f(t)dt\right)^q . \end{aligned} \tag{3.10}$$

Then problem (1.1) *has at least one solution which minimizes the functional* φ *on* $W_T^{1,p}$.

Theorem 1.4. *Suppose that* (A), $(I3)$, $(F4)$, *and that*

(I4) *There exist* $a_{ij}, b_{ij} > 0$ *and* $\beta_{ij} \in (0,1), \gamma \in (0,1)$ *such that*

$$|I_{ij}(t)| \le a_{ij} + b_{ij}|t|^{\gamma\beta_{ij}}, \quad \text{for every } t \in \mathbb{R}, i \in A, j \in B; \tag{3.11}$$

(F6)

$$\begin{aligned} &\limsup_{|x|\to+\infty} |x|^{-p} \int_0^T F(t,x)dt \\ &< -\frac{2^p(p+1)T}{q(q+1)(p-1-2^{p-1}(\frac{T}{q+1})^{p/q}\int_0^T f(t)dt)} \left(\int_0^T f(t)dt\right)^q. \end{aligned} \tag{3.12}$$

Then problem (1.1) *has at least one solution in* $W_T^{1,p}$.

Remark 1.2. When $I_{ij} \equiv 0, p = 2$, problem (1.1) degenerates to the corresponding ones for second order ordinary differential systems, Theorem 1.3 holds and generalize the previous work [25, 26].

For the sake of convenience, we denote

$$M_1 = \int_0^T f(t)dt, \quad M_2 = \int_0^T g(t)dt.$$
$$a = \max_{i\in A, j\in B} a_{ij}, \quad b = \max_{i\in A, j\in B} b_{ij}.$$

Proof of Theorem 1.1. By (F2), we can choose an $a_1 > \left(\frac{T}{q+1}\right)^{1/q}$ such that

$$\liminf_{|x|\to+\infty} |x|^{-q\alpha} \int_0^T F(t,x)dt > \frac{2^{q\alpha}a_1^q}{q} M_1^q. \tag{3.13}$$

It follows from (2.4), (2.5) and Young inequality that

$$\begin{aligned} &\left|\int_0^T (F(t,u(t)) - F(t,\bar{u}))dt\right| \\ &= \left|\int_0^T \int_0^1 (\nabla F(t,\bar{u} + s\tilde{u}(t)), \tilde{u}(t))dsdt\right| \\ &\le \int_0^T \int_0^1 f(t)\,|\bar{u} + s\tilde{u}(t)|^\alpha\,|\tilde{u}(t)|\,dsdt + \int_0^T \int_0^1 g(t)\,|\tilde{u}(t)|\,dsdt \\ &\le 2^\alpha \int_0^T f(t)\,(|\bar{u}|^\alpha + |\tilde{u}(t)|^\alpha)\,|\tilde{u}(t)|\,dt + \int_0^T g(t)\,|\tilde{u}|\,dt \\ &\le 2^\alpha \left(|\bar{u}|^\alpha \|\tilde{u}\|_\infty + \|\tilde{u}\|_\infty^{\alpha+1}\right) \int_0^T f(t)dt + \|\tilde{u}\|_\infty \int_0^T g(t)dt \end{aligned} \tag{3.14}$$

$$
\begin{aligned}
&= 2^{\alpha}M_1|\bar{u}|^{\alpha}\|\tilde{u}\|_{\infty} + 2^{\alpha}M_1\|\tilde{u}\|_{\infty}^{\alpha+1} + M_2\|\tilde{u}\|_{\infty} \\
&\le \frac{1}{pa_1^p}\|\tilde{u}\|_{\infty}^p + \frac{2^{q\alpha}a_1^q}{q}M_1^q|\bar{u}|^{q\alpha} + 2^{\alpha}M_1\|\tilde{u}\|_{\infty}^{\alpha+1} + M_2\|\tilde{u}\|_{\infty} \\
&\le \frac{T^{p/q}}{pa_1^p(q+1)^{p/q}}\|\dot{u}\|_{L^p}^p + \frac{2^{q\alpha}a_1^q}{q}M_1^q|\bar{u}|^{q\alpha} + 2^{\alpha}\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}M_1\|\dot{u}\|_{L^p}^{\alpha+1} \\
&\quad + \left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}\|_{L^p}.
\end{aligned}
$$

Hence we have by (I1) and (3.14)

$$
\begin{aligned}
\varphi(u) &= \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T [F(t,u(t)) - F(t,\bar{u})]\,dt + \int_0^T F(t,\bar{u})dt + \phi(u) \\
&\ge \left(\frac{1}{p} - \frac{T^{p/q}}{pa_1^p(q+1)^{p/q}}\right)\|\dot{u}\|_{L^p}^p \\
&\quad -2^{\alpha}\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}M_1\|\dot{u}\|_{L^p}^{\alpha+1} - \left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}\|_{L^p} \\
&\quad +(|\bar{u}|^p)^{q\alpha/p}\left(|\bar{u}|^{-q\alpha}\int_0^T F(t,\bar{u})dt - \frac{2^{q\alpha}a_1^q}{q}M_1^q\right).
\end{aligned} \tag{3.15}
$$

In Sobolev space $W_T^{1,p}$, for $u \in W_T^{1,p}$, $\|u\| \to \infty$ if and only if $(|\bar{u}|^p + \|\dot{u}\|_{L^p}^p)^{1/p} \to \infty$, (F2) and (3.15) show that $\varphi(u) \to +\infty$ as $\|u\| \to \infty$. Similar to the proof of Lemma 3.1 in [28], φ is weakly lower semi-continuous on $W_T^{1,p}$, by Theorem 1.1 and Corollary 1.1 in [8], φ has a minimum point on $W_T^{1,p}$, which is a critical point of φ, we complete the proof of Theorem 1.1.

Proof of Theorem 1.2. Suppose that $\{U_n\} \subset W_T^{1,p}$ is a (PS) sequence of φ. In a way similar to the proof of Theorem 1.1, we have

$$
\begin{aligned}
&\left|\int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt\right| \\
&\le 2^{\alpha}M_1|\bar{u}_n|^{\alpha}\|\tilde{u}_n\|_{\infty} + 2^{\alpha}M_1\|\tilde{u}_n\|_{\infty}^{\alpha+1} + M_2\|\tilde{u}_n\|_{\infty} \\
&\le \frac{1}{pa_2^p}\|\tilde{u}_n\|_{\infty}^p + \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar{u}_n|^{q\alpha} + 2^{\alpha}M_1\|\tilde{u}_n\|_{\infty}^{\alpha+1} + M_2\|\tilde{u}_n\|_{\infty} \\
&\le \frac{T^{p/q}}{pa_2^p(q+1)^{p/q}}\|\dot{u}_n\|_{L^p}^p + \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar{u}_n|^{q\alpha} \\
&\quad +2^{\alpha}\left(\frac{T}{q+1}\right)^{(\alpha+1)/q}M_1\|\dot{u}_n\|_{L^p}^{\alpha+1} \\
&\quad + \left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}_n\|_{L^p}.
\end{aligned} \tag{3.16}
$$

By (3.16), we can choose $a_2 > \left(\frac{T}{q+1}\right)^{1/q}$ such that

$$\begin{aligned}&\left|\int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t))dt\right| \\ &\leq \left(\frac{1}{p} + \delta_1\right) \|\dot{u}_n\|_{L^p}^p + \frac{2^{q\alpha} a_2^q}{q} M_1^q |\bar{u}_n|^{q\alpha} + M_3.\end{aligned} \tag{3.17}$$

for all u_n, where M_3 is a positive constant dependent of the arbitrary positive number δ_1.

By (I2) and Lemma 2.1, we have

$$\begin{aligned}&\left|\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u_n^i(t))\tilde{u}_n^i(t)\right| \\ &\leq \sum_{j=1}^{m}\sum_{i=1}^{N}(a_{ij} + b_{ij}|u_n^i(t)|^{\gamma\beta_{ij}})|\tilde{u}_n^i(t)| \\ &\leq \sum_{j=1}^{m}\sum_{i=1}^{N}(a_{ij} + b_{ij}|\bar{u}_n^i(t) + \tilde{u}_n^i(t)|^{\gamma\beta_{ij}})|\tilde{u}_n^i(t)| \\ &\leq amN\|\tilde{u}_n\|_\infty + b\sum_{j=1}^{m}\sum_{i=1}^{N} 2^\alpha(|\bar{u}_n|^{\gamma\beta_{ij}} + \|\tilde{u}_n\|_\infty^{\gamma\beta_{ij}})\|\tilde{u}_n\|_\infty \\ &\leq amN\left(\frac{T}{q+1}\right)^{1/q}\|\dot{u}_n\|_{L^p} + \frac{2^\alpha b}{q}\sum_{j=1}^{m}\sum_{i=1}^{N}\beta_{ij}|\bar{u}_n|^{q\gamma} \\ &\quad +2^\alpha b\sum_{j=1}^{m}\sum_{i=1}^{N}\frac{q-\beta_{ij}}{q}\|\tilde{u}_n\|_\infty^{\frac{q}{q-\beta_{ij}}} + 2^\alpha b\sum_{j=1}^{m}\sum_{i=1}^{N}\|\tilde{u}_n\|_\infty^{\gamma\beta_{ij}+1} \\ &\leq amN\left(\frac{T}{q+1}\right)^{1/q}\|\dot{u}_n\|_{L^p} + \frac{2^\alpha b}{q}\sum_{j=1}^{m}\sum_{i=1}^{N}\beta_{ij}|\bar{u}_n|^{q\gamma} \\ &\quad +2^\alpha b\sum_{j=1}^{m}\sum_{i=1}^{N}\frac{q-\beta_{ij}}{q}\left[\left(\frac{T}{q+1}\right)^{p/q}\int_0^T|\dot{u}_n|^p dt\right]^{\frac{q}{p(q-\beta_{ij})}} \\ &\quad +2^\alpha b\sum_{j=1}^{m}\sum_{i=1}^{N}\left[\left(\frac{T}{q+1}\right)^{p/q}\int_0^T|\dot{u}_n|^p dt\right]^{\frac{\gamma\beta_{ij}+1}{p}},\end{aligned}$$

which follows there exists $\delta_2 > 0$ such that

$$\left|\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u_n^i(t))\tilde{u}_n^i(t)\right| \le \delta_2\|\dot{u}_n\|_{L^p}^p + \frac{2^\alpha b}{q} mN|\bar{u}_n|^{q\gamma} + M_4$$

for all u_n, where M_4 is a positive constant dependent of the arbitrary positive number δ_2.

Hence we get

$$\begin{aligned}
\|\tilde{u}_n\| &\ge \langle \varphi'(u_n), \tilde{u}_n\rangle \\
&= \|\dot{u}_n\|_{L^p}^p + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t))dt + \sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u_n^i(t))\tilde{u}_n^i(t) \\
&\ge \left(\frac{p-1}{p} - \delta_1\right)\|\dot{u}_n\|_{L^p}^p - \frac{2^{q\alpha}a_2^q}{q} M_1^q |\bar{u}_n|^{q\alpha} \\
&\quad -\delta_2\|\dot{u}_n\|_{L^p}^p - \frac{2^\alpha b}{q} mN|\bar{u}_n|^{q\gamma} - M_3 - M_4.
\end{aligned} \tag{3.18}$$

On the other hand, by (2.5), we have

$$\|\tilde{u}_n\| \le \left[1 + \frac{T^p\Theta(p,q)}{(q+1)^{p/q}}\right]^{1/p} \|\dot{u}_n\|_{L^p} \le \delta_3\|\dot{u}_n\|_{L^p}^p + M_5, \tag{3.19}$$

It follows from (3.18) and (3.19) that there exists $M_6 > 0$ dependent of $\delta_1, \delta_2, \delta_3$ such that

$$\begin{aligned}
\|\dot{u}_n\|_{L^p}^p \le\ & \frac{2^{q\alpha}a_2^q}{q\left(\frac{p-1}{p} - \delta_1 - \delta_2 - \delta_3\right)} M_1^q|\bar{u}_n|^{q\alpha} \\
& + \frac{2^\alpha bmN}{q\left(\frac{p-1}{p} - \delta_1 - \delta_2 - \delta_3\right)}|\bar{u}_n|^{q\gamma} + M_6.
\end{aligned} \tag{3.20}$$

It follows from (3.17) that

$$\begin{aligned}
&\left|\int_0^T (F(t, u_n(t)) - F(t, \bar{u}_n))dt\right| \\
&= \left|\int_0^T \int_0^1 (\nabla F(t, \bar{u}_n + s\tilde{u}_n(t)), \tilde{u}_n(t))dsdt\right| \\
&\le \left(\frac{1}{p} + \delta_1\right)\|\dot{u}_n\|_{L^p}^p + \frac{2^{q\alpha}a_2^q}{q} M_1^q|\bar{u}_n|^{q\alpha} + M_3.
\end{aligned} \tag{3.21}$$

Combining with (2.2), (3.20), (3.21) and (I3), we have

$$
\begin{aligned}
\varphi(u_n) &= \frac{1}{p}\|\dot{u}_n\|_{L^p}^p + \int_0^T [F(t,u_n(t)) - F(t,\bar{u}_n)]\,dt + \int_0^T F(t,\bar{u}_n)dt + \phi(u_n) \\
&\le \left(\frac{2}{p} + \delta_1\right)\|\dot{u}_n\|_{L^p}^p + \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar{u}_n|^{q\alpha} + \int_0^T F(t,\bar{u}_n)dt + M_3 \\
&\le \Bigg[\frac{(2+p\delta_1)2^{q\alpha}a_2^q}{pq\left(\frac{p-1}{p} - \delta_1 - \delta_2 - \delta_3\right)}M_1^q + \frac{2^{q\alpha}a_2^q}{q}M_1^q \\
&\quad + |\bar{u}_n|^{-q\alpha}\int_0^T F(t,\bar{u}_n)dt\Bigg]|\bar{u}_n|^{q\alpha} + \frac{(2+p\delta_1)2^{\alpha}bmN}{pq\left(\frac{p-1}{p} - \delta_1 - \delta_2 - \delta_3\right)}|\bar{u}_n|^{q\gamma} + M_7
\end{aligned}
$$

for some positive constant M_7 dependent of $\delta_1, \delta_2, \delta_3$.

We claim that $\{|\bar{u}_n|\}$ is bounded. In fact, if $\{|\bar{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary, $|\bar{u}_n| \to +\infty, \quad n \to +\infty$.

Let

$$G(\delta_1, \delta_2, \delta_3) = \frac{(2 + p\delta_1)}{(\frac{p-1}{p} - \delta_1 - \delta_2 - \delta_3)}$$

when $\delta_1, \delta_2, \delta_3$ are small enough, it is easy to see that $G(\delta_1, \delta_2, \delta_3)$ is monotone increasing for every variable. Furthermore, we have

$$\lim_{(\delta_1,\delta_2,\delta_3)\to(0^+,0^+,0^+)} G(\delta_1, \delta_2, \delta_3) = \frac{2p}{p-1}.$$

It follows from (F3) that

$$\varphi(u_n) \to -\infty, \quad n \to \infty.$$

which contradicts the boundedness of $\{\varphi(u_n)\}$. Hence $\{|\bar{u}_n|\}$ is bounded, it follows from (3.19) and Lema 2.1 that $\{u_n\}$ is bounded in $W_T^{1,p}$, going if necessary to a subsequence, we can assume that

$$u_n \rightharpoonup u_0 \quad \text{in} \quad W_T^{1,p}, \tag{3.22}$$

by Proposition 1.2 in [8], we have

$$u_n \to u_0 \quad \text{in} \quad C([0,T], \mathbb{R}^N). \tag{3.23}$$

It follows from (2.3) and the Hölder inequality that

$$
\begin{aligned}
&\langle \psi'(u_n) - \psi'(u_0), u_n - u_0 \rangle \\
=\ & \int_0^T |\dot{u}_n(t)|^{p-2}(\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_0(t))dt \\
& - \int_0^T |\dot{u}_0(t)|^{p-2}(\dot{u}_0(t), \dot{u}_n(t) - \dot{u}_0(t))dt \\
& - \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t)), u_n(t) - u_0(t))dt \\
=\ & \|u_n\|^p + \|u_0\|^p - \int_0^T |\dot{u}_n(t)|^{p-2}(\dot{u}_n(t), \dot{u}_0(t))dt \\
& - \int_0^T |\dot{u}_0(t)|^{p-2}(\dot{u}_0(t), \dot{u}_n(t))dt \\
& - \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t)), u_n(t) - u_0(t))dt \\
\geq\ & \|u_n\|^p + \|u_0\|^p - \int_0^T |\dot{u}_n(t)|^{p-1}|\dot{u}_0(t)|dt - \int_0^T |\dot{u}_0(t)|^{p-1}|\dot{u}_n(t)|dt \\
& - \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t)), u_n(t) - u_0(t))dt \\
\geq\ & \|u_n\|^p + \|u_0\|^p - \left(\int_0^T |\dot{u}_0(t)|^p dt\right)^{1/p} \left(\int_0^T |\dot{u}_n(t)|^p dt\right)^{1/q} \\
& - \left(\int_0^T |\dot{u}_n(t)|^p dt\right)^{1/p} \left(\int_0^T |\dot{u}_0(t)|^p dt\right)^{1/q} \\
& - \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t)), u_n(t) - u_0(t))dt \\
\geq\ & \|u_n\|^p + \|u_0\|^p - \left(\int_0^T \left[|\dot{u}_0(t)|^p + |u_0(t)|^p\right] dt\right)^{1/p} \\
& \quad \left(\int_0^T \left[|\dot{u}_n(t)|^p + |u_n(t)|^p dt\right] dt\right)^{1/q} \\
& - \left(\int_0^T \left[|\dot{u}_n(t)|^p + |u_n(t)|^p\right] dt\right)^{1/p} \left(\int_0^T \left[|\dot{u}_0(t)|^p + |u_0(t)|^p\right] dt\right)^{1/q} \\
& - \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t)), u_n(t) - u_0(t))dt \\
=\ & \|u_n\|^p + \|u_0\|^p - \|u_0\|\|u_n\|^{p-1} - \|u_n\|\|u_0\|^{p-1} \\
& - \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t)), u_n(t) - u_0(t))dt \\
=\ & \left(\|u_n\|^{p-1} - \|u_0\|^{p-1}\right)\left(\|u_n\| - \|u_0\|\right) \\
& - \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t)), u_n(t) - u_0(t))dt.
\end{aligned}
\tag{3.24}
$$

It follows from (2.3) and (3.24) that

$$
\begin{aligned}
&\langle \varphi'(u_n) - \varphi'(u_0), u_n - u_0 \rangle \\
\geq\ & \left(\|u_n\|^{p-1} - \|u_0\|^{p-1} \right) (\|u_n\| - \|u_0\|) \\
& - \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t)), u_n(t) - u_0(t)) dt \\
& - \sum_{j=1}^{m} \sum_{i=1}^{N} (I_{ij}(u_n^i(t_j)) - I_{ij}(u_0^i(t_j)))(u_n^i(t_j)) - u_0^i(t_j))).
\end{aligned} \tag{3.25}
$$

From (3.22)-(3.25), (A) and the continuity of I_{ij}, it follows that $\|u_n\| \to \|u\|$ in $W_T^{1,p}$. Thus, φ satisfies the P.S. condition.

In order to use the saddle point theorem ([12], Theorem 4.6]), we only need to verify the following conditions:

(A_1) $\varphi(x) \to -\infty$ as $|x| \to \infty$ in $\mathbb{R}^N$.

(A_2) $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ in $\tilde{W}_T^{1,p}$, where $\tilde{W}_T^{1,p} = \{u \in W_T^{1,p} \mid \bar{u} = 0\}$.

In fact, by (3.6), we get

$$
\int_0^T F(t, x) dt \to -\infty \quad \text{ as } \ |x| \to \infty \ \text{ in } \ \mathbb{R}^N. \tag{3.26}
$$

From (I3) and (3.26), we have

$$
\varphi(x) = \int_0^T F(t, x) dt + \phi(x) \to -\infty \quad \text{ as } \ |x| \to \infty \ \text{ in } \ \mathbb{R}^N.
$$

Thus (A_1) is easy to verify.

Next, for all $u \in \tilde{W}_T^{1,p}$, by (F1) and Lemma 2.1, we have

$$
\begin{aligned}
& \left| \int_0^T [F(t, u(t)) - F(t, 0)] dt \right| \\
=\ & \left| \int_0^T \int_0^1 (\nabla F(t, su(t)), u(t)) ds dt \right| \\
\leq\ & \int_0^T f(t) |u(t)|^{\alpha+1} dt + \int_0^T g(t) |u(t)| dt \\
\leq\ & M_1 \|u\|_\infty^{\alpha+1} + M_2 \|u\|_\infty \\
\leq\ & \left(\frac{T}{q+1} \right)^{(\alpha+1)/q} M_1 \|\dot{u}\|_{L^p}^{\alpha+1} + \left(\frac{T}{q+1} \right)^{1/q} M_2 \|\dot{u}\|_{L^p}.
\end{aligned} \tag{3.27}
$$

It derives from (I2) that

$$
\begin{aligned}
|\phi(u)| &= |\sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt| \\
&\leq \sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} (a_{ij}+b_{ij}|t|^{\gamma\beta_{ij}})dt \\
&\leq amN\|u\|_\infty + b\sum_{j=1}^{m}\sum_{i=1}^{N}\|u\|_\infty^{\gamma\beta_{ij}+1} \\
&\leq amN\left(\frac{T}{q+1}\right)^{1/q}\|\dot{u}\|_{L^p} + b\sum_{j=1}^{m}\sum_{i=1}^{N}\left(\frac{T}{q+1}\right)^{\frac{\gamma\beta_{ij}+1}{q}}\|\dot{u}\|_{L^p}^{\frac{\gamma\beta_{ij}+1}{q}}.
\end{aligned}
\tag{3.28}
$$

It follows from (2.2), (3.27) and (3.28) that

$$
\begin{aligned}
&\varphi(u) \\
&= \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T [F(t,u(t)) - F(t,0)]dt + \int_0^T F(t,0)dt + \phi(u) \\
&\geq \frac{1}{p}\|\dot{u}\|_{L^p}^p - \left(\frac{T}{q+1}\right)^{(\alpha+1)/q} M_1\|\dot{u}\|_{L^p}^{\alpha+1} - \left(\frac{T}{q+1}\right)^{1/q} M_2\|\dot{u}\|_{L^p} \\
&\quad -amN\left(\frac{T}{q+1}\right)^{1/q}\|\dot{u}\|_{L^p} - b\sum_{j=1}^{m}\sum_{i=1}^{N}\left(\frac{T}{q+1}\right)^{\frac{\gamma\beta_{ij}+1}{q}}\|\dot{u}\|_{L^p}^{\frac{\gamma\beta_{ij}+1}{q}} \\
&\quad +\int_0^T F(t,0)dt
\end{aligned}
\tag{3.29}
$$

for all $u \in \tilde{W}_T^{1,p}$. By Lemma 2.1, $\|u\| \to \infty$ in $\tilde{W}_T^{1,p}$ if and only if $\|\dot{u}\|_{L^p} \to \infty$. So we obtain $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ in $\tilde{W}_T^{1,p}$ from (3.29), i.e. (A_2) is verified. The proof of Theorem 1.2 is complete.

Proof of Theorem 1.3. By (f) and (F5), we can choose an $a_3 \in \mathbb{R}$ such that

$$
a_3 > \left(\frac{T^{p/q}}{(q+1)^{p/q} - 2^{p-1}pM_1T^{p/q}}\right)^{1/p} > 0,
\tag{3.30}
$$

and

$$
\liminf_{|x|\to+\infty} |x|^{-p}\int_0^T F(t,x)dt > \frac{2^p a_3^q}{q}M_1^q.
\tag{3.31}
$$

It follows from (F4), Lemma 2.1 and Young inequality that

$$
\begin{aligned}
&\left|\int_0^T (F(t,u(t)) - F(t,\bar{u}))dt\right| \\
&= \left|\int_0^T \int_0^1 (\nabla F(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t))dsdt\right| \\
&\le \int_0^T \int_0^1 f(t)\,|\bar{u} + s\tilde{u}(t)|^{p-1}\,|\tilde{u}(t)|\,dsdt + \int_0^T \int_0^1 g(t)\,|\tilde{u}(t)|\,dsdt \\
&\le 2^{p-1}\int_0^T f(t)\left(|\bar{u}|^{p-1} + |\tilde{u}(t)|^{p-1}\right)|\tilde{u}(t)|\,dt + \int_0^T g(t)\,|\tilde{u}(t)|\,dt \\
&\le 2^{p-1}(|\bar{u}|^{p-1}\,\|\tilde{u}\|_\infty + \|\tilde{u}\|_\infty^p)\int_0^T f(t)dt + \|\tilde{u}\|_\infty \int_0^T g(t)dt \\
&= 2^{p-1}M_1|\bar{u}|^{p-1}\|\tilde{u}\|_\infty + 2^{p-1}M_1\|\tilde{u}\|_\infty^p + M_2\|\tilde{u}\|_\infty \\
&\le \frac{1}{pa_3^p}\|\tilde{u}\|_\infty^p + \frac{2^p a_3^q}{q}M_1^q|\bar{u}|^p + 2^{p-1}M_1\|\tilde{u}\|_\infty^p + M_2\|\tilde{u}\|_\infty \\
&\le \frac{T^{p/q}}{pa_3^p(q+1)^{p/q}}\|\dot{u}\|_{L^p}^p + \frac{2^p a_3^q}{q}M_1^q|\bar{u}|^p \\
&\quad + 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q}\|\dot{u}\|_{L^p}^p + \left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}\|_{L^p} \\
&= \left(\frac{T^{p/q}}{pa_3^p(q+1)^{p/q}} + 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q}\right)\|\dot{u}\|_{L^p}^p \\
&\quad + \frac{2^p a_3^q}{q}M_1^q|\bar{u}|^p + \left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}\|_{L^p}.
\end{aligned} \tag{3.32}
$$

Hence we have by (I1) and (3.32)

$$
\begin{aligned}
\varphi(u) &= \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T [F(t,u(t)) - F(t,\bar{u})]\,dt + \int_0^T F(t,\bar{u})dt + \phi(u) \\
&\ge \left(\frac{1}{p} - \frac{T^{p/q}}{pa_3^p(q+1)^{p/q}} - 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q}\right)\|\dot{u}\|_{L^p}^p \\
&\quad - \left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}\|_{L^p} \\
&\quad + |\bar{u}|^p\left(|\bar{u}|^{-p}\int_0^T F(t,\bar{u})dt - \frac{2^p a_3^q}{q}M_1^q\right).
\end{aligned} \tag{3.33}
$$

As $\|u\| \to \infty$ if and only if $(|\bar{u}|^p + \|\dot{u}\|_{L^p})^{1/p} \to \infty$, the above inequality and (3.31) and (3.33) imply that $\varphi(u) \to +\infty$ as $\|u\| \to \infty$. Similar to the proof of Theorem 1.1, φ has a minimum point on $W_T^{1,p}$, which is a critical point of φ. The proof of Theorem 1.3 is complete.

Proof of Theorem 1.4. Suppose that $\{U_n\} \subset W_T^{1,p}$ is a (PS) sequence of φ. In a way similar to the proof of Theorem 1.3, we have

$$\begin{aligned}&\left|\int_0^T (\nabla F(t,u_n(t)),\tilde{u}_n(t))dt\right|\\ &\le \left(\frac{T^{p/q}}{pa_4^p(q+1)^{p/q}}+2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q}\right)\|\dot{u}_n\|_{L^p}^p\\ &\quad+\frac{2^pa_4^q}{q}M_1^q|\bar{u}_n|^p+\left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}_n\|_{L^p},\end{aligned}$$

we can choose $a_4 > \left(\frac{T}{q+1}\right)^{1/q}$ such that

$$\begin{aligned}&\left|\int_0^T (\nabla F(t,u_n(t)),\tilde{u}_n(t))dt\right|\\ &\le \left(\frac{1}{p}+2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q}\right)\|\dot{u}_n\|_{L^p}^p+\frac{2^pa_4^q}{q}M_1^q|\bar{u}_n|^p+\left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot{u}_n\|_{L^p},\end{aligned}$$

which means that

$$\begin{aligned}&\left|\int_0^T (\nabla F(t,u_n(t)),\tilde{u}_n(t))dt\right|\\ &\le \left(\frac{1}{p}+2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q}+\delta_1'\right)\|\dot{u}_n\|_{L^p}^p+\frac{2^pa_4^q}{q}M_1^q|\bar{u}_n|^p+M_3'.\end{aligned}$$

for all u_n, where M_3' is a positive constant dependent of the arbitrary positive number δ_1'.

By a fashion similar to the proofs of Theorem 1.2, we have

$$\begin{aligned}\|\tilde{u}_n\| &\ge \langle\varphi'(u_n),\tilde{u}_n\rangle\\ &= \|\dot{u}_n\|_{L^p}^p+\int_0^T(\nabla F(t,u_n(t)),\tilde{u}_n(t))dt+\sum_{j=1}^m\sum_{i=1}^N I_{ij}(u_n^i(t))\tilde{u}_n^i(t)\\ &\ge \left(\frac{p-1}{p}-2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q}-\delta_1'\right)\|\dot{u}_n\|_{L^p}^p\\ &\quad-\frac{2^pa_4^q}{q}M_1^q|\bar{u}_n|^p-\delta_2'\|\dot{u}_n\|_{L^p}^p-\frac{2^{p-1}b}{p}mN|\bar{u}_n|^{p\gamma}-M_4'.\end{aligned} \tag{3.34}$$

On the other hand, by Lemma 2.1, we have

$$\|\tilde{u}_n\| \le \left[1+\frac{T^p\Theta(p,q)}{(q+1)^{p/q}}\right]^{1/p}\|\dot{u}_n\|_{L^p} \le \delta_3'\|\dot{u}_n\|_{L^p}^p+M_5', \tag{3.35}$$

It follows that there exists $M_6' > 0$ dependent of δ_1', δ_2' and δ_3' such that

$$
\begin{aligned}
\|\dot{u}_n\|^p \le\ & \frac{2^p a_4^q}{q\left(\frac{p-1}{p} - 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q} - \delta_1' - \delta_2' - \delta_3'\right)} M_1^q |\bar{u}_n|^p \\
& + \frac{2^{p-1}bmN}{p\left(\frac{p-1}{p} - 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q} - \delta_1' - \delta_2' - \delta_3'\right)} |\bar{u}_n|^{p\gamma} + M_7'.
\end{aligned}
\tag{3.36}
$$

In a way similar to the proof of Theorem 1.1, we have

$$
\begin{aligned}
& \left|\int_0^T (F(t,u_n(t)) - F(t,\bar{u}_n))dt\right| \\
= & \left|\int_0^T \int_0^1 (\nabla F(t, \bar{u}_n + s\tilde{u}_n(t)), \tilde{u}(t))dsdt\right| \\
\le & \left(\frac{1}{p} + 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q} + \delta_1'\right)\|\dot{u}_n\|_{L^p}^p + \frac{2^p a_4^q}{q} M_1^q |\bar{u}_n|^p + M_3'.
\end{aligned}
\tag{3.37}
$$

Combining with (2.2), (3.36) and (3.37), we have

$$
\begin{aligned}
\varphi(u_n) =\ & \frac{1}{p}\|\dot{u}_n\|_{L^p}^p + \int_0^T \left[F(t,u_n(t)) - F(t,\bar{u}_n)\right] dt + \int_0^T F(t,\bar{u}_n)dt + \phi(u_n) \\
\le\ & \left(\frac{2}{p} + 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q} + \delta_1'\right)\|\dot{u}_n\|_{L^p}^p \\
& + \frac{2^p a_4^q}{q} M_1^q |\bar{u}_n|^p + \int_0^T F(t,\bar{u}_n)dt + M_3' \\
\le\ & \left[\frac{\left(2 + p\delta_1' + 2^{p-1}pM_1\left(\frac{T}{q+1}\right)^{p/q}\right)2^p a_4^q}{pq\left(\frac{p-1}{p} - 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q} - \delta_1' - \delta_2' - \delta_3'\right)} M_1^q \right. \\
& \left. + \frac{2^p a_4^q}{q} M_1^q + |\bar{u}_n|^{-p}\int_0^T F(t,\bar{u})dt\right] |\bar{u}_n|^p \\
& + \frac{2^{p-1}bmN}{p\left(\frac{p-1}{p} - 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q} - \delta_1' - \delta_2' - \delta_3'\right)} |\bar{u}_n|^{p\gamma} + M_7'
\end{aligned}
$$

for some positive constant M_7' dependent of δ_1', δ_2' and δ_3'.

We claim that $\{|\bar{u}_n|\}$ is bounded. In fact, if $\{|\bar{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary, $|\bar{u}_n| \to +\infty, \quad n \to +\infty$.

Let

$$H(\delta_1', \delta_2', \delta_3') = \frac{2 + p\delta_1' + 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q}}{\frac{p-1}{p} - 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q} - \delta_1' - \delta_2' - \delta_3'}$$

when $\delta_1', \delta_2', \delta_3'$ are small enough, it is easy to see that $H(\delta_1', \delta_2', \delta_3')$ is monotone increasing for every variable. Furthermore, we have

$$\lim_{(\delta_1',\ \delta_2',\ \delta_3')\to(0^+,\ 0^+,\ 0^+)} H(\delta_1', \delta_2', \delta_3') = \frac{2p + 2^{p-1}pM_1\left(\frac{T}{q+1}\right)^{p/q}}{p - 1 - 2^{p-1}M_1\left(\frac{T}{q+1}\right)^{p/q}}.$$

It follows from (3.12) that

$$\varphi(u_n) \to -\infty, \quad n \to \infty.$$

which contradicts the boundedness of $\{\varphi(u_n)\}$. Hence $\{|\bar{u}_n|\}$ is bounded, going if necessary to a subsequence, we can assume that

$$u_n \rightharpoonup u_0 \quad \text{in} \ W_T^{1,p}, \tag{3.38}$$

Similar to the proof of Theorem 1.2, we can easily verify that φ satisfies the P.S. condition, we only need to verify (A_1) and (A_2). It is easy to verify (A_1) by (I3) and (F6). In what follows, we verify that (A_2) holds also. For all $u \in \tilde{W}_T^{1,p}$, by (1.12) and Sobolev's inequality, we have

$$\begin{aligned}
&\left|\int_0^T [F(t, u(t)) - F(t, 0)]dt\right| \\
&= \left|\int_0^T \int_0^1 (\nabla F(t, su(t)), u(t))dsdt\right| \\
&\leq \frac{1}{p}\int_0^T f(t)|u(t)|^p dt + \int_0^T g(t)|u(t)|dt \\
&\leq \frac{M_1}{p}\|u\|_\infty^p + M_2\|u\|_\infty \\
&\leq \frac{M_1\left(\frac{T}{q+1}\right)^{p/q}}{p}\|\dot{u}\|_{L^p}^p + \left(\frac{T}{q+1}\right)^{1/q} M_2\|\dot{u}\|_{L^p}.
\end{aligned} \tag{3.39}$$

Similar to the proof in (3.28), It derives from (I2) that

$$\begin{aligned}
|\phi(u)| &= |\sum_{j=1}^m \sum_{i=1}^N \int_0^{u^i(t_j)} I_{ij}(t)dt| \\
&\leq amN\left(\frac{T}{q+1}\right)^{1/q} \|\dot{u}(t)\|_{L^p} \\
&\quad + b\sum_{j=1}^m \sum_{i=1}^N \left(\frac{T}{q+1}\right)^{\frac{\gamma\beta_{ij}+1}{q}} \|\dot{u}(t)\|_{L^p}^{\frac{\gamma\beta_{ij}+1}{q}}.
\end{aligned} \tag{3.40}$$

It follows from (f), (2.2), (3.39) and (3.40) that

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T [F(t,u(t)) - F(t,0)]dt + \int_0^T F(t,0)dt + \phi(u) \\ &\geq \frac{1}{p}\|\dot{u}\|_{L^p}^p - \frac{M_1\left(\frac{T}{q+1}\right)^{p/q}}{p}\|\dot{u}\|_{L^p}^p - \left(\frac{T}{q+1}\right)^{1/q} M_2\|\dot{u}\|_{L^p} \\ &\quad -amN\left(\frac{T}{q+1}\right)^{1/q}\|\dot{u}(t)\|_{L^p} \\ &\quad -b\sum_{j=1}^{m}\sum_{i=1}^{N}\left(\frac{T}{q+1}\right)^{\frac{\gamma\beta_{ij}+1}{q}}\|\dot{u}(t)\|_{L^p}^{\frac{\gamma\beta_{ij}+1}{q}} + \int_0^T F(t,0)dt\end{aligned} \tag{3.41}$$

for all $u \in \tilde{W}_T^{1,p}$. $\|u\| \to \infty$ in $\tilde{W}_T^{1,p}$ if and only if $\|\dot{u}\|_{L^p} \to \infty$. So we obtain $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ in $\tilde{W}_T^{1,p}$ from (3.41), i.e. (A_2) is verified. The proof of Theorem 1.4 is complete.

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. Let $T=2,\ N=3,\ t_1=1,\ p=\frac{3}{2},\ q=3$, consider the second-order p-Laplacian systems with impulsive effects

$$\begin{cases} \dfrac{d}{dt}\left(|\dot{u}(t)|^{1/2}\right) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) - u(2) = \dot{u}(0) - \dot{u}(2) = 0, \\ \Delta \dot{u}^i(1) = \dot{u}^i(1^+) - \dot{u}^i(1^-) = (u^i(1))^{\frac{1}{3}}, i = 1,2,3, \end{cases} \tag{4.1}$$

let

$$F(t,x) = (0.5T - t)|x|^{10/7} + \left(\frac{2}{3}T^2 - t^2\right)|x|^{9/7} + (h(t),x), \tag{4.2}$$

$I_{ij}(t) = t^{\frac{1}{3}}, \alpha = \frac{3}{7}$. It is easy to see that

$$\begin{aligned}|\nabla F(t,x)| &\leq \frac{10}{7}|0.5T - t|\,|x|^{3/7} + \frac{9}{7}\left|\frac{2}{3}T^2 - t^2\right||x|^{2/7} + |h(t)| \\ &\leq \frac{10}{7}(|0.5T - t| + \varepsilon)|x|^{3/7} + \frac{T^6}{\varepsilon^2} + |h(t)|\end{aligned}$$

The above shows (3.2) holds with $\alpha = 3/7$ and

$$f(t) = \frac{10}{7}\left(|0.5T - t| + \varepsilon\right), \quad g(t) = \frac{T^6}{\varepsilon^2} + |h(t)|, \tag{4.3}$$

and

$$\begin{aligned}
&\frac{2^{q\alpha}T}{q(q+1)}\left(\int_0^T f(t)dt\right)^q \\
&= \frac{2^{\frac{9}{7}}T}{3\times 4}\int_0^T\left(\frac{10}{7}\left(|0.5T-t|+\varepsilon\right)dt\right)^3 \\
&\le \left(\frac{10}{7}\right)^3\frac{T^4}{3}\left(\frac{T^3}{64}+\frac{3T^2}{16}\varepsilon+\frac{3T^3}{4}\varepsilon^2+\varepsilon^3\right).
\end{aligned}$$

If $T^4 < 64\times\left(\frac{7}{10}\right)^3 = 21.952$, we choose $\varepsilon > 0$ sufficient small such that

$$\liminf_{|x|\to+\infty}|x|^{-3\alpha}\int_0^T F(t,x)dt = \frac{T^3}{3} > \left(\frac{10}{7}\right)^3\frac{T^4}{3}\left(\frac{T^3}{64}+\frac{3T^2}{16}\varepsilon+\frac{3T^3}{4}\varepsilon^2+\varepsilon^3\right).$$

This shows that (3.3) holds. By Theorem 1.1, problem (4.1) has at least one solution.

Example 4.2. Let $T = 0.3,\ N = 5,\ t_1 = 0.2,\ p = \frac{3}{2},\ q = 3$, consider the second-order Hamiltonian systems with impulsive effects

$$\text{(4.4)}\qquad \begin{cases} \dfrac{d}{dt}\left(|\dot u(t)|^{1/2}\dot u(t)\right) = \nabla F(t,u(t)), \quad \text{a.e. } t\in[0,\pi], \\ u(0)-u(0.3) = \dot u(0)-\dot u(0.3) = 0, \\ \Delta \dot u^i(0.2) = \dot u^i(0.2^+)-\dot u^i(0.2^-) = I_{i1}(u^i(0.2))^{\frac{1}{9}}, i = 1,2,3,4,5, \end{cases}$$

let

$$\text{(4.5)}\qquad F(t,x) = (0.5T-t)|x|^{10/7}+\left(\tfrac{1}{4}T^2-t^2\right)|x|^{9/7}+(h(t),x),$$

$I_{i1}(t) = -t^{\frac{1}{7}}, \alpha = 3/7, \beta_{i1} = 1/3,\ h\in L^1([0,T],\mathbb{R}^N)$. It is easy to see that

$$\begin{aligned}
|\nabla F(t,x)| &\le \frac{10}{7}\left(|0.5T-t|+\varepsilon\right)|x|^{3/7}+\frac{T^6}{\varepsilon^2}+|h(t)|. \\
f(t) &= \frac{10}{7}\left(|0.5T-t|+\varepsilon\right)|x|^{3/7},\ g(t) = \frac{T^6}{\varepsilon^2}+|h(t)|.
\end{aligned}$$

$$\text{(4.6)}\qquad \begin{aligned}
|x|^{-3\alpha}\int_0^T F(t,x)dt &= |x|^{-9/7}\int_0^T\Big[(0.5T-t)|x|^{10/7} \\
&\quad +\left(\frac{1}{4}T^2-t^2\right)|x|^{9/7}+(h(t),x)\Big]dt \\
&= -T^3/12+\left(\int_0^T h(t)dt, |x|^{-9/7}x\right).
\end{aligned}$$

Similar to the computation in Example 4.1, it is easy to verify that all the conditions of Theorem 1.2 hold, by Theorem 1.2, problem (4.4) has at least one solution.

Example 4.3. Let $T = 0.6, N = 3, t_1 = 0.5, p = q = 2$, consider the second-order Hamiltonian systems with impulsive effects

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(0.6) = \dot{u}(0) - \dot{u}(0.6) = 0, \\ \Delta \dot{u}^i(1) = \dot{u}^i(0.5^+) - \dot{u}^i(0.5^-) = (u^i(0.5))^{\frac{1}{3}}, i = 1, 2, 3. \end{cases} \tag{4.7}$$

Let

$$F(t, x) = (0.6T - t)|x|^2 - t|x|^{3/2} + (h(t), x), \tag{4.8}$$

where $h \in L^1([0, T], \mathbb{R}^N)$, $I_{ij}(t) = t^{\frac{1}{3}}$. It is easy to see that

$$\begin{aligned} |\nabla F(t, x)| &\leq 2|0.6T - t||x| + \frac{3t}{2}|x|^{1/2} + |h(t)| \\ &\leq 2\left(|0.6T - t| + \varepsilon\right)|x| + \frac{T^2}{2\varepsilon} + |h(t)| \end{aligned}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, where $\varepsilon > 0$. The above shows (1.12) holds with

$$f(t) = 2\left(|0.6T - t| + \varepsilon\right), \qquad g(t) = \frac{T^2}{2\varepsilon} + |h(t)|. \tag{4.9}$$

Observe that

$$\begin{aligned} |x|^{-2}\int_0^T F(t, x)dt &= |x|^{-2}\int_0^T \left[(0.6T - t)|x|^2 - t|x|^{3/2} + (h(t), x)\right] dt \\ &= 0.1T^2 - 0.5T^2|x|^{-1/2} + \left(\int_0^T h(t)dt, |x|^{-2}x\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_0^T f(t)dt &= 2\int_0^T \left(|0.6T - t| + \varepsilon\right)dt = 0.52T^2 + 2\varepsilon T, \\ \left(\int_0^T f(t)dt\right)^2 &= \left(0.52T^2 + 2\varepsilon T\right)^2 = 0.2704T^4 + 2.08\varepsilon T^3 + 4\varepsilon^2 T^2, \end{aligned}$$

and

$$\frac{3T^2\int_0^T f^2(t)dt}{2\pi^2\left(12 - T\int_0^T f(t)dt\right)} = \frac{T^3(1.12T^2 + 6.24\varepsilon T + 12\varepsilon^2)}{2\pi^2[12 - T^2(0.52T + 2\varepsilon)]}.$$

If $T^3 < 0.7$, we choose $\varepsilon > 0$ sufficient small such that

$$\int_0^T f(t)dt = 0.52T^2 + 2\varepsilon T < \frac{3}{4T}$$

and

$$\begin{aligned}\liminf_{|x|\to+\infty} |x|^{-2}\int_0^T F(t,x)dt &= 0.1T^2 \\ &> \frac{2T(0.2704T^4 + 2.08\varepsilon T^3 + 4\varepsilon^2 T^2)}{3 - 4T(0.52T^2 + 2\varepsilon T)}.\end{aligned}$$

These show that (3.8)-(3.10) hold. By Theorem 1.3, problem (4.7) has at least one solution.

Example 4.4. Let $T = 0.5, N = 5, t_1 = 0.4$, consider the second-order Hamiltonian systems with impulsive effects

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(0.5) = \dot{u}(0) - \dot{u}(0.5) = 0, \\ \Delta \dot{u}^i(0.4) = \dot{u}^i(0.4^+) - \dot{u}^i(0.4^-) \\ = I_{ij}(u^i(0.4)), i = 1, 2, ..., N; j = 1, 2, ..., m, \end{cases} \tag{4.10}$$

$$F(t,x) = (0.4T - t)|x|^2 + t|x|^{3/2} + (h(t), x), \tag{4.11}$$

where $h \in L^1([0,T], \mathbb{R}^N)$. It is easy to see that

$$\begin{aligned}|\nabla F(t,x)| &\le 2|0.4T - t||x| + \frac{3t}{2}|x|^{1/2} + |h(t)| \\ &\le 2\left(|0.4T - t| + \varepsilon\right)|x| + \frac{T^2}{2\varepsilon} + |h(t)|\end{aligned}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, where $\varepsilon > 0$. The above shows (1.12) holds with

$$f(t) = 2\left(|0.4T - t| + \varepsilon\right), \qquad g(t) = \frac{T^2}{2\varepsilon} + |h(t)|. \tag{4.12}$$

Observe that

$$\begin{aligned}|x|^{-2}\int_0^T F(t,x)dt &= |x|^{-2}\int_0^T \Big[(0.4T - t)|x|^2 + t|x|^{3/2} + (h(t), x)\Big]\,dt \\ &= -0.1T^2 + 0.5T^2|x|^{-1/2} + \left(\int_0^T h(t)dt, |x|^{-2}x\right).\end{aligned}$$

Similar to the computation in Example 4.3, it is easy to verify that all the conditions of Theorem 1.4 hold, by Theorem 1.4, problem (4.10) has at least one solution.

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