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THE BANACH ALGEBRA $\mathcal{F}(S,T)$ AND ITS AMENABILITY OF COMMUTATIVE FOUNDATION $*$-SEMIGROUPS S AND T

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Abstract. In the present paper we shall first introduce the notion of the algebra $\mathcal{F}(S,T)$ of two topological $*$-semigroups S and T in terms of bounded and weakly continuous $*$-representations of S and T on Hilbert spaces. In the case where both S and T are commutative foundation $*$-semigroups with identities it is shown that $\mathcal{F}(S,T)$ is identical to the algebra of the Fourier transforms of bimeasures in $BM(S^*,T^*)$, where S^* $(T^*$, respectively) denotes the locally compact Hausdorff space of all bounded and continuous $*$-semicharacters on $S(T,$ respectively) endowed with the compact open topology. This result has enabled us to make the bimeasure Banach space $BM(S^*,T^*)$ into a Banach algebra. It is also shown that the Banach algebra $\mathcal{F}(S,T)$ is amenable and $K\big(\sigma(\overline{\mathcal{F}(S,T)})\big)$ is a compact topological group, where $\sigma(\overline{\mathcal{F}(S,T)})$ denotes the spectrum of the commutative Banach algebra $\overline{\mathcal{F}(S,T)}$ as a closed subalgebra of wap $(S\times T)$, the Banach algebra of weakly almost periodic continuous functions on $S\times T$.

0. Preliminaries

For a locally compact Hausdorff space X, we let $L^\infty(X)$, $C_b(X)$, $C_0(X)$ be the spaces of complex valued and bounded functions on X which are respectively, Borel measurable, continuous, continuous with limit zero at infinity. The supremum norm on each of these spaces will be denoted by $\|\cdot\|_\infty$. If X and Y are locally compact Hausdorff spaces, we write $V_0(X,Y)=C_0(X)\widehat{\otimes}C_0(Y)$, the projective tensor product of $C_0(X)$ and $C_0(Y)$. Then the space $BM(X,Y)$ may be identified with the dual Banach space of $V_0(X,Y)$. The elements of $BM(X,Y)$ are called the *bimeasures* on $X\times Y$. It is well-known [7] that corresponding to every $u\in BM(X,Y)$ there exist regular probability Borel measures λ_X on X and λ_Y on Y and $C>0$ such that

$$(1)\qquad |\langle f\otimes g,u\rangle|\le C\|f\|_2\|g\|_2 \quad \big(f\in C_0(X), g\in C_0(Y)\big)$$

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where the L^2-norms refer to $L^2(X, \lambda_X)$ and $L^2(Y, \lambda_Y)$, respectively. Let $|||u||| = \inf\{C : (1)$ holds for some $\lambda_X, \lambda_Y\}$. Then there is a universal constant K_G such that

$$\|u\| \le |||u||| \le K_G\|u\| \quad \big(u \in BM(X,Y)\big). \tag{2}$$

The measures λ_X, λ_Y are called a *Grothendieck measure pair* for u. Moreover, corresponding to every $u \in BM(X,Y)$ there is a unique extension of u to $L^\infty(X)\hat{\otimes}L^\infty(Y)$ such that for every pair λ_X, λ_Y of Grothendieck measures for u with C as in (1)

$$|\langle f \otimes g, u\rangle| \le C\|f\|_2\|g\|_2 \quad \big(f \in L^\infty(X), g \in L^\infty(Y)\big), \tag{3}$$

(cf. Corollary 1.3 of [7]). Recall that the support of a bimeasure u on $X \times Y$ is the smallest closed set F in $X \times Y$ for which $\langle f, u\rangle = 0$ for all $f \in V_0(X,Y)$ such that $f \equiv 0$ on a neighbourhood of F. Note that the bimeasures with compact support are dense in $BM(X,Y)$ (see [7, Lemma 1.4]).

Throughout this paper S and T will denote locally compact, Hausdorff and jointly continuous topological semigroups. A continuous mapping $* : S \to S$ is called an *involution* on S if $x^{**} = x$ and $(xy)^* = y^*x^*$ $(x, y \in S)$. A topological semigroup with an involution is called a *topological $*$-semigroup*. A homomorphism π of a topolical $*$-semigroup S into the unit ball of $B(H)$ (the C^*-algebra of bounded linear operators on a Hilbert space H) is called a *$*$-representation* if $\pi(x^*) = \pi(x)^*$ for all $x \in S$, where $\pi(x)^*$ denotes the adjoint operator to $\pi(x)$. A $*$-representation $\pi : S \to B(H)$ is called *weakly continuous* (*strongly continuous*, respectively) if the mapping $s \mapsto \langle \pi(s)\xi, \eta\rangle$ of S into $\mathbb{C}$ $(s \mapsto \pi(s)\xi$ of S into H, respectively) is continuous (norm continuous, respectively) for every $\xi, \eta \in H$. A one dimensional representation is called a *semicharacter*. We denote by $\widetilde{S}$ (respectively, S^*) the space of continuous semicharacters (respectively, continuous $*$-semicharacters) on S. That is a $\chi \in \hat{S}$ if $\chi : S \to \mathbb{C}$ is continuous, $|\chi(x)| \le 1$ and $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$, and $\chi \in X^*$ if $\chi \in \widehat{S}$ and $\chi(x^*) = \overline{\chi(x)}$ $(x \in X)$. A function $f \in C_b(S)$ is called *weakly almost periodic* if $R_S f = \{R_s f : s \in S\}$ is relatively weakly compact in $C_b(S)$, where for every $s \in S$, the function $R_s f$ is defined by $R_s f(x) = f(xs)$ $(x \in S)$. The space of all weakly almost periodic continuous functions on S will be denoted by $\mathrm{wap}\,(S)$.

Recall that on a Hausdorff locally compact topological semigroup S the space of all measures μ in $M(S)$ (the Banach algebra of all regular complex bounded measures on S with total variation norm) for which the mappings: $x \mapsto |\mu| * \delta_x$ and $x \mapsto \delta_x * |\mu|$ $(\delta_x$ denotes by $M_a(S)$ (see [1, 2, 6]). It is well known that $M_a(S)$ is a closed two sided L-ideal of $M(S)$. A Hausdorff locally compact topological semigroup S is called a *foundation semigroup* if S coincides with the closure of $\bigcup\{\mathrm{supp}\,(\mu) : \mu \in M_a(S)\}$. It is well known that if S is a foundation semigroup with an identity then for every $\mu \in M_a(S)$ both the mappings: $x \to \delta_x * \mu$ and $x :\to \mu * \delta_x$ from S into $M_a(S)$ are norm continuous (cf. [12]). We also recall that if S is a commutative foundation semigroup, then the Gelfand space $\widehat{M_a(S)}$ of $M_a(S)$ with the Gelfand topology is homeomorphic with $\widehat{S}$ when $\widehat{S}$ is endowed with the compact open topology. In particular $\widehat{S}$ with the compact open topology defines a locally compact Hausdorff space. Moreover, for every μ in $M_a(S)$ the Gelfand transform $\widehat{\mu}$ is given by the equation $\widehat{\mu}(\chi) = \int_S \chi(s)d\mu(s)$ $(\chi \in \widehat{S})$ (see [1]). We also note that if S is a foundation $*$-semigroup, then with the involution given by $\mu^*(f) = \int \overline{f(x^*)d\mu(x)}$ $\big(f \in C_0(S)\big)$, both $M(S)$ and $M_a(S)$ define Banach $*$-algebras. If S is a commutative foundation $*$-semigroup, then it is clear the $\widehat{\mu}\big|_{S^*}$ (the restriction of μ to S^*) belongs to $C_0(S^*)$ $\big(\mu \in M_a(S)\big)$.

1. The Banach Algebra $\mathcal{F}(S,T)$ of Commutative Foundation $*$-Semigroups S and T

We start with the following definition.

Definition 1.1. Let S, T be two Hausdorff locally compact topological $*$-semigroups. We denote by $\mathcal{F}(S,T)$ the set of functions $f : S \times T \to \mathbb{C}$ such that

$$f(s,t) = \langle \pi_1(s)\xi, \pi_2(t)\eta \rangle \quad \big((s,t) \in S \times T\big), \tag{4}$$

where π_1 (respectively, π_2) defines a continuous $*$-representation of S (respectively, T) by bounded operators on some Hilbert space H and some vectors $\xi, \eta \in H$.

In the following result, $F(S \times T)$ denotes the Fourier-Stieljes algebra of $S \times T$ defined by Lau in [10].

Lemma 1.2. (i) *For any two Hausdorff locally compact topological $*$-semigroups S and T, $\mathcal{F}(S,T)$ defines an algebra of bounded functions on $S \times T$.*

(ii) *If f is as in* (4) *and both π_1 and π_2 are strongly continuous representations, then $f \in$ wap $(S \times T)$.*

(iii) $F(S \times T) \subseteq \mathcal{F}(S,T)$.

Proof. The proof of (i) is clear. To prove (ii) we assume that there exist two strongly continuous $*$-representations π_1 of S and π_2 of T by bounded operators on a Hilbert space H such that for some vectors $\xi, \eta \in H$

$$f(s,t) = \langle \pi_1(s)\xi, \pi_2(t)\eta \rangle, \quad \big((s,t) \in S \times T).$$

We first show that $f \in C_b(S \times T)$. To this end, we suppose that $\big((s_\alpha, t_\alpha)\big)_{\alpha \in I}$ is a net in $S \times T$ converging to $(s,t) \in S \times T$.

Given $\varepsilon > 0$, by the strong continuity of π_1 and π_2 there exists $\alpha_0 \in I$ such that for all $\alpha \geq \alpha_0$

$$\|\pi_1(s_\alpha)\xi - \pi_1(s)\xi\| < \varepsilon \quad \text{and} \quad \|\pi_2(t_\alpha)\eta - \pi_2(t)\eta\| < \varepsilon.$$

Then for all $\alpha \geq \alpha_0$

$$\begin{aligned}
&|\langle \pi_1(s_\alpha)\xi, \pi_2(t_\alpha)\eta\rangle - \langle \pi_1(s)\xi, \pi_2(t)\eta\rangle| \\
&\leq |\langle \pi_1(s_\alpha)\xi, \pi_2(t_\alpha)\eta\rangle - \langle \pi_1(s)\xi, \pi_2(t_\alpha)\eta\rangle| \\
&\quad + |\langle \pi_1(s)\xi, \pi_2(t_\alpha)\eta\rangle - \langle \pi_1(s)\xi, \pi_2(t)\eta\rangle| \\
&\leq \|\pi_1(\sigma_\alpha)\xi - \pi_1(s)\xi\| \, \|\eta\| + \|\pi_2(t_\alpha)\eta - \pi_2(t)\eta\| \, \|\xi\| \\
&< \varepsilon(\|\eta\| + \|\xi\|).
\end{aligned}$$

Thus f is continuous on $S\times T$. It is also clear that $\|f\|_\infty \leq \|\xi\| \, \|\eta\|$. So $f \in C_b(S\times T)$. To prove that f belongs to wap $(S \times T)$ we define $U : S \times T \to B(H \otimes H)$ by

$$U_{(s,t)}(\xi' \otimes \eta') = \pi_1(s)\xi' \otimes \pi_2(t)\eta' \quad (\xi' \otimes \eta' \in H \otimes H),$$

where $H \otimes H$ denotes the Hilbert space tensor product of H by itself. Since π_1 and π_2 are weakly continuous and for every $\xi', \xi'', \eta', \eta'' \in H$, and $(s,t) \in S \times T$

$$\langle U_{(s,t)}(\xi' \otimes \eta'), \xi'' \otimes \eta''\rangle = \langle \pi_1(s)\xi', \xi''\rangle \, \langle \pi_2(t)\eta', \eta''\rangle,$$

we infer that I defines a weakly continuous $*$-representation of $S \times T$ by bounded operators on $H \otimes H$. So for every $x \in H \otimes H$ the set $U_{S\times T}(x) = \{U_{(s,t)}x : (s,t) \in S \times T\}$ is relatively weakly compact in $H \times H$. Define

$$V : H \otimes H \to C_b(S \times T) \quad \text{by} \quad \big(V(\xi' \otimes \eta')\big)(s,t) = \langle \pi_1(s)\xi', \pi_2(t)\eta'\rangle$$

$(\xi' \otimes \eta' \in H \otimes H, (s,t) \in S \times T)$. Then $V(\xi' \otimes \eta') \in C_b(S \times T)$ and $\|V(\xi' \otimes \eta')\|_\infty \leq \|\xi'\| \, \|\eta'\|$. Thus V defines a bounded linear operator. So by Theorem V.3.15, p. 422 of [5] V is continuous when $H \otimes H$ and $C_b(S \times T)$ have the weak topology. So V maps weakly compact sets onto weakly compact sets. Since $V\big(U_{S\times T}(x)\big) = R_{S\times T}(Vx)$, it follows that $R_{S\times T}(Vx)$ is relatively weakly compact, for every $x \in H \otimes H)$. Therefore $f = V(\xi \otimes \eta) \in$ wap $(S \times T)$.

(iii) We only need to choose $\pi_2(t) = I$ $(t \in T)$, where I is the identity operator on a Hilbert space and then applying Theorem 3.2 of [10]. ■

Lemma 1.3. *Let S be a foundation $*$ semigroup with an identity. Then every weakly continuous $*$-representation of S by bounded operators on a Hilbert space is strongly continuous.*

Proof. Let π be a continuous $*$-representation of S by bounded operators on a Hilbert space H. Then by Theorem 2.4 of [8] the equation

$$\langle \widetilde{\pi}(\mu)\xi, \eta\rangle = \int_S \langle \pi(x)\xi, \eta\rangle d\mu(x), \quad \big(\mu \in M_a(S), \xi, \eta \in H\big)$$

defines a $*$-representation of the Banach $*$-algebra $M_a(S)$ by bounded operators on H such that $\pi(x)T\mu = T_{\delta_x} * \mu$ $\big(x \in S, \mu \in M_a(S)\big)$. We claim that $\xi \in \overline{\{\widetilde{\pi}(\mu)\xi, \mu \in M_a(S)\}}$.

Suppose that this is not the case, then there exists $\eta \in H$ such that $\langle \xi, \eta \rangle \neq 0$, but $\langle \widetilde{\pi}(\mu)\xi, \eta \rangle = 0$ for all $\mu \in M_a(S)$. Thus $\int_S \langle \pi(x)\xi, \eta \rangle d\mu(x) = 0$ for all $\mu \in M_a(S)$. Since the mapping: $S \to \mathbb{C}$ given by: $x \mapsto \langle \pi(x)\xi, \eta \rangle$ is continuous on S, from Lemma 2.2 of [8] we conclude that $\langle \pi(x)\xi, \eta \rangle = 0$ for all $x \in S$. Since $\pi(e) = I$, where I denotes the identity operator on H and e denotes the identity of S, it follows that $\langle \xi, \eta \rangle = 0$. This contradiction proves our claim. To prove the strong continuity of π we suppose $\xi \in H$ and $\varepsilon > 0$ are given. Then there exists $\mu \in M_a(S)$ such that $\|\xi - \widetilde{\pi}(\mu)\xi\| < \varepsilon$. Let x_0 be fixed in S. Then by the norm continuity of the mapping: $x \to \delta_x * \mu$ from S into $M_a(S)$ one can find a neighbourhood V of x_0 such that $\|\delta_x * \mu - \delta_{x_0} * \mu\| < \varepsilon$ for all $x \in V$. Thus

$$\begin{aligned}
\|\pi(x)\xi - \pi(x_0)\xi\| &\leq \|\pi(x)\xi - \pi(x)\widetilde{\pi}(\mu)(\xi)\| \\
&\quad + \|\pi(x)\widetilde{\pi}(\mu)\xi - \pi(x_0)T_\mu\xi\| \\
&\leq \|\xi - \widetilde{\pi}(\mu)\xi\| + \|\widetilde{\pi}(\delta_x * \mu)(\xi) - \widetilde{\pi}(\delta_{x_0} * \mu)(\xi)\| \\
&\leq \varepsilon + \|\delta_x * \mu - \delta_{x_0} * \mu\| \; \|\xi\| \\
&< \varepsilon(1 + \|\xi\|).
\end{aligned}$$

That is π is strongly continuous. ■

A combination of Lemmas 1.2 and 1.3 yields the following result.

Theorem 1.4. *Let S and T be two foundation topological $*$-semigroups with identities. Then $\mathcal{F}(S,T) \subseteq$ wap $(S \times T)$.*

Before turning to the next lemma, we need to introduce the C^*-algebra $C^*(S)$ of a foundation $*$-semigroup S with an identity. To do this, we first recall that for any foundation $*$-semigroup S with an identity the Banach $*$-algebra $M_a(S)$ has a bounded approximate identity (cf. [12] [Proposition 5.16]). Since by Theorem 2.4 of [8] the equation

$$\langle \widetilde{\pi}(\mu)\xi, \eta \rangle = \int_S \langle \pi(s)\xi, \eta \rangle d\mu(s) \quad \big(\xi, \eta \in H, \mu \in M_a(S)\big)$$

defines a one-to-one correspondence between the continuous non-degenerate $*$-representations π of S by bounded operators on Hilbert spaces H and the $*$-representations of the Banach $*$-algebra $M_a(S)$, so if for every $\mu \in M_a(S)$ we let $\|\mu\|'$ to denote the supremum of all $\|\widetilde{\pi}(\mu)\|$ where π is a continuous non-degenerate $*$-representation of S by bounded operators on some Hilbert space, then we have

$$\|\mu * \mu^*\|' = \|\mu\|'^2 \quad \text{and} \quad \|\mu\|' \leq \|\mu\| \quad \text{for every} \quad \mu \in M_a(S).$$

Putting $I^0 = \{\mu \in M_a(S) : \|\mu\|' = 0\}$, then I^0 defines a closed ideal of $M_a(S)$. The completion of $M_a(S)/I^0$ with respect to $\| \cdot \|'$ defines a C^*-algebra which we denote it by $C^*(S)$. Indeed, $C^*(S)$ is the enveloping C^*-algebra of $M_a(S)$ (cf. 2.7.2 of [4]).

Remark 1.5. For the rest of this paper if S is any commutative foundation $*$-semigroup S with an identity then for every $\mu \in M_a(S)$ we shall denote $\widetilde{\mu}\big|_{S^*}$ again by $\widehat{\mu}$.

Lemma 1.6. *Let S be a commutative foundation $*$-semigroup with an identity. Then for every $\mu \in M_a(S)$, $\|\mu\|' = \|\widehat{\mu}\|_\infty = \sup\{|\widehat{\mu}(x)| : \chi \in S^*\}$. Furthermore, $C^*(S) \approx C_0(S^*)$.*

Proof. Let π be a non-degenerate continuous $*$-representation of S by operators on a Hilbert space H. Then A the closure of $\{\widetilde{\pi}(\mu) : \mu \in M_a(S)\}$ in $B(H)$ defines a commutative C^*-algebra. Let $\sigma(A)$ denote the maximal ideal space of A. For every $\tau \in \sigma(A)$ we define $\widetilde{\tau}$ on $M_a(S)$ by $\widetilde{\tau}(\mu) = \tau\big(\widetilde{\pi}(\mu)\big)$ $\big(\mu \in M_a(S)\big)$. So $|\widetilde{\tau}(\mu)| \leq \|\pi(\mu)\| \leq \|\mu\|$. Thus $\widetilde{\tau} \in \sigma\big(M_a(S)\big)$. Moreover, $\widetilde{\tau}(\mu^*) = \overline{\tau(\mu)}$ $\big(\mu \in M_a(S)\big)$. That is $\widetilde{\tau}$ is a $*$-homorphism on $M_a(S)$. Define χ_τ by $\chi_\tau(x) = \frac{\tau(\nu * \delta_x)}{\tau(\nu)}$ $(x \in S)$, where ν is some measure in $M_a(S)$ with $\tau(\nu) \neq 0$. Then

$$\chi_\tau(x^*) = \frac{\tau(\nu * \delta_{x^*})}{\tau(\nu)} = \frac{\overline{\tau(\nu^* * \delta_x)}}{\overline{\tau(\nu^*)}} = \overline{\chi_\tau(x)} \quad (x \in S).$$

That is $\chi_{\widetilde{\tau}} \in S^*$, by Theorem 2.5.3 of [6]. For every $\mu \in M_a(S)$

$$\begin{aligned}
\|\widetilde{\pi}(\mu)\| &= \sup\ \big\{\big|\tau\big(\widetilde{\pi}(\mu)\big)\big| : \tau \in \widehat{A}\big\} \\
&= \sup\ \big\{\big|\widetilde{\tau}(\mu)\big| : \mu \in M_a(S)\big\} \\
&= \sup\ \big\{\big|\int_S \chi_{\widetilde{\tau}}(x) d\mu(x)\big| : \mu \in M_a(S)\big\} \\
&\leq \sup\ \big\{\big|\widehat{\mu}(\chi)\big| : x \in S^*\big\} \\
&= \|\widehat{\mu}|_\infty.
\end{aligned}$$

■

Thus $\|\mu\|' \leq \|\widehat{\mu}\|_\infty$. Now the equality of $\|\mu\|' = \|\widehat{\mu}\|_\infty$ follows from Corollary 1.2.5 of [11]. That is $C^*(S) \approx C_0(\widehat{S})$.

Lemma 1.7. *Let S be a commutative foundation $*$-semigroup with an identity e. Suppose that $\mathcal{U}$ is a compact neighbourhood base of e. Then the following are valid:*

(i) *For every $U \in \mathcal{U}$ and every $\varepsilon \in (0,1)$ the set*

$$\widehat{U}_\varepsilon = \{\chi \in S^* : |\chi(x) - 1| < \varepsilon \quad \textit{for all} \quad x \in U\}$$

is open in S^*.

(ii) *For every $\varepsilon \in (0,1)$,*

$$S^* = \cup\{\widehat{U}_\varepsilon : U \in \mathcal{U}\}.$$

Proof. (i) Let $U \in \mathcal{U}$, $\varepsilon \in (0,1)$ and $\chi_0 \in \widehat{U}_\varepsilon$. Put $\delta = \inf\{\varepsilon - |\chi_0(x) - 1| : x \in U\}$. Since U is compact, it follows that $\delta > 0$. It is easy to see that the neighbourhood $\{\chi \in S^* : |\chi(x) - \chi_0(x)| < \varepsilon$ for all $x \in U\}$ of χ_0 is contained in $\widehat{U}_\varepsilon$.

(ii) Let $\chi_0 \in S^*$ and $\varepsilon \in (0,1)$, then from the continuity of χ_0 at e it follows that $\{x \in S : |\chi_0(x) - 1| < \varepsilon\}$ is a neighbourhood of e and therefore it contains some $U \in \mathcal{U}$. Thus $\chi_0 \in \widetilde{U}_\varepsilon$. This completes the proof of (ii). ∎

Definition 1.8. Let S and T be two commutative foundation $*$-semigroups with identities. For every $u \in BM(S^*, T^*)$ we define $\widehat{u} : S \times T \to \mathbb{C}$ by

$$\widehat{u}(s,t) = \langle \widetilde{s} \otimes \widetilde{t}, u\rangle \quad \big((s,t) \in S \times T\big), \tag{7}$$

where $\widetilde{s} : S^* \to \mathbb{C}$ and $\widetilde{t} : T \to \mathbb{C}$ are given by $\widetilde{s}(\chi) = \chi(s)$ $(\chi \in S^*)$ and $\widetilde{t}(\gamma) = \gamma(t)$ $(\gamma \in T^*)$. From (3) and (2) it follows that (7) makes sense and

$$\|\widehat{u}\|_\infty \le K_G \|u\|_{BM} \quad \big(u \in BM(S^*, T^*)\big).$$

We are now in a position to prove the first main result of the paper.

Theorem 1.9. *Let S and T be two commutative foundation $*$-semigroups with identities e_S and e_T, respectively. Then the following properties are valid:*

(i) *If $u \in BM(S^*, T^*)$, then $\widehat{u} \in \mathcal{F}(S,T)$.*

(ii) *If $f \in \mathcal{F}(S,T)$, then there exists a unique $u \in BM(S^*, T^*)$ such that $f = \widehat{u}$.*

(iii) *If $f \in \mathcal{F}(S,T)$ is represented as in* (4) *and $u \in BM(S^*, T^*)$ is such that $f = \widehat{u}$, then $\|u\|_{BM} \le \|\xi\| \, \|\eta\|$.*

Proof. (i) We may assume that $u \neq 0$. Let λ_1, λ_2 be the Grothendieck measure pair for u. For every $h \in L^2(S^*, \lambda_1)$ and $g \in L^2(T^*, \lambda_2)$ we have

$$|\langle h \otimes g, u\rangle| \le K_G \|h\|_2 \|g\|_2.$$

So there is an operator $\theta : L^2(S^*, \lambda_1) \to L^2(T^*, \lambda_2)$ such that for every $h \in L^2(S^*, \lambda_1)$ and $g \in L^2(T^*, \lambda_2)$

$$\langle h \otimes g, u\rangle = \langle \theta h, \overline{g}\rangle.$$

Define $\pi_1 : S \to B_1\big(L^2(S^*, \lambda_1)\big)$ by $\pi_1(s)h = \tilde{s}h\big(h \in L^2(S^*, \lambda_1)\big)$ and $\pi_2 : T \to B_1\big(L^2(T^*, \lambda_2)\big)$ by $\pi_2(t)g = \widetilde{t^*}g$ $\big(g \in L^2(T^*, \lambda_2)\big)$. By Proposition 4.4 of [2] π_1 and π_2 define continuous $*$-representations of S and T, respectively. Furthermore, for every $(s,t) \in S \times T$

$$\widehat{u}(s,t) = \langle \widetilde{s} \otimes \widetilde{t}, u\rangle = \langle \theta(\widetilde{s}), \widetilde{t^*}\rangle = \langle \theta\big(\pi(s)\mathbf{1}_S, \pi_2(t)\mathbf{1}_T\rangle,$$

where $\mathbf{1}_S$ and $\mathbf{1}_T$ denote the functions which are identically one on S and T, respectively. Let $H = L^2(S^*, \lambda_1) \oplus L^2(T^*, \lambda_2)$ and let $\widetilde{\theta}$ denote the extension of θ to H with the matrix $\begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix}$. Let $C = \|\widetilde{\theta}\|$ and W be a unitary dilation of $C^{-1}\widetilde{\theta}$ on the Hilbert space H_1 containing H (cf. p. 16 of [13]). Writing

$$H_1 = L^2(S^*, \lambda_1) \oplus L^2(T^*, \lambda_2) \oplus H^{\perp},$$

and putting $\pi_1' = \pi_1 \oplus I \oplus I$, $\pi_2' = W^*(I \oplus \pi_2 \oplus I)W$, $\xi = (C \cdot \mathbf{1}_S, 0, 0)$ and $\eta = W^*(0, \mathbf{1}_T, 0)$, then in H_1 we have

$$\begin{aligned} \widehat{u}(s,t) &= \langle CW\big(\pi_1(s)\mathbf{1}_S, 0, 0\big), \big(0, \pi_2(t)\mathbf{1}_T, 0\big)\rangle \\ &= \langle C\big(\pi_1(s)\mathbf{1}_S, 0, 0\big), W^*\big(0, \pi_2(t)\mathbf{1}_T, 0\big)\rangle \\ &= \langle \pi_1'(s)\xi, \pi_2'(t)\eta\rangle \quad \big((s,t) \in S \times T\big). \end{aligned}$$

That is; $\widehat{u} \in \mathcal{F}(S,T)$.

(ii) - (iii). Let $f \in \mathcal{F}(S,T)$. Then there exist two continuous $*$-representations π_1 of S and π_2 of T by bounded operators on some Hilbert space H such that for some vectors $\xi, \eta \in H$

$$f(s,t) = \langle \pi_1(s)\xi, \pi_2(t)\eta\rangle, \quad \big((s,t) \in S \times T\big).$$

For every $\mu \in M_a(S)$ and $\nu \in M_a(T)$ we define

$$\langle \widehat{\mu} \otimes \widehat{\nu}, u\rangle = \int_S \int_T \langle \pi_1(s)\xi, \pi_2(t)\eta\rangle d\mu(s) d\nu(t).$$

Thus by Lemma 1.6

$$\begin{aligned} |\langle \widehat{\mu} \otimes \widehat{\nu}, u\rangle| &= \Big| \int_S \int_T \langle \pi_1(s)\xi, \pi_2(t)\eta\rangle d\mu(s) d\nu(t)\Big| \\ &= \Big|\langle \int_S \pi_1(s)\xi d\mu(s), \int_T \pi_2(t)\eta d\nu(t)\rangle\Big| \\ &= |\langle \widetilde{\pi}(\mu)\xi, \widetilde{\pi}_2(\nu)\eta\rangle| \\ &\le \|\widetilde{\pi}(\mu)\| \, \|\widetilde{\pi}_2(\nu)\| \, \|\xi\| \, \|\eta\| \\ &\le \|\mu\|' \, \|\nu\|' \, \|\xi\| \, \|\eta\| \\ &= \|\widehat{\mu}\|_\infty \|\widehat{\nu}\|_\infty \|\xi\| \, \|\eta\|. \end{aligned}$$

Using the fact that $\{\widehat{\mu} : \mu \in M_a(S)\}$ is dense in $C_0(S^*)$ and $\{\widehat{\nu} : \nu \in M_a(T)\}$ is dense in $C_0(T^*)$, one can extend u (uniquely) to a bimeasure on $S^* \times T^*$ which again we denote it by u such that

$$\|u\|_{BM} \le \|\xi\| \, \|\eta\|.$$

This yielding (iii) once (ii) is proven.

Now we extend u to $L^\infty(S^*,\lambda_1)\widehat{\otimes}L^\infty(T^*,\lambda_2)$ so that it satisfies the inequality (3). We prove that $f=\widehat{u}$. To see this we choose fixed compact neighbourhood bases $\mathcal{U}$ and $\mathcal{V}$ of e_S and e_T, respectively. The collection $\mathcal{U}\times\mathcal{V}=\{U\times V : U\in\mathcal{U}, V\in\mathcal{V}\}$ with the order inclusion form a directed set (i.e. for $U_1\times V_1$ and $U_2\times V_2$ in $\mathcal{U}\times\mathcal{V}$, $U_1\times V_1\leq U_2\times V_2$ if $U_! \supseteq U_2$ and $V_1\supseteq V_2$). For every $U\times V\in\mathcal{U}\times\mathcal{V}$ we choose positive measures $\mu_{U\times V}$ in $M_a(S)$ and $\nu_{U\times V}$ in $M_a(T)$ such that

$$\mu_{U\times V}(S\backslash U)=0,\quad \nu_{U\times V}(T\backslash V)=0 \quad\text{and}\quad \|\mu_{U\times V}\|=1=\|\nu_{U\times V}\|.$$

For every fixed $(s_0,t_0)\in S\times T$ and every two compact subsets F of S^* and K of T^* we prove that $\widehat{\mu_{U\times V}*\delta_{s_0}}\otimes\widehat{\nu_{U\times V}*\delta_{t_0}}\to\widetilde{s_0}\times\widetilde{t_0}$ uniformly on $F\times K$. To prove this we suppose that $0<\varepsilon<1$ is given. Then by Lemma 1.7 we can find $U^0\times V^0$ in $\mathcal{U}\times\mathcal{V}$ such that $F\subseteq\widehat{U}^0_{(\varepsilon)}$ and $K\subseteq\widehat{V}^0_{(\varepsilon)}$. Now for all $(U\times V)\geq U^0\times V^0$ and every $\chi\in F$ and $\gamma\in K$ we have

$$\begin{aligned}
&|\widehat{\mu_{U\times V}*\delta_{s_0}}(\chi)\widehat{\nu_{U\times V}*\delta_{t_0}}(\gamma)-\widetilde{s}_0(\chi)\widetilde{t}_0(\gamma)|\\
&=\Big|\int_S\chi(s)d\mu_{U\times V}*\delta_{s_0}(s)\int_T\gamma(t)d\nu_{U\times V}*\delta_{t_0}(t)-\chi(s_0)\gamma(t_0)\Big|\\
&=\Big|\int_S\int_T[\chi(ss_0)\gamma(tt_0)-\chi(s_0)\gamma(t_0)]d\mu_{U\times V}(s)d\nu_{U\times V}(t)\Big|\\
&\leq\int_U\int_V|\chi(s)\gamma(t)-1|d\mu_{U\times V}(s)d\nu_{U\times V}(t)\\
&<\int_U\int_V 2\varepsilon d\mu_{U\times V}(s)d\nu_{U\times V}(t)\\
&=2\varepsilon.
\end{aligned}$$

That is $\widehat{\mu_{U\times V}*\delta_{s_0}}\otimes\widehat{\nu_{U\times V}*\delta_{t_0}}$ converges uniformly on $F\times K$ to $\widetilde{s_0}\otimes\widetilde{t_0}$. Let $\xi',\nu'\in H$ and $g(s,t)=\langle\pi_1(s)\xi',\pi_2(t)\eta'\rangle$ for every $(s,t)\in S\times T$. By Theorem 1.4, g is continuous at (e_S,e_T). So for every $\varepsilon>0$ there exists $U_1\times V_1\in\mathcal{U}\times\mathcal{V}$ such that

$$|g(s,t)-g(e_S,e_T)|<\varepsilon\quad\big((s,t)\in\mathcal{U}_1\times\mathcal{V}_1\big). \tag{5}$$

For all $U\times V\geq\mathcal{U}_1\times\mathcal{V}_1$ by (5) we have

$$\begin{aligned}
&\Big|\int_S\int_T\langle\pi_1(s)\xi',\pi_2(t)\eta'\rangle d\mu_{U\times V}(s)d\nu_{U\times V}(t)-\langle\xi',\eta'\rangle\Big|\\
&=\Big|\int_U\int_V\big(\pi_1(s)\xi',\pi_2(t)\eta'\rangle-\langle\xi',\eta'\rangle\big)d\mu_{U\times V}(s)d\nu_{U\times V}(t)\Big|\\
&\leq\int_U\int_V|g(s,t)-g(e_S,e_T)|d\mu_{U\times V}(s)d\nu_{U\times V}(t)\\
&<\varepsilon.
\end{aligned}$$

Thus for every $\xi', \eta' \in H$

$$\lim_{U\times V} \int_S \int_T \langle \pi_1(s)\xi', \pi_2(t)\eta'\rangle d\mu_{U\times V}(s) d\nu_{U\times V}(t) = \langle \xi', \eta'\rangle. \tag{6}$$

Suppose $\varepsilon > 0$ is given, then as in the proof of Lemma 1.4 of [7] we can find a bimeasure w in $BM(S^*, T^*)$ and two compact subsets $F_0 \subseteq S^*$ and $K_0 \subseteq T^*$ such that

$$\operatorname{supp}(w) \subseteq F_0 \times K_0 \quad \text{and} \quad \|u - w\|_B < \varepsilon. \tag{7}$$

Since $\widehat{\mu_{U\times V} * \delta_{s_0}} \otimes \widehat{\nu_{U\times V} * \delta_{t_0}}$ converging uniformly on $F_0 \times K_0$ to $\widetilde{s}_0 \times \widetilde{t}_0$, there exists $U_1 \times V_1 \in \mathcal{U} \times \mathcal{V}$ such that for all $U \times V \geq U_1 \times V_1$,

$$\|\widehat{\mu_{U\times V} * \delta_{s_0}} \otimes \widehat{\nu_{U\times V} * \delta_{t_0}} - \widetilde{s}_0 \otimes \widetilde{t}_0\|_{F_0\times K_0} < \varepsilon, \tag{8}$$

where $\| \quad \|_{F_0\times K_0}$ denotes the sup-norm on $F_0 \times K_0$.

Thus for all $U \times V \geq V_0 \times V_0$

$$\begin{aligned}
&|\langle u, \widetilde{s}_0 \otimes \widetilde{t}_0\rangle - \langle u, \widehat{\mu_{U\times V} * \delta_{s_0}} \otimes \widehat{\nu_{U\times V} * \delta_{t_0}}\rangle| \\
&\quad \leq |\langle (u - w), (\widetilde{s}_0 \times \widetilde{t}_0 - \widehat{\mu_{U\times V} * \delta_{s_0}} \otimes \widehat{\nu_{U\times V} * \delta_{t_0}})\rangle| \\
&\qquad + |\langle w, (\widetilde{s}_0 \times \widetilde{t}_0 - \widehat{\mu_{U\times V} * \delta_{s_0}} \otimes \widehat{\nu_{U\times V} * \delta_{t_0}})\rangle| \\
&\quad < \|u - w\|_{BM} \|\widetilde{s}_0 \times \widetilde{t}_0 - \widehat{\mu_{U\times V} * \delta_{s_0}} \otimes \widehat{\nu_{U\times V} * \delta_{t_0}}\|_\infty \\
&\qquad + \|w\|_{BM} \|\widetilde{s}_0 \times \widetilde{t}_0 - \widehat{\mu_{U\times V} * \delta_{s_0}} \otimes \widehat{\nu_{U\times V} * \delta_{t_0}}\|_{F_0\times K_0} \\
&\quad < 2\|u - w\|_{BM} + \varepsilon \|w\|_{BM} \\
&\quad < \varepsilon(2 + \|w\|_{BM}),
\end{aligned}$$

by (7) and (8). That is

$$\lim_{U\times V} \langle u, \widehat{\mu_{U\times V} * \delta_{s_0}} \otimes \widehat{\nu_{U\times V} * \delta_{t_0}}\rangle = \langle u, \widetilde{s}_0 \otimes \widetilde{t}_0\rangle. \tag{9}$$

On the other hand an application of (6) shows that

$$\begin{aligned}
&\lim_{U\times V} \langle u, \widehat{\mu_{U\times V} * \delta_{s_0}} \otimes \widehat{\nu_{U\times V} * \delta_{t_0}}\rangle \\
&\quad = \lim_{U\times V} \int_S \int_T \langle \pi_1(s)\xi, \pi_2(t)\eta\rangle d\mu_{U\times V} * \delta_{s_0}(s) d\nu_{U\times V} * \delta_{t_0}(t) \\
&\quad = \lim_{U\times V} \int_S \int_T \langle \pi_1(ss_0)\xi, \pi_2(tt_0)\eta\rangle d\mu_{U\times V}(s) d\nu_{U\times V}(t) \\
&\quad = \lim_{U\times V} \int_S \int_T \langle \pi_1(s)\big(\pi(s_0)\xi\big), \pi_2(t)\big(\pi_2(t_0)\eta\big)\rangle d\mu_{U\times V}(s) d\nu_{U\times V}(t) \\
&\quad = \langle \pi_1(s_0)\xi, \pi_2(t_0)\eta\rangle.
\end{aligned} \tag{10}$$

From (9) and (10) it follows that

$$\langle u, \widetilde{s}_0 \otimes \widetilde{t}_0 \rangle = f(s_0, t_0).$$

Since (s_0, t_0) was an arbitrary element of $S \times T$, we conclude that

$$\widehat{u}(s,t) = f(s,t) \quad \big((s,t) \in S \times T\big).$$

To prove the uniqueness part of theorem, we suppose that $\widehat{u}_1 = \widehat{u}_2$ for some $u_1, u_2 \in BM(S^*, T^*)$. Let $\lambda_{11}, \lambda_{12}$ be the Grothendieck measure pair for u_i $(i = 1, 2)$. Put $\lambda_i = \frac{1}{2}(\lambda_{1i} + \lambda_{2i})$ $(i = 1, 2)$. It is easy to see that

$$L^2(S^*, \lambda_1) = L^2(S^*, \lambda_{11}) \cap L^2(S^*, \lambda_{21}),$$

and

$$L^2(T^*, \lambda_2) = L^2(T^*, \lambda_{12}) \cap L^2(T^*, \lambda_{22}).$$

Let $\langle \widetilde{s}, s \in S \rangle$ denote the subalgebra of $C_b(S^*)$ generated by the set $\{\widetilde{s} : s \in S\}$. Then by Corollary A.4, p. 175 of [6], $\langle \widetilde{s} = s \in S \rangle$ is dense in $L^2(S^*, \lambda_1)$. Similarly, $\langle \widetilde{t} : t \in T \rangle$ is dense in $L^2(T^*, \lambda_2)$. So $u_1 = u_2$ on $L^2(S^*, \lambda_1) \widehat{\otimes} L^2(T^*, \lambda_2)$. In particular, $u_1 = u_2$ on $C_0(S^*) \widehat{\otimes} C_0(T^*)$. This completes the proof. ■

We shall define in the next theorem the measure algebra structure $BM(S^*, T^*)$ extending the measure algebra structure of $M(S^* \times T^*)$ when S and T are commutative.

Theorem 1.10. *Let S and T be two commutative foundation $*$-semigroups with identities. For every u, $v \in BM(S^*, T^*)$ let $u * v \in BM(S^*, T^*)$ be defined by $(u * v)\widehat{\ } = \widehat{u}\,\widehat{v}$. Then $\big(BM(S^*, T^*), *\big)$ defines a commutative convolution Banach algebra with*

$$\|u * v\|_{BM} \leq K_G^2 \|u\|_{BM} \|v\|_{BM}.$$

Moreover, $M(S^* \times T^*)$ is a subalgebra of $BM(S^*, T^*)$.

Proof. For $u \in BM(S^*, T^*)$, let λ_1, λ_2 be Grothendieck pair measures for u. Let $\theta, \pi_1', \pi_2', \xi, w$ and C be as in the proof of Theorem 1.8. Using (1) and (2) we conclude that for every $h \in C_0(S^*)$ and $g \in C_0(T^*)$

$$|\langle \theta h, g \rangle| = |\langle h \otimes g, u \rangle| \leq K_G \|u\|_{BM} \|h\|_2 \|g\|_2.$$

Thus $\|\theta\| \leq K_G \|u\|_{BM}$. Therefore $\|\xi\| = |C| = \|\theta\| \leq K_G \|u\|_{BM}$. Since

$$\|\eta\| = \|W^*\| \leq 1 \quad \text{and} \quad u(\widetilde{s} \otimes \widetilde{t}) = \langle \pi_1'(s)\xi, \pi_2'(t)\eta \rangle \big((s,t) \in S \times T\big),$$

we infer that

$$\|u\|_{BM} \leq \|\xi\| \; \|\eta\| \leq K_G \|u\|_{BM}. \tag{11}$$

Let $v \in BM(S^*, T^*)$. So there exist a continuous $*$-representation π_1'' of S (π_2'' of T, respectively) by operators on a Hilbert space H_1 (H_2, respectively) such that for some vector $\xi'' \in H_1 (\eta'' \in H_2$, respectively)

$$(\widetilde{s} \otimes \widetilde{t}, v) = \langle \pi_1''(s)\xi'', \pi_2''(t)\eta'' \rangle \quad \big((s,t) \in S \times T\big).$$

Applying (11) to v we obtain

$$\|v\|_{BM} \leq \|\xi''\| \; \|\eta''\| \leq K_G \|v\|_{BM}. \tag{12}$$

So for every $(s,t) \in S \times T$

$$\begin{aligned} \langle \widetilde{s} \otimes \widetilde{t}, u * v \rangle &= \widehat{u}(s,t)\widehat{v}(s,t) \\ &= \langle \pi_1'(s)\xi, \pi_2'(t)\eta \rangle \; \langle \pi_1''(s)\xi'', \pi''(t)\eta'' \rangle \\ &= \langle (\pi_1' \otimes \pi_1'')(s)(\xi \otimes \xi''), (\pi_2' \otimes \pi_2'')(t)(\eta \otimes \eta'') \rangle, \end{aligned} \tag{13}$$

with $\xi \otimes \xi''$, $\eta \otimes \eta'' \in H_1 \otimes H_2$. Using (11), (12) and (13), we obtain

$$\begin{aligned} \|u * v\|_{BM} &\leq \|\xi \otimes \xi''\| \; \|\eta \otimes \eta''\| \\ &= (\|\xi\| \; \|\eta\|)(\|\xi'' \otimes \eta''\|) \\ &\leq K_G^2 \|u\|_{BM} \|v\|_{BM}. \end{aligned}$$

To show that the algebra structure of $BM(S^*, T^*)$ extends the algebra structure of $M(S^* \times T^*)$, we first note that since S, T are foundation $*$-semigroups with identity, then both S^* and T^* define locally compact topological $*$-semigroups under the compact open topology and the pointwise multiplcation.

Identifying $S^* \times T^*$ with $(S \times T)^*$ and noting that $S \times T$ defines a foundation $*$-semigroup with identity, whenever endowed with the compact open topology (cf. [1]), we conclude that $M(S^* \times T^*) \approx M\big((S \times T)^*\big)$. Using our version of Bochner's Theorem in [9] with the aid of Lemma 1.6 and Proposition 3.4 of [10] we conclude that $F(S \times T)$ is isometric isomorphic with $M(S^* \times T^*)$. Since $M(S^* \times T^*)$ is a subalgebra of $BM(S^*, T^*)$, we infer that $F(S \times T)$ is indeed a subalgebra of $BM(S^*, T^*)$. This completes the proof of the theorem. ■

2. Amenability of $\mathcal{F}(S,T)$ of Two Commutative Foundation $*$-Semigroup S and T

The aim of the present section is to prove that for any two commutative foundation $*$-semigroups S and T with identities, the algebra $\mathcal{F}(S,T)$ as a subalgebra of $\mathcal{B}(S \times T)$ (the space of all bounded complex-valued functions on $S \times T$) is amenable if and only if $K\big(\sigma(\overline{\mathcal{F}(S,T)})\big)$ is a compact topological group, where $K\big(\sigma(\overline{\mathcal{F}(S,T)})\big)$ denotes the minimal ideal of $\sigma(\overline{\mathcal{F}(S,T)})\big)$. We first need to recall some notation form [3].

Definition 2.1. Let S be a semigroup and $\mathcal{F}$ be a linear subspace of $\mathcal{B}(S)$, the space of all bounded complex-valued factions on S. A mean of $\mathcal{F}$ is a linear function μ on $\mathcal{F}$ with the property that

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s) \quad (f \in \mathcal{F}_r),$$

where $\mathcal{F}_r$ denotes the set of all real-valued functions in $\mathcal{F}$. The set all means of $\mathcal{F}$ is denoted by $\mathcal{M}(\mathcal{F})$. If $\mathcal{F}$ is an algebra then $\mu \in \mathcal{M}(\mathcal{F})$ is called *multiplicative* if

$$\mu(fg) = \mu(f)\mu(g) \quad (f, g \in \mathcal{F}).$$

The set of all multiplicative means on $\mathcal{F}$ is called the *spectrum* of $\mathcal{F}$ and will be denoted by $\sigma(\mathcal{F})$.

Definition 2.2. A subset $\mathcal{F} \subseteq \mathcal{B}(S)$ is called *right* (respectively, left) *translation invariant* if $r_s f \in \mathcal{F}$ (respectively, $\ell_s f \in \mathcal{F}$) for every $s \in S$ and $f \in \mathcal{F}$. If $\mathcal{F}$ is both left translation invariant and right translation invariant, then it is called *translation invariant.*

Definition 2.3. For a translation invariant linear subspace $\mathcal{F}$ of $\mathcal{B}(S)$ and $\mu \in \mathcal{F}^*$, the *left introversion operator determined by* μ is the mapping $T_\mu : \mathcal{F} \to \mathcal{B}(S)$ defined by

$$(T_\mu f)(s) = \mu(\ell_s f) \quad (f \in \mathcal{F}, s \in S).$$

The *right introversion operator determined by* μ is the mapping $U_\mu : \mathcal{F} \to \mathcal{B}(S)$ defined by

$$(U_\mu f)(s) = \mu(r_s f) \quad (f \in \mathcal{F}, s \in S).$$

Definition 2.4. Let $\mathcal{F}$ be a conjugate closed, translation invariant, linear subspace (respectively, subalgebra) of $\mathcal{B}(S)$ containing the constant functions. $\mathcal{F}$ is said to be *left introverted* (respectively, *left m-introverted*) if $T_\mu\mathcal{F} \subseteq \mathcal{F}$ for $\mu \in M(\mathcal{F})$ [respectively, $\mu \in \sigma(\mathcal{F})$]. *Right introversion* and *right m-introversion* are defined similarly. $\mathcal{F}$ is said to be *introverted* (respectively, *m-introverted*) if is both left and right introverted (respectively, left and right m-introverted).

Definition 2.5. An *admissible subspace* of $\mathcal{B}(S)$ is a norm closed, conjugate closed, translation invariant, left introverted subspace of $\mathcal{B}(S)$ containing the constant functions. An m-admissible subalgebra of $\mathcal{B}(S)$ is a translation invariant, left m-introverted C^*-subalgebra of $\mathcal{B}(S)$ containing the constant functions.

Definition 2.6. Let $\mathcal{F}$ be a left (respectively, right) translation invariant, conjugate closed, linear subspace of $\mathcal{B}(S)$ containing the constant functions. A member μ of $\mathcal{F}^*$ is said to be *left* (respectively, *right*) *invariant* if, $\mu(\ell_s f) = \mu(f)$ [respectively, $\mu(r_s f) = \mu(f)$] for all $s \in S$ and all $f \in \mathcal{F}$. $\mathcal{F}$ is said to be *left* (respectively, *right*) *amenable*, if it has a left (respectively, right) invariant mean. If $\mathcal{F}$ is translation invariant, then $\mathcal{F}$ is called *amenable* if it is both left and right amenable.

The following is the first result of this section.

Lemma 2.1. *Let S and T be commutative foundation $*$-semigroups with identities. Then $\mathcal{F}(S,T)$ is a conjugation closed and translation invariant subalgebra of* $\text{wap}\,(S\times T)$ *which also contains the constant functions. Furthermore, $\|\ell_{\,(s,t)}f\|_{BM} \leq \|f\|_{BM}$ for every $(s,t)\in S\times T$, and $\|f\|_\infty \leq \|f\|_{BM}$ for all $f\in\mathcal{F}(S,T)$.*

Proof. By Lemma 1.2, $\mathcal{F}(S,T)$ is a subalgebra of $\text{wap}\,(S\times T)$. Let $f\in\mathcal{F}(S,T)$ and u_f be the unique element in $BM(S^*,T^*)$ such that $f=\widehat{u}_f$. That is $f(s,t)=\langle\widetilde{t}\otimes\widetilde{s},u_f\rangle\ \big((s,t)\in S\times T\big)$. Let $\widetilde{u}_f$ denote the bimeasure defined on $S^*\times T^*$ by $\widetilde{u}_f(g,h)=u_f(h,g)\ \big(g\in C_0(S^*),h\in C_0(T^*)\big)$. Then there exist two continuous $*$-representations π_1 on S and π_2 of T by bounded operators on a Hilbert space H with some two vectors $\xi,\eta\in H$ such that $\langle\widetilde{t}\otimes\widetilde{s},\widetilde{u}_g\rangle=\langle\pi_1(t)\xi,\pi_2(s)\eta\rangle\big((s,t)\in S\times T\big)$. Thus,

$$\begin{aligned} f(s,t)&=\langle\widetilde{s}\otimes\widetilde{t},u_f\rangle=\langle\widetilde{t}\otimes\widetilde{s},\widetilde{u}_f\rangle\\ &=\langle\pi_1(t)\xi,\pi_2(s)\eta\rangle\big((s,t)\in S\times T\big).\end{aligned}$$

So $\overline{f(s,t)}=\langle\pi_2(s)\eta,\pi_1(t)\xi\rangle\big((s,t)\in S\times T\big)$. Thus $\overline{f}\in\mathcal{F}(S,T)$. It is also clear that $\mathcal{F}(S,T)$ is translation invariant. Since $F(S\times T)\subseteq\mathcal{F}(S,T)$, from Theorem 3.2 of [10] it follows that $\mathcal{F}(S,T)$ contains the constant functions. It is also clear that $\mathcal{F}(S,T)$ is translation invariant. Let $(s_0,t_0)\in S\times T$ by fixed. Since $\|(\widetilde{s}_0g)\otimes\widetilde{t}_0h\|_\infty\leq\|g\|_\infty\|h\|_\infty$ for every $g\in C_0(S^*)$ and $h\in C_0(T^*)$, it follows that $\|(\widetilde{s}_0\otimes\widetilde{t}_0)k\|_{V_0}\leq\|k\|_{V_0}$ for every $k\in V_0(S,T)$. Thus $\|\ell_{(s_0,t_0)}f\|_{BM}\leq\|f\|_{BM}$ for every $f\in\mathcal{F}(S,T)$. To see that $\|f\|_\infty\leq\|f\|_{BM}$ for every $f\in\mathcal{F}(S,T)$, we first note that if u is any bimeasure on $S^*\times T^*$ with compact support, then for every $(s,t)\in S\times T$ the function $\widetilde{s}\otimes\widetilde{t}$ agrees on support of u with a function $g\otimes h$ in $V_0(S^*,T^*)$ with $\|g\otimes h\|_\infty\leq 1$. Thus $|\widehat{u}(\widetilde{s}\otimes\widetilde{t})|\leq\|u\|_{BM}$. So $\|\widehat{u}\|_\infty\leq\|u\|_{BM}$. Let w be any bimeasure in $BM(S^*,T^*)$. Given $\varepsilon>0$, by Lemma 1.4 of [7] there exists a bimeasure w in $BM(S^*,T^*)$ with compact support such that $\|w-u\|_{BM}<\varepsilon$. Hence $\|u\|_{BM}\leq\|u-w\|_{BM}+\|w\|_B$. So

$$\begin{aligned}\|\widehat{w}\|_\infty&\leq\|\widehat{w-u}\|_\infty+\|\widehat{u}\|_\infty\leq K_G\|w-u\|_{BM}+\|u\|_{BM}\\ &<(K_G+1)\|w-u\|_{BM}+\|w\|_B\\ &<(K_G+1)\varepsilon+\|w\|_{BM}.\end{aligned}$$

Letting $\varepsilon\to 0$, we conclude that $\|\widehat{w}\|_\infty\leq\|w\|_{BM}$. That is $\|f\|_\infty\leq\|f\|_{BM}$ for every $f\in\mathcal{F}(S,T)$. ∎

We close this paper with the following theorem which characterizes the amenability of $\mathcal{F}(S,T)$ as a subalgebra of $\mathcal{B}(S\times T)$ of two commutative foundation $*$-semigroups S and T with identities.

Theorem 2.8. *Let S and T be two commutative foundation $*$-semigroups with identities. Let $\overline{\mathcal{F}(S,T)}$ denotes the sup-norm closure of $\mathcal{F}(S,T)$ in* wap $(S \times T)$. *Then $\overline{\mathcal{F}(S,T)}$ is an m-admissible subalgebra of* wap $(S \times T)$. *Moreover, $\overline{\mathcal{F}(S,T)}$ is amenable and $K\big(\sigma(\overline{\mathcal{F}(S,T)})\big)$ is a topological group, where $K\big(\sigma(\overline{\mathcal{F}(S,T)})\big)$ denotes the minimal ideal of $\sigma(\overline{\mathcal{F}(S,T)})$.*

Proof. By Lemma 2.7, $\mathcal{F}(S,T)$ is a conjugation closed and translation invariant subalgebra of wap $(S \times T)$. Since $BM(S^*, T^*)$ is commutative, so is amenable. Hence $\mathcal{F}(S,T)$ is amenable. Let m be an invariant mean on $\mathcal{F}(S,T)$. By the Hahn-Banach theorem we can extend m to a mean $\widetilde{m}$ on $\overline{\mathcal{F}(S,T)}$. It is clear that $\widetilde{m}$ defines an invariant mean on $\overline{\mathcal{F}(S,T)}$. So $\overline{\mathcal{F}(S,T)}$ is amenable. Clearly, $\overline{\mathcal{F}(S,T)}$ is a norm closed, conjugate closed and translation invariant subalgebra of wap $(S \times T)$ which also contains the constant functions. By Corollary 4.2.7 of [3] it is introverted and hence is an m-admissible subalgebra of wap $(S \times T)$. So by Theorem 4.2.12 of [3] $K\big(\sigma(\overline{\mathcal{F}(S,T)})\big)$ is a compact topological group. ■

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