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ON GENERALIZED DERIVATIONS OF PRIME AND SEMIPRIME RINGS

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Abstract. Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative. We also examine the case where R is a semiprime ring.

1. Introduction

In all that follows, unless stated otherwise, R will be an associative ring, $Z(R)$ the center of R, Q its Martindale quotient ring. The center of Q, denoted by C, is called the extended centroid of R. For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for the commutator $xy-yx$ and anti-commutator $xy+yx$, respectively. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular d is an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$.

In [6], Bresar introduced the definition of generalized derivation: an additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y+xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Hence, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier (i.e., an additive mapping satisfying $f(xy) = f(x)y$ for all $x, y \in R$). Basic examples are derivations and generalized inner derivations (i.e., mappings of type $x \longrightarrow ax + xb$ for some $a, b \in R$). We refer to call such mappings generalized inner derivations for the reason they present a generalization of the concept of inner derivations (i.e., mappings of the form $x \longrightarrow ax - xa$ for some $a \in R$).

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In [13], Hvala studied generalized derivations in the context of algebras on certain norm spaces. The related object we need to mention is the right Utumi quotient ring U of ring R (sometimes, as in [5], U is called the maximal right ring of quotient). In [16], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \longrightarrow U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense left ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole of U. Lee obtained the following: every generalized derivation F on a dense left ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. This result will be used in the sequel to prove our theorems. More related results about derivations and generalized derivations can be found in [3, 4, 11] and [12].

In [1, Theorem 4.1], Ashraf and Rehman proved that if R is a prime ring, I a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In [2, Theorem 1], Argaç and Inceboz generalized the above result as following: Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer, if R admits a derivation d with the property $(d(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative. In [21, Theorem 2.3], Quadri et al., discussed the commutativity of prime rings with generalized derivations. More precisely, Quadri et al., proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative.

The present paper is then motivated by [2] and [21]. Explicitly we shall prove the following:

Theorem A. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative.*

Theorem B. *Let R be a semiprime ring and n a fixed positive integer. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in R$, then R is commutative.*

We are now in a position to prove our main results.

2. The Case: R a Prime Ring

Theorem 2.1. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative.*

Proof. Since R is a prime ring and F is a generalized derivation of R, by Lee [16], $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. By the given hypothesis we have now $x \circ y = (a(x \circ y) + d(x \circ y))^n = (a(x \circ y) + d(x)y + xd(y) + d(y)x + yd(x))^n$ for all $x, y \in I$. By our hypothesis $d \neq 0$. By Kharchenko [15], we divide the proof into two cases:

Case 1. Let d be an outer derivation of U, then I satisfies the polynomial identity $(a(x \circ y) + sy + xt + tx + ys)^n = x \circ y$ for all $x, y, s, t \in I$. In particular, for $y = 0$, I satisfies the blended component $(xt + tx)^n = 0$ for all $x, t \in I$. If $CharR \neq 2$, then $(2x^2)^n = 0$ for all $x \in I$. This is a contradiction by Xu [22]. If $CharR = 2$, then $(xt + tx)^n = 0 = [x, t]^n$ and by Herstein [14], we have $I \subseteq Z(R)$, and so R is commutative by Mayne [19].

Case 2. Let now d be the inner derivation induced by an element $q \in Q$, that is $d(x) = [q, x]$ for all $x, y \in U$. It follows that $(a(x \circ y) + [q, x]y + x[q, y] + [q, y]x + y[q, x])^n = x \circ y$ for all $x, y \in I$. By a theorem due to Chuang [8], I and Q satisfy the same generalized polynomial identities (GPIs), we have $(a(x \circ y) + [q, x]y + x[q, y] + [q, y]x + y[q, x])^n = x \circ y$ for all $x, y \in Q$. In case center C of Q is infinite, we have $(a(x \circ y) + [q, x]y + x[q, y] + [q, y]x + y[q, x])^n = x \circ y$ for all $x, y \in Q \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \bigotimes_C \overline{C}$ are prime and centrally closed [10], we may replace R by Q or $Q \bigotimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e. $RC = R$) which is either finite or algebraically closed and $(a(x \circ y) + [q, x]y + x[q, y] + [q, y]x + y[q, x])^n = x \circ y$ for all $x, y \in R$. By Martindale [20], RC (and so R) is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D.

Assume that $dimV_D \geq 3$.

First of all, we want to show that v and qv are linearly D-dependent for all $v \in V$. Since if $qv = 0$ then $\{v, qv\}$ is D-dependent, suppose that $qv \neq 0$. If v and qv are D-independent, since $dimV_D \geq 3$, then there exists $w \in V$ such that v, qv, w are also linearly independent. By the density of R, there exists $x, y \in R$ such that: $xv = 0, xqv = w, xw = v; yv = 0, yqv = 0, yw = v$. These imply that $(-1)^n v = (a(x \circ y) + [q, x]y + x[q, y] + [q, y]x + y[q, x])^n v = (x \circ y)v = xyv + yxv = 0$, a contradiction. So we conclude that v and qv are linearly D-dependent for all $v \in V$.

Our next goal is to show that there exists $b \in D$ such that $qv = vb$ for all $v \in V$. Note that the arguments in [7] are still valid in the present situation. For the sake of completeness and clearness we prefer to present it. In fact, choose $v, w \in V$ linearly independent. Since $dimV_D \geq 3$, then there exists $u \in V$ such that $\{u, v, w\}$ is linearly independent. Then $b_u, b_v, b_w \in D$ such that $qu = ub_u$, $qv = vb_v$, $qw = wb_w$, that is $q(u + v + w) = ub_u + vb_v + wb_w$. Moreover $q(u + v + w) = (u + v + w)b_{u+v+w}$ for a suitable $b_{u+v+w} \in D$. Then $0 =$

$u(b_{u+v+w} - b_u) + v(b_{u+v+w} - b_v) + w(b_{u+v+w} - b_w)$ and because u, v, w are linearly independent, $b_u = b_v = b_w = b_{u+v+w}$, that is b does not depend on the choice of v. Hence now we have $qv = vb$ for all $v \in V$.

Now for $r \in R$, $v \in V$, we have $(rq)v = r(qv) = r(vb) = (rv)b = q(rv)$, that is $[q, R]V = 0$. Since V is a left faithful irreducible R-module, hence $[q, R] = 0$, i.e. $q \in Z(R)$ and so $d = 0$, a contradiction.

Therefore $dimV_D$ must be ≤ 2. In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [18], it follows that there exists a suitable filed F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies the same GPI as R.

Assume $k \geq 3$, by the same argument as in the above, we can get a contradiction. If $k = 1$, then it is clear that R is commutative. Thus we may assume that $R \subseteq M_2(F)$, where $M_2(F)$ satisfies $(a(x \circ y) + [q, x]y + x[q, y] + [q, y]x + y[q, x])^n = x \circ y$. Denote e_{ij} the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. Let $x \circ y = e_{21} \circ e_{11} = e_{21}$. In this case we have $(ae_{21} + qe_{21} - e_{21}q)^n = e_{21}$. Right multiplying by e_{21}, we get $(-1)^n(e_{21}q)^n e_{21} = (ae_{21} + qe_{21} - e_{21}q)^n e_{21} = e_{21}e_{21} = 0$. Set $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By calculation we find that $(-1)^n \begin{pmatrix} 0 & 0 \\ q_{12}^n & 0 \end{pmatrix} = 0$, which implies that $q_{12} = 0$. Similarly we can see that $q_{21} = 0$. Therefore q is diagonal in $M_2(F)$. Let $f \in Aut(M_2(F))$. Since $(f(a)[f(x), f(y)] + [[f(q), f(x)], f(y)] + [f(x), [f(q), f(y)]])^n = [f(x), f(y)]$ so $f(q)$ must be a diagonal matrix in $M_2(F)$. In particular, let $f(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$, then $f(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $M_2(F)$, which leads to $d = 0$, a contradiction. This completes the proof of the theorem.

The following example shows that the primeness condition in the above theorem can not be omitted.

Example 2.1. Let S be any ring and $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$. Let $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in S \right\}$ be a nonzero ideal of R and we define a map $F : R \to R$ by $F(x) = 2e_{11}x - xe_{11}$. Then it is easy to see that F is a generalized derivation associated with a nonzero derivation $d(x) = [e_{11}, x]$. It is straightforward to check that F satisfies the property: $(F(x \circ y))^n = x \circ y$ for all $x, y \in I$. However, R is not commutative.

3. The Case: R a Semiprime Ring

Theorem 3.1 *Let R be a semiprime ring and n a fixed positive integer. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in R$, then R is commutative.*

Proof. Since R is semiprime and F is a generalized derivation of R, by Lee

[16], $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. We are given that $(a(x \circ y) + d(x \circ y))^n = x \circ y$ for all $x, y \in R$. By Lee [16,], R and U satisfy the same differential identities, then $(a(x \circ y) + d(x \circ y))^n = x \circ y$ for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. Since U is a B-algebra orthogonal complete [15] and MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. For all $\overline{x}, \overline{y} \in \overline{U}$, $(\overline{a}(\overline{x} \circ \overline{y}) + \overline{d}(\overline{x} \circ \overline{y}))^n = \overline{x} \circ \overline{y}$. It is obvious that $\overline{U}$ is prime. Therefore, by Theorem 2.1, we have $\overline{U}$ is commutative, i.e., $[\overline{U}, \overline{U}] = \overline{0}$. This implies that, for any maximal ideal M of B, $[U, U] \subseteq MU$. Consequently, $[U, U] \subseteq \bigcap MU$, where MU runs over all prime ideals of U. Therefore $[U, U] = 0$ since $\bigcap MU = 0$. In particular, R is commutative.

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