

θ -MONOTONE OPERATORS AND θ -CONVEX FUNCTIONS

Szilárd László

Abstract. In this paper we introduce a new monotonicity concept for multivalued operators, respectively, a new convexity concept for real valued functions, which generalize several monotonicity, respectively, convexity notions already known in literature. We present some fundamental properties of the operators having this monotonicity property. We show that if such a monotonicity property holds locally then the same property holds globally on the whole domain of the operator. We also show that these two new concepts are closely related. As an immediate application we furnish some surjectivity results in finite dimensional spaces.

1. INTRODUCTION

The concept of monotonicity for multivalued operators defined on a Banach space and taking values in its dual has imposed itself (see for example [2, 3, 4], [14, 15]), due to its importance, and has influenced some other branches of mathematics, such as differential equations, economics, engineering, management science, probability theory, etc. Due to the celebrated result of Rockafellar, which claims that the subdifferential of a proper, convex and lower semicontinuous function is a maximal monotone operator, the concepts of monotonicity alongside with convexity were subjects of a dynamical evolution reflected in a number of new concepts - which represent extensions of the classical assumption of monotonicity and convexity without the loss of valuable properties (see for instance [5, 8, 9, 16, 20 and the references therein).

In the present paper we introduce the concept of θ -monotonicity for operators and the concept of θ -convexity for real valued functions. These concepts contain as particular case several monotonicity, respectively, convexity notions already known in literature. We also establish some fundamental properties of operators

Received August 14, 2010, accepted January 17, 2011.

Communicated by Juan Enrique Martinez-Legaz.

2010 *Mathematics Subject Classification*: 47H05, 26A51, 26B25, 49J50.

Key words and phrases: Generalized monotone operator, Maximal monotonicity, Locally monotone operator, Generalized convex function.

having this monotonicity property. The concept of a maximal θ -monotone operator is also introduced, and it is shown that such an operator has convex and closed values. Further we are going to analyze some conditions which ensure that the local θ -monotonicity property of an operator provides the global θ -monotonicity property for that operator. Via some examples it is shown that the θ -monotonicity is more general than most of monotonicity properties known in literature, while an example of a θ -monotone operator is given, which is not even quasimonotone. An analytical condition on the function θ that ensures, beside some extra requirements, the θ -convexity of a real valued function is also established. It is shown that the θ -convexity property of a function is more general than the majority of the convexity properties known in literature, while an example of a θ -convex function is given, which is not even quasiconvex.

In what follows we introduce the concept of θ -monotonicity for an operator. Let X be a real Banach space with its dual denoted by X^* and let $T : X \longrightarrow 2^{X^*}$ be a multivalued operator. We denote by $D(T) = \{x \in X : Tx \neq \emptyset\}$ its domain and by $R(T) = \bigcup_{x \in D(T)} Tx$ its range. The graph of the operator T is the set $G(T) = \{(x, u) \in X \times X^* : u \in Tx\}$. Let $\theta : X \times X \longrightarrow \mathbb{R}$ be a given function with the property that $\theta(x, y) = \theta(y, x)$ for all $x, y \in X$.

Definition 1.1. We say that T is θ -monotone, if

$$(1) \quad \langle u - v, x - y \rangle \geq \theta(x, y)\|x - y\| \text{ for all } (x, u), (y, v) \in G(T).$$

T is called strictly θ -monotone if in (1) equality holds only for $x = y$.

To this respect single-valued θ -monotone operators are those θ -monotone operators $T : X \longrightarrow 2^{X^*}$, which satisfy the condition $\text{card}(Tx) = 1$, for all $x \in D(T)$. It can be easily observed that the concept of θ -monotonicity generalizes several concepts of monotonicity known in literature.

If $\theta(x, y) = 0$ for all $x, y \in D(T)$ we obtain the concept of *Minty-Browder monotonicity*, respectively the concept of *strict Minty-Browder monotonicity* (see [2, 3, 14, 15]), i.e.

$$\langle u - v, x - y \rangle \geq 0 \text{ for all } (x, u), (y, v) \in G(T),$$

respectively,

$$\langle u - v, x - y \rangle > 0 \text{ for all } (x, u), (y, v) \in G(T), x \neq y.$$

If $\theta(x, y) = r\|x - y\|$ for all $x, y \in D(T)$, where $r > 0$, we obtain the concept of *strong monotonicity* (see for instance [24]), i.e.

$$\langle u - v, x - y \rangle \geq r\|x - y\|^2 \text{ for all } (x, u), (y, v) \in G(T).$$

If $\theta(x, y) = f(\|x - y\|)$ for all $x, y \in D(T)$, $x \neq y$ and $\theta(x, x) = 0$ for all $x \in D(T)$, where $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing function, with $\lim_{t \downarrow 0} f(t) = 0$ and $\lim_{t \rightarrow \infty} f(t) = \infty$, then the θ -monotonicity becomes the *uniform monotonicity* (see for instance [12]), i.e.

$$\langle u - v, x - y \rangle \geq f(\|x - y\|)\|x - y\| \text{ for all } (x, u), (y, v) \in G(T), x \neq y.$$

If $\theta(x, y) = -\epsilon$ for all $x, y \in D(T)$, where $\epsilon > 0$, we obtain the concept of ϵ -monotonicity (see [10, 17]), i.e.

$$\langle u - v, x - y \rangle \geq -\epsilon\|x - y\| \text{ for all } (x, u), (y, v) \in G(T).$$

If $\theta(x, y) = -C\|x - y\|^{\gamma-1}$ for all $x, y \in D(T)$, where $C > 0$ and $\gamma > 1$, we obtain the concept of γ -paramonotonicity (see [11]), i.e.

$$\langle u - v, x - y \rangle \geq -C\|x - y\|^\gamma \text{ for all } (x, u), (y, v) \in G(T).$$

For $\gamma = 2$, hence for $\theta(x, y) = -C\|x - y\|$ for all $x, y \in D(T)$, where $C > 0$, the γ -paramonotonicity becomes the C -relaxed monotonicity (see for instance [24]), i.e.

$$\langle u - v, x - y \rangle \geq -C\|x - y\|^2 \text{ for all } (x, u), (y, v) \in G(T).$$

If $\theta(x, y) = -\min\{\sigma(x), \sigma(y)\}$, for all $x, y \in D(T)$ and $\theta(x, y) = 0$ otherwise, where $\sigma : D(T) \rightarrow (0, \infty)$ is a given function, we obtain the concept of *premonotonicity*, introduced in [9], i.e.

$$\langle u - v, x - y \rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\| \text{ for all } (x, u), (y, v) \in G(T).$$

The paper is organized as follows. In Section 2 we establish some fundamental properties of the operators having the θ -monotonicity property. We provide some conditions that ensure their local boundedness. We show that under some circumstances the inverse of a θ -monotone operator is also θ -monotone. In Section 3 we introduce the concept of a maximal θ -monotone operator. We show that these operators have as values closed and convex subsets of X^* , and that, if the function θ is continuous, then their graph is demi-closed. In Section 4 we introduce the concept of a locally θ -monotone operator. Also here, we give a sufficient condition involving θ , guaranteeing that the local θ -monotonicity property of an operator provides the global θ -monotonicity property for that operator. Further, an analytical condition involving the function θ is given which ensures the local θ -monotonicity of an operator. Via some examples it is shown that the concept of θ -monotonicity is larger than most of the monotonicity notions known in literature. In Section 5 we introduce the notion of a θ -convex, respectively, weak θ -convex function and

show that, under some circumstances, a differentiable function is a θ -convex if and only if its differential is a 2θ -monotone operator. Also here an example of θ -convex function that is not even quasiconvex, is given. In Section 6 we present some applications of our results obtaining some surjectivity results in finite dimensional spaces. We conclude the paper by underlying some possible further related research.

2. ON SOME PROPERTIES OF θ -MONOTONE OPERATORS

In this section we present some properties of multivalued θ -monotone operators. As a main result of the section we show, in a Hilbert space context, that under some mild requirements imposed on the function θ , a θ -monotone operator is locally bounded. We also establish a condition on the function θ that ensures that the inverse of a θ -monotone operator is θ -monotone too.

The next result gives a sufficient condition for the θ -monotonicity property of an operator.

Proposition 2.1. *Let $T : X \rightarrow 2^{X^*}$ be an operator with bounded values (i.e. for all $x \in X$ Tx is a bounded set). Then T is a θ -monotone operator, with $\theta(x, y) = -2 \max\{\sup_{u \in Tx} \|u\|, \sup_{v \in Ty} \|v\|\}$ when $x, y \in D(T)$ and $\theta(x, y) = 0$ otherwise.*

Proof. Easily can be observed that for all $(x, y) \in D(T) \times D(T)$ the function $\theta(x, y) = -2 \max\{\sup_{u \in Tx} \|u\|, \sup_{v \in Ty} \|v\|\}$ is finite, since T has bounded values. Hence, θ is well defined, and obviously $\theta(x, y) = \theta(y, x)$.

Since $\langle u, y - x \rangle \leq \|u\| \|y - x\| \leq \max\{\sup_{u \in Tx} \|u\|, \sup_{v \in Ty} \|v\|\} \|x - y\|$, for all $(x, u) \in G(T)$, $y \in D(T)$ we have $-\langle u, y - x \rangle \geq -\|u\| \|y - x\| \geq -\max\{\sup_{u \in Tx} \|u\|, \sup_{v \in Ty} \|v\|\} \|x - y\|$, for all $(x, u) \in G(T)$, $y \in D(T)$. Hence, $\langle v - u, y - x \rangle = \langle v, y - x \rangle - \langle u, y - x \rangle \geq \langle v, y - x \rangle - \max\{\sup_{u \in Tx} \|u\|, \sup_{v \in Ty} \|v\|\} \|x - y\| \geq -\|v\| \|y - x\| - \max\{\sup_{u \in Tx} \|u\|, \sup_{v \in Ty} \|v\|\} \|x - y\| \geq -\max\{\sup_{u \in Tx} \|u\|, \sup_{v \in Ty} \|v\|\} \|y - x\| - \max\{\sup_{u \in Tx} \|u\|, \sup_{v \in Ty} \|v\|\} \|x - y\|$, for all $(x, u), (y, v) \in G(T)$.

Therefore, $\langle v - u, y - x \rangle \geq -2 \max\{\sup_{u \in Tx} \|u\|, \sup_{v \in Ty} \|v\|\} \|y - x\| = \theta(x, y) \|x - y\|$ for all $(x, u), (y, v) \in G(T)$. ■

The next result ensures under some circumstances the θ -monotonicity of the sum of two θ -monotone operators.

Proposition 2.2. *Let $T_1 : X \rightarrow 2^{X^*}$, respectively, $T_2 : X \rightarrow 2^{X^*}$ be a θ_1 -monotone, respectively, a θ_2 -monotone operator. Suppose that $D(T_1) \cap D(T_2) \neq \emptyset$. Then for all $p_1, p_2 \in \mathbb{R}_+$, the operator $T : D(T_1) \cap D(T_2) \rightarrow 2^{X^*}$, $T(x) = p_1 T_1(x) + p_2 T_2(x)$ is a θ -monotone operator, with $\theta(x, y) = p_1 \theta_1(x, y) + p_2 \theta_2(x, y)$, for all $x, y \in D(T_1) \cap D(T_2)$.*

Proof. For $x, y \in D(T_1) \cap D(T_2)$, we have $\langle u_1 - v_1, x - y \rangle \geq \theta_1(x, y)\|x - y\|$, for all $u_1 \in T_1x, v_1 \in T_1y$, respectively, $\langle u_2 - v_2, x - y \rangle \geq \theta_2(x, y)\|x - y\|$, for all $u_2 \in T_2x, v_2 \in T_2y$. By multiplying the first inequality with $p_1 \geq 0$, respectively the second inequality with $p_2 \geq 0$ and then summing we obtain:

$$\langle (p_1u_1 + p_2u_2) - (p_1v_1 + p_2v_2), x - y \rangle \geq (p_1\theta_1(x, y) + p_2\theta_2(x, y))\|x - y\|,$$

for all $u_1 \in T_1x, u_2 \in T_2x, v_1 \in T_1y, v_2 \in T_2y$, therefore $\langle u - v, x - y \rangle \geq \theta(x, y)\|x - y\|$, for all $u \in Tx, v \in Ty$. ■

The next result gives us a sufficient condition for the θ -monotonicity property of an operator.

Proposition 2.3. *If the operator $T : X \longrightarrow 2^{X^*}$ is globally bounded then T is θ -monotone with $\theta(x, y) = -2M$, where the number M is defined by $M \geq \|u\|$, for all $u \in R(T)$.*

Proof. Indeed, for all $(x, u), (y, v) \in G(T)$ we have $\langle u - v, x - y \rangle \geq -\|u - v\|\|x - y\|$, and from the triangle inequality we obtain $\langle u - v, x - y \rangle \geq -(\|u\| + \|v\|)\|x - y\| \geq -2M\|x - y\|$. ■

Remark 2.4. The preceding results were first established in [9] for premonotone operators in the setting of finite dimensional spaces, and the proof of each one of them is an adaptation of the other one used in [9].

Recall that the operator $T : X \longrightarrow 2^{X^*}$ is *locally bounded* in $x \in X$, if there exists a neighborhood $U \subseteq X$ of x , such that the set $T(U)$ is a bounded subset of X^* .

Let $f : X \longrightarrow \mathbb{R}$ be a function. We say that f is *lower semicontinuous* in $x \in X$, if for every $\epsilon > 0$ there exists a neighborhood $U \subseteq X$ of x , such that $f(x) - \epsilon \leq f(y)$ for all $y \in U$. Equivalently, this can be expressed as $\liminf_{y \rightarrow x} f(y) \geq f(x)$. We say that f is lower semicontinuous on $U \subseteq X$ if f is lower semicontinuous in every $x \in U$.

The next result provides, in a finite dimensional Hilbert space context, the local boundedness of a θ -monotone operator in the interior of its domain. This is a major result, since the θ -monotonicity property of the operators is a weaker condition than the Minty-Browder monotonicity, and still one of the fundamental property of the Minty-Browder monotone operators remains true.

Theorem 2.5. *Let $T : \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^n}$ be a θ -monotone operator. If the function $\theta(\cdot, y)$ is lower semicontinuous on $\text{int}(D(T))$ for all $y \in \text{int}(D(T))$, then T is locally bounded in the interior of its domain $D(T)$.*

Proof. First we prove that for every $x, y \in D(T)$ we have

$$\sup_{u \in Tx} \langle u, y - x \rangle \leq \left(\inf_{v \in Ty} \|v\| - \theta(x, y) \right) \|x - y\|.$$

Since T is θ -monotone, we get $\langle u - v, x - y \rangle \geq \theta(x, y) \|x - y\|$, for all $(x, u), (y, v) \in G(T)$ or, equivalently, $\langle u, y - x \rangle \leq \langle v, y - x \rangle - \theta(x, y) \|x - y\| \leq \|v\| \|x - y\| - \theta(x, y) \|x - y\|$, for all $v \in Ty$. Hence, we have $\langle u, y - x \rangle \leq (\inf_{v \in Ty} \|v\| - \theta(x, y)) \|x - y\|$, for all $u \in Tx$, therefore $\sup_{u \in Tx} \langle u, y - x \rangle \leq (\inf_{v \in Ty} \|v\| - \theta(x, y)) \|x - y\|$.

Suppose that T is not locally bounded on $\text{int } D(T)$. Then there exists an $x \in \text{int } D(T)$, and a sequence $(x_k) \subseteq D(T)$, such that $x_k \rightarrow x, k \rightarrow \infty$, and a sequence $(u_k), u_k \in Tx_k, k \geq 1$ such that $\|u_k\| \rightarrow \infty, k \rightarrow \infty$. Let $\delta > 0$ such that $\overline{B}(x, \delta) \subset D(T)$, where $\overline{B}(x, \delta)$ is the closed ball with center x and radius δ . Let $v_k = \frac{u_k}{\|u_k\|}, k \geq 1$. Then the sequence v_k is bounded, and let be \bar{v} one of its cluster points. Let be now $y = x + \delta\bar{v}$. It results that $y \in \overline{B}(x, \delta)$.

Since $\sup_{u \in Tx} \langle u, y - x \rangle \leq (\inf_{v \in Ty} \|v\| - \theta(x, y)) \|x - y\|$ for all $x, y \in D(T)$, we have $\langle u_k, (x + \delta\bar{v}) - x_k \rangle \leq (\inf_{v \in T(x + \delta\bar{v})} \|v\| - \theta(x_k, x + \delta\bar{v})) \|x_k - (x + \delta\bar{v})\|$ for all $k \geq 1$. Dividing by $\|u_k\|$ we obtain

$$(2) \quad \begin{aligned} & \langle v_k, (x + \delta\bar{v}) - x_k \rangle \\ & \leq \frac{(\inf_{v \in T(x + \delta\bar{v})} \|v\| - \theta(x_k, x + \delta\bar{v})) \|x_k - (x + \delta\bar{v})\|}{\|u_k\|} \text{ for all } k \geq 1. \end{aligned}$$

Since the sequence (v_k) is bounded, it has a subsequence (v_{k_j}) which converge to \bar{v} . Hence we have

$$(3) \quad \begin{aligned} & \langle v_{k_j}, (x + \delta\bar{v}) - x_{k_j} \rangle \\ & \leq \frac{(\inf_{v \in T(x + \delta\bar{v})} \|v\| - \theta(x_{k_j}, x + \delta\bar{v})) \|x_{k_j} - (x + \delta\bar{v})\|}{\|u_{k_j}\|} \text{ for all } j \geq 1. \end{aligned}$$

Easily can be realized that in this case we have

$$(4) \quad \begin{aligned} & \liminf_{j \rightarrow \infty} \langle v_{k_j}, x_{k_j} - (x + \delta\bar{v}) \rangle \\ & \geq \liminf_{j \rightarrow \infty} \frac{(\theta(x_{k_j}, x + \delta\bar{v}) - \inf_{v \in T(x + \delta\bar{v})} \|v\|) \|x_{k_j} - (x + \delta\bar{v})\|}{\|u_{k_j}\|}. \end{aligned}$$

Since $\theta(\cdot, x + \delta\bar{v})$ is lower semicontinuous in x , and $\|u_{k_j}\| \rightarrow \infty, j \rightarrow \infty$ we obtain that $\langle \bar{v}, -\delta\bar{v} \rangle \geq 0$, hence $\delta \leq 0$, which is a contradiction. ■

Remark 2.6. The previous result was first established in [9] for premonotone operators, and the technique used in its proof is an adaptation of the one used in [9].

The author wishes to thank to an anonymous referee who pointed out, that the proof of Theorem 2.5 does not remain valid in an arbitrary infinite dimensional Hilbert space, due to the fact that in this case the sequence $\left\{ \frac{u_k}{\|u_k\|} \right\}$, $k \geq 1$, where $\{u_k\}$ is an unbounded sequence, does not necessarily have a cluster point.

The following proposition is obvious.

Proposition 2.7. *Let $T : X \longrightarrow 2^{X^*}$ be a θ -monotone operator and $\theta' : D(T) \times D(T) \longrightarrow \mathbb{R}$ a function satisfying $\theta'(x, y) = \theta'(y, x)$ and $\theta'(x, y) \leq \theta(x, y)$ for all $x, y \in D(T)$. Then T is a θ' -monotone operator.*

At this point we present a results involving the inverse of a θ -monotone operator. Let X be a reflexive real Banach space. Recall that for an operator $T : X \longrightarrow 2^{X^*}$ its inverse is defined as $T^{-1} : X^* \longrightarrow 2^X$, $T^{-1}u = \{x \in X : u \in Tx\}$. It is obvious that $D(T^{-1}) = R(T)$ and $(u, x) \in G(T^{-1})$ if and only if $(x, u) \in G(T)$. In [19] the concept of a ρ -semimonotone operator is introduced, and it is shown that an operator is ρ -semimonotone if and only if its inverse is ρ -semimonotone. Unfortunately, we cannot give a similar result for θ -monotone operators, however under some conditions the inverse of a θ -monotone operator is θ' -monotone.

Let T be a θ -monotone operator. If $\inf \{\theta(x, y)\|x - y\| : x, y \in D(T)\} > -\infty$ we introduce the function $\theta_{-1} : R(T) \times R(T) \longrightarrow \mathbb{R}$

$$(5) \quad \theta_{-1}(u, v) = \begin{cases} \frac{\inf \{\theta(x, y)\|x - y\| : x \in T^{-1}u, y \in T^{-1}v\}}{\|u - v\|}, & \text{if } u \neq v, \\ 0, & \text{if } u = v. \end{cases}$$

We have the following result.

Proposition 2.8. *Let X be a reflexive real Banach space and $T : X \longrightarrow 2^{X^*}$ a θ -monotone operator. If $\inf_{x, y \in D(T)} \theta(x, y)\|x - y\| > -\infty$, then $T^{-1} : X^* \longrightarrow 2^X$ is a θ_{-1} -monotone operator, where θ_{-1} is defined by (5).*

Proof. Since T is θ -monotone, we have $\langle u - v, x - y \rangle \geq \theta(x, y)\|x - y\|$ for all $(x, u), (y, v) \in G(T)$, consequently, $\langle x - y, u - v \rangle \geq \inf \{\theta(x, y)\|x - y\| : x \in T^{-1}u, y \in T^{-1}v\} = \theta_{-1}(u, v)\|u - v\|$ for all $(u, x), (v, y) \in G(T^{-1})$, $u \neq v$, and $\langle x - y, u - v \rangle = 0 = \theta_{-1}(u, v)\|u - v\|$ when $u = v$, hence T^{-1} is θ_{-1} monotone. ■

The next corollary is an easy consequence of the previous result.

Corollary 2.9. *Let X be a reflexive real Banach space and $T : X \longrightarrow 2^{X^*}$ an ϵ -montone operator. If $D(T)$ is bounded, then T^{-1} is an ϵ' -monotone operator, with $\epsilon' = -\epsilon \sup \{\|x - y\| : x, y \in D(T)\}$.*

3. MAXIMAL θ -MONOTONE OPERATORS

In this section the concept of maximal θ -monotone operator is considered. It is shown that a maximal θ -monotone operator has convex and closed images and that, under some circumstances its graph is $\|\cdot\| \times bdw^*$ -closed, where by bdw^* we denote weak*-convergence for bounded nets. Finally, for a single-valued operator, we present some conditions that ensure its maximal θ -monotonicity. This result is a generalization of a well-known result established for the classical Minty-Browder monotonicity.

Definition 3.1. Let $T : X \longrightarrow 2^{X^*}$ be a θ -monotone operator. One says that T is maximal θ -monotone, if for every operator $T' : X \longrightarrow 2^{X^*}$, which is θ -monotone with $G(T) \subseteq G(T')$, one has $T = T'$.

The next result ensures that every θ -monotone operator can be extended to a maximal θ -monotone one.

Proposition 3.2. Every θ -monotone operator has a maximal θ -monotone extension.

Proof. Let $T : X \longrightarrow 2^{X^*}$ be a θ -monotone operator. In view of Zorn's lemma it suffices to check that an increasing chain of θ -monotone operators $\{T_i\}_{i \in J}$ whose graph contain $G(T)$ has an upper bound. Let $\tilde{T}x = \bigcup_{i \in J} T_i x$, for all $x \in X$. Then \tilde{T} is an upper bound since for all $(x, u), (y, v) \in G(\tilde{T})$ there exists $i \in J$ such that $(x, u), (y, v) \in G(T_i)$, therefore $\langle u - v, x - y \rangle \geq \theta(x, y)\|x - y\|$. ■

Definition 3.3. Two pairs $(x, u), (y, v) \in X \times X^*$ are θ -monotonically related if

$$\langle u - v, x - y \rangle \geq \theta(x, y)\|x - y\|.$$

The following results provide a necessary and sufficient condition for maximal θ -monotonicity of operators.

Proposition 3.4. A θ -monotone operator $T : X \longrightarrow 2^{X^*}$ is maximal θ -monotone if and only if whenever a pair $(x, u) \in X \times X^*$ is θ -monotonically related to all pairs $(y, v) \in G(T)$, it holds that $u \in Tx$.

Proof. Let T' be a θ -monotone operator such that $G(T) \subseteq G(T')$. Then any pair $(x', u') \in G(T')$ is θ -monotonically related to all pairs $(y, v) \in G(T)$, and from the assumption of the proposition we obtain that $u' \in Tx'$, so that $G(T') \subseteq G(T)$. Therefore $T' = T$, hence T is maximal θ -monotone.

For the converse statement it is enough to observe that for a given pair (x', u'^*) which is θ -monotonically related to all pairs $(y, v) \in G(T)$ the operator

$$T'x = \begin{cases} Tx' \cup \{u'\} & \text{if } x = x' \\ Tx & \text{otherwise,} \end{cases}$$

is a θ -monotone operator as well that $G(T) \subseteq G(T')$. Since T is maximal θ -monotone we get $G(T') = G(T)$, hence $(x', u') \in G(T)$. ■

Proposition 3.5. *A θ -monotone operator T is maximal θ -monotone if and only if, for every θ' -monotone operator T' , with $G(T) \subseteq G(T')$ and $\theta(x, y) \leq \theta'(x, y)$ for all $x, y \in D(T')$, one has $T = T'$.*

Proof. Let T be a maximal θ -monotone operator. Since $\theta(x, y) \leq \theta'(x, y)$ for all $x, y \in D(T')$ according to Proposition 3.4 the operator T' is θ -monotone. But $G(T) \subseteq G(T')$ hence due to the maximal θ -monotonicity of T we have $T = T'$.

Conversely, since in this case every θ' -operator T' is θ -monotone as well, and $G(T) \subseteq G(T')$ implies $T = T'$ we obtain that T is maximal θ -monotone. ■

The next result provides the convexity and closedness of the images of a maximal θ -monotone operator.

Theorem 3.6. *Let $T : X \rightarrow 2^{X^*}$ be a maximal θ -monotone operator. Then Tx is convex and closed for all $x \in D(T)$.*

Proof. Let be $x \in D(T)$, and $u, v \in Tx$. We have:

$$\langle u - w, x - y \rangle \geq \theta(x, y) \|x - y\|, (\forall)(y, w) \in G(T), \text{ and}$$

$$\langle v - w, x - y \rangle \geq \theta(x, y) \|x - y\|, (\forall)(y, w) \in G(T).$$

Let be $t_1, t_2 \in [0, 1]$, with $t_1 + t_2 = 1$. We will show that the convex combination of u and v is contained in Tx . Adding the inequalities $\langle t_1(u - w), x - y \rangle \geq t_1\theta(x, y) \|x - y\|$, $(\forall)(y, w) \in G(T)$, and $\langle t_2(v - w), x - y \rangle \geq t_2\theta(x, y) \|x - y\|$, $(\forall)(y, w) \in G(T)$, we obtain $\langle (t_1u + t_2v) - w, x - y \rangle \geq \theta(x, y) \|x - y\|$, $(\forall)(y, w) \in G(T)$, and according to the maximal θ -monotonicity of T , we obtain that $(x, t_1u + t_2v) \in G(T)$, which shows that Tx is convex.

Now, let $x \in D(T)$, $u_k \in Tx$, $k = 1, 2, \dots$ such that $u_k \rightarrow u$, $k \rightarrow \infty$. We will show that $u \in Tx$. We have $\langle u_k - v, x - y \rangle \geq \theta(x, y) \|x - y\|$, $(\forall)(y, v) \in G(T)$, and taking the limit $k \rightarrow \infty$ we obtain $\langle u - v, x - y \rangle \geq \theta(x, y) \|x - y\|$, $(\forall)(y, v) \in G(T)$. According to maximal θ -monotonicity of T , we obtain that $(x, u) \in G(T)$, therefore $u \in Tx$, and this shows that Tx is closed. ■

By bdw^* we denote weak*-convergence for bounded nets and hence include all weak*-convergent sequences. It is known that $\langle \cdot, \cdot \rangle$ is $\|\cdot\| \times bdw^*$ continuous (see [1]). The next result ensures, in the case when θ is continuous, the $\|\cdot\| \times bdw^*$ closedness of the graph of a maximal θ -monotone operator.

Proposition 3.7. *Let $T : X \rightarrow 2^{X^*}$ be a maximal θ -monotone operator. If the function $\theta(\cdot, y) : X \rightarrow \mathbb{R}$ is continuous on $\overline{D(T)}$ for every $y \in D(T)$, then $G(T)$ is $\|\cdot\| \times bdw^*$ -closed.*

Proof. Let $(x_\alpha, u_\alpha) \in G(T)$ be a bounded net such that $x_\alpha \longrightarrow x$, $u_\alpha \rightharpoonup^* u$, where \rightharpoonup^* denotes the convergence in the weak* topology of X^* . We have $\langle u_\alpha - v, x_\alpha - y \rangle \geq \theta(x_\alpha, y) \|x_\alpha - y\|$, $(\forall)(y, v) \in G(T)$.

Since $\langle \cdot, \cdot \rangle$ is $\|\cdot\| \times bdw^*$ continuous, by taking the limit we obtain that $\langle u - v, x - y \rangle \geq \theta(x, y) \|x - y\|$, $(\forall)(y, v) \in G(T)$, and from the maximal θ -monotonicity of T results that $(x, u) \in G(T)$, which shows that $G(T)$ is $\|\cdot\| \times bdw^*$ -closed. ■

The $\|\cdot\| \times \|\cdot\|$ closedness of the graph of a maximal θ -monotone operator holds under the same assumptions.

Proposition 3.8. *Let $T : X \longrightarrow 2^{X^*}$ be a maximal θ -monotone operator. If the function $\theta(\cdot, y) : X \longrightarrow \mathbb{R}$ is lower semicontinuous on $\overline{D(T)}$ for every $y \in D(T)$, then $G(T)$ is $\|\cdot\| \times \|\cdot\|$ -closed.*

Proof. Let $(x_n, u_n) \in G(T)$ be a sequence such that $x_n \longrightarrow x$, $u_n \longrightarrow u$. We have $\langle u_n - v, x_n - y \rangle \geq \theta(x_n, y) \|x_n - y\|$, $(\forall)(y, v) \in G(T)$.

Hence, $\langle u - v, x - y \rangle = \liminf_{n \rightarrow \infty} \langle u_n - v, x_n - y \rangle \geq \liminf_{n \rightarrow \infty} \theta(x_n, y) \|x_n - y\| \geq \theta(x, y) \|x - y\|$, $(\forall)(y, v) \in G(T)$. From the maximal θ -monotonicity of T results that $(x, u) \in G(T)$, which shows that $G(T)$ is $\|\cdot\| \times \|\cdot\|$ -closed. ■

Let $A : X \longrightarrow X^*$ be a single-valued operator. Recall that A is said to be *hemicontinuous* at $x \in X$, if for all $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, $t_n \longrightarrow 0$, $(n \longrightarrow \infty)$ and $y \in X$, we have $A(x + t_n y) \rightharpoonup^* Ax$, $(n \longrightarrow \infty)$, where " \rightharpoonup^* " denotes the convergence with respect to the *weak** topology of X^* .

Recall that a function $g : X \longrightarrow \mathbb{R}$ is called *radially continuous* at $x \in X$, if $\lim_{t \searrow 0} g(x + tz) = g(x)$ for all $z \in X$. In what follows we give a sufficient condition that furnishes the maximal θ -monotonicity of a single valued operator.

Proposition 3.9. *Let $T : X \longrightarrow X^*$ be a hemicontinuous single-valued θ -monotone operator, with $D(T) = X$. Assume further that $\theta(x, x) \geq 0$ and $\theta(x, \cdot)$ is radially continuous for all $x \in X$. Then T is maximal θ -monotone.*

Proof. Let be $(x, u) \in X \times X^*$ such that $\langle u - Ty, x - y \rangle \geq \theta(x, y) \|x - y\|$ for all $y \in X$. We need to prove that $u = Tx$.

Let $\lambda > 0$ and $z \in X$ arbitrary and $y = x - \lambda z$. We have $\langle u - T(x - \lambda z), \lambda z \rangle \geq \theta(x, x - \lambda z) \|\lambda z\|$ that divided by λ leads to $\langle u - T(x - \lambda z), z \rangle \geq \theta(x, x - \lambda z) \|z\|$. Taking the limit $\lambda \searrow 0$ and using the hemicontinuity of T respective the radial continuity of θ we obtain

$$\langle u - Tx, z \rangle \geq \theta(x, x) \|z\| \geq 0.$$

Since $z \in X$ was arbitrary chosen we obtain that $u = Tx$. ■

4. LOCALLY θ -MONOTONE OPERATORS

In this section we introduce the local θ -monotonicity concept of a multivalued operator. Further we give under some conditions involving the function θ , a sufficient condition that ensures the θ -monotonicity of an operator. We present next the concept of local θ -monotonicity, respectively, of local central θ -monotonicity for operators.

Definition 4.1. Let $T : X \longrightarrow 2^{X^*}$ be an operator. One says that T is locally θ -monotone, respectively, locally central θ -monotone, if for all $z \in D(T)$ there exists an open neighborhood $U_z \subseteq X$ of z , such that

$$(6) \quad \langle u - v, x - y \rangle \geq \theta(x, y)\|x - y\|, \text{ for all } x, y \in U_z \cap D(T), u \in Tx, v \in Ty$$

respectively

$$(7) \quad \langle u - v, x - z \rangle \geq \theta(x, z)\|x - z\|, \text{ for all } x \in U_z \cap D(T), u \in Tx, v \in Tz.$$

The notion of strict local θ -monotonicity, respectively, the notion of strict local central θ -monotonicity is obtained if in (6), respectively in (7) we have equality only for $x = y$, respectively, for $x = z$.

A local monotonicity concept, the so called *submonotonicity* (see [6, 7]), is well-known in literature. Recall that the operator T is called submonotone, if for all $z \in D(T)$ and all $\epsilon > 0$ there exists $\delta > 0$, such that

$$\langle u - v, x - y \rangle \geq -\epsilon\|x - y\| \text{ for all } x, y \in B(z, \delta), u \in Tx, v \in Ty,$$

where $B(z, \delta)$ denotes the open ball with center z and radius δ .

The next result provides sufficient condition for the submonotonicity of a locally θ -monotone operator.

Proposition 4.2. *If $T : X \longrightarrow 2^{X^*}$ is a locally θ -monotone operator with an open domain $D(T)$, where θ is lower semicontinuous in (x, x) in the topolog of the norm $\|(\cdot, \cdot)\| = \sqrt{\|\cdot\|^2 + \|\cdot\|^2}$ of $X \times X$, and $\theta(x, x) \geq 0$ for all $x \in D(T)$, then T is submonotone.*

Proof. We have to prove that for all $\epsilon > 0$ and all $z \in D(T)$, there exists $\delta > 0$, such that for all $x, y \in B(z, \delta) \subseteq D(T)$ we have $\langle u - v, x - y \rangle \geq -\epsilon\|x - y\|$, for every $u \in Tx, v \in Ty$. Let $z \in D(T)$. Since T is locally θ -monotone in z we only need to prove, that for all $\epsilon > 0$ there exists $\delta > 0$ such that $\theta(x, y) \geq -\epsilon$ for all $x, y \in B(z, \delta) \subseteq U_z$, where U_z is an open neighborhood of z where θ -monotonicity holds. Let $\epsilon > 0$. Due the lower semicontinuity of θ in (z, z) we obtain, that there exists $\delta_1 > 0$ such that $\theta(x, y) \geq \theta(z, z) - \epsilon$ when $\|(x, y) - (z, z)\| \leq \delta_1$. Then, there exists $\delta > 0$ such that $0 < \delta < \frac{\delta_1}{2}$ and $B(z, \delta) \times B(z, \delta) \subset B((z, z), \delta_1)$, hence for all $x, y \in B(z, \delta)$ we have $\|(x, y) - (z, z)\| \leq \delta_1$, thus $\theta(x, y) \geq -\epsilon$. ■

In view of Definition (4.1), we say that the operator $T : X \longrightarrow 2^{X^*}$ is a locally strong monotone operator, (respectively, a locally Minty-Browder monotone operator, a locally C -relaxed monotone operator, a locally γ -paramonoton operator), if for all $z \in D(T)$ there exists an open neighborhood $U_z \subseteq X$ of z , such that the restriction of T on $U_z \cap D(T)$ is a strongly monotone operator, (respectively, a Minty-Browder monotone operator, a C -relaxed monotone operator, a γ -paramonoton operator).

Since the function $\theta(x, y) = C\|x - y\|^{\gamma-1}$, $C, \gamma \in \mathbb{R}$, $\gamma > 1$ is lower semi-continuous and $\theta(x, x) = 0$, we conclude that any locally θ -monotone operator with $\theta(x, y) = C\|x - y\|^{\gamma-1}$ is actually submonotone. In particular, a locally strong monotone operator, a locally Minty-Browder monotone operator, a locally C -relaxed monotone operator or a locally γ -paramonoton operator is submonotone.

In what follows we present a condition which ensures that the local θ -monotonicity of an operator provides the θ -monotonicity for that operator. We denote by $(x, y) = \{x + t(y - x) : t \in (0, 1)\}$, respectively by $[x, y] = \{x + t(y - x) : t \in [0, 1]\}$ the open line segment with endpoints x and y , respectively the closed line segment with endpoints x and y . We need the following definition.

Definition 4.3. Let $D \leq X$ be convex. One says that the function θ has the (m) property on D , if $\theta(x, z) + \theta(z, y) \geq \theta(x, y)$ for all $z \in (x, y)$, $x, y \in D$, $x \neq y$.

One can easily observe, that $\theta(x, y) = C\|x - y\|$, $C \in \mathbb{R}$, has the (m) property (satisfied with equality), even more, the function $\theta(x, y) = \rho(x, y) + C\|x - y\|$, where $\rho : X \times X \longrightarrow \mathbb{R}_+$ is a semi-metric and $C \in \mathbb{R}$, has the (m) property on every convex subset $D \leq X$.

If $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a function with the property $f(ts) \geq tf(s)$ for all $t \in (0, 1)$ and all $s \in \mathbb{R}$, then $\theta(x, y) = f(\|x - y\|)$ has the (m) property on every convex subset $D \leq X$.

If $\theta(x, y) = C\|x - y\|^{\gamma-1}$, with $C, \gamma \in \mathbb{R}$, $\gamma > 1$, then θ has the (m) property, on every convex subset $D \leq X$, if and only if $C = 0$ or $C > 0$ and $\gamma \in (1, 2]$ or $C < 0$ and $\gamma \geq 2$.

Remark 4.4. Let $D \geq X$ be convex. If the function $\theta(x, \cdot) : D \longrightarrow \mathbb{R}$ is concave and $\theta(x, x) \geq 0$ for all $x \in D$, then θ has the (m) property on D .

Indeed, for $t \in (0, 1)$ we have $\theta(x, x + t(y - x)) \geq (1 - t)\theta(x, x) + t\theta(x, y) \geq t\theta(x, y)$ and $\theta(y, x + t(y - x)) \geq (1 - t)\theta(y, x) + t\theta(y, y) \geq (1 - t)\theta(y, x)$. Adding these two relations and taking into account the fact that θ is symmetric we obtain the desired result.

The next result provides a sufficient condition for the θ -monotonicity of an operator. To this aim we introduce an order relation on the segment $[x, y]$ as follows: for $z_1, z_2 \in [x, y]$, $z_1 = x + t_1(y - x)$, $z_2 = x + t_2(y - x)$ we have $z_1 \leq z_2$, if and only if $t_1 \leq t_2$. Obviously $z_1 < z_2$ if and only if $t_1 < t_2$.

Theorem 4.5. *Let $T : X \longrightarrow 2^{X^*}$ be a locally central θ -monotone operator, having a convex domain $D(T)$. If the function θ has the (m) property on $D(T)$, then T is θ -monotone.*

Proof. Let be $x, y \in D(T)$, $x \neq y$. We have to show that $\langle u - v, y - x \rangle \geq \theta(y, x)\|y - x\|$ for all $u \in Tx$, $v \in Ty$. Since $D(T)$ is convex we have $[x, y] \subseteq D(T)$. Obviously $[x, y] \subseteq \bigcup_{z \in [x, y]} U_z$, where $U_z \subseteq D$ is an open ball with center z such that $\langle u - w, x - z \rangle \geq \theta(x, z)\|x - z\|$, for all $x \in U_z \cap D(T)$ and for all $u \in Tx$, $w \in Tz$. Due to the compactness of $[x, y]$, from the open cover $\bigcup_{z \in [x, y]} U_z$ it may be extracted a finite subcover. Let $\bigcup_{i=1}^n U_{z_i}$, $z_1 \leq z_2 \leq \dots \leq z_n$ be a finite subcover, minimal in the sense that none of the sets U_{z_i} , $i = 1, 2, \dots, n$, may be omitted, such that the property $\bigcup_{j=1, j \neq i}^n U_{z_j}$ is a cover of $[x, y]$ remains true.

Obviously $U_{z_i} \cap U_{z_{i+1}} \cap (x, y) \neq \emptyset$, for all $i = 1, 2, \dots, n - 1$.

Let us consider $x_i \in U_{z_i} \cap U_{z_{i+1}} \cap (x, y)$, such that $z_i \leq x_i \leq z_{i+1}$ with $z_{i+1} \in (x_i, x_{i+1})$ for $i = 1, 2, \dots, n - 1$, where $x_n = y$. Then $x \leq z_1 \leq x_1$, $x, x_1 \in U_{z_1}$ and by the assumption of the theorem we obtain $\langle v_1 - u, z_1 - x \rangle \geq \theta(z_1, x)\|z_1 - x\|$ for all $v_1 \in Tz_1$, $u \in Tx$, and $\langle u_1 - v_1, x_1 - z_1 \rangle \geq \theta(x_1, z_1)\|x_1 - z_1\|$ for all $u_1 \in Tx_1$, $v_1 \in Tz_1$. Since $z_1 \in (x, x_1)$ there exists $\delta \in (0, 1)$ such that $z_1 = x + \delta(x_1 - x)$, and replacing this in the above inequalities and simplifying with δ respectively with $1 - \delta$ we obtain $\langle v_1 - u, x_1 - x \rangle \geq \theta(z_1, x)\|x_1 - x\|$ for all $v_1 \in Tz_1$, $u \in Tx$ and $\langle u_1 - v_1, x_1 - x \rangle \geq \theta(x_1, z_1)\|x_1 - x\|$, for all $u_1 \in Tx_1$, $v_1 \in Tz_1$. By fixing a $v_1 \in Tz_1$ and summing we obtain $\langle u_1 - u, x_1 - x \rangle \geq (\theta(z_1, x) + \theta(x_1, z_1))\|x_1 - x\|$, for all $u_1 \in Tx_1$, $u \in Tx$, and due to the (m) property of θ we get $\langle u_1 - u, x_1 - x \rangle \geq \theta(x_1, x)\|x_1 - x\|$ for all $u_1 \in Tx_1$, $u \in Tx$.

Since $x_1 \in (x, y)$ we have $x_1 = x + t_1(y - x)$ for some $t_1 \in (0, 1)$, and simplifying with t_1 the previous relation becomes $\langle u_1 - u, y - x \rangle \geq \theta(x_1, x)\|y - x\|$ for all $u_1 \in Tx_1$, $u \in Tx$.

In the same way we obtain that $\langle u_2 - u_1, y - x \rangle \geq \theta(x_2, x_1)\|y - x\|$ for all $u_2 \in Tx_2$, $u_1 \in Tx_1$, and continuing the procedure finally we get $\langle v - u_{n-1}, y - x \rangle \geq \theta(y, x_{n-1})\|y - x\|$ for all $v \in Ty$, $u_{n-1} \in Tx_{n-1}$. By fixing $u_1 \in Tx_1$, $u_2 \in Tx_2, \dots, u_{n-1} \in Tx_{n-1}$, then summing term by term we obtain

$$\langle v - u, y - x \rangle \geq \sum_{i=1}^n \theta(x_i, x_{i-1})\|y - x\|,$$

for all $v \in Ty$, $u \in Tx$, where $x_0 = x$, $x_n = y$. Since θ has the (m) property we have $\sum_{i=1}^n \theta(x_i, x_{i-1}) \geq \theta(x, y) = \theta(y, x)$, hence $\langle v - u, y - x \rangle \geq \theta(y, x)\|y - x\|$

for all $v \in Ty$, $u \in Tx$, and the proof is complete. \blacksquare

In view of Definition (4.1), we say that the operator $T : X \rightarrow 2^{X^*}$ is a locally central strong monotone operator, (respectively, a locally central Minty-Browder monotone operator, a locally central C -relaxed monotone operator), if for all $z \in D(T)$ there exists an open neighborhood $U_z \subseteq X$ of z , such that for every $x \in U_z \cap D(T)$ and all $u \in Tx$, $v \in Tz$ we have $\langle u - v, x - z \rangle \geq C\|x - z\|^2$, where $C > 0$, (respectively, $\langle u - v, x - z \rangle \geq 0$, $\langle u - v, x - z \rangle \geq -C\|x - z\|^2$, where $C > 0$).

Remark 4.6. Since $\theta(x, y) = C\|x - y\|$, $C \in \mathbb{R}$ has the (m) property, the locally central strongly monotone, the locally central Minty-Browder monotone, respectively the locally central C -relaxed monotone operators are strongly monotone, Minty-Browder monotone, respectively, C -relaxed monotone on D .

Remark 4.7. Obviously we obtain the same result if in Theorem 4.5 we replace the condition of locally central θ -monotonicity by the condition of strict locally central θ -monotonicity, and the conclusion of θ -monotonicity of the operator T , by the conclusion of strict θ -monotonicity of the operator.

The next corollary provides the equivalence among the local central θ -monotonicity, local θ -monotonicity and θ -monotonicity of an operator under some circumstances.

Corollary 4.8. *If $T : X \rightarrow 2^{X^*}$ is an operator having a convex domain $D(T)$ and if θ has the (m) property on $D(T)$, then the local θ -monotonicity property, the local central θ -monotonicity property and the θ -monotonicity property of the operator T are equivalent.*

The next result provides a sufficient condition for the θ -monotonicity property of a single-valued operator in a Hilbert space. Let H be a Hilbert space. If the operator $T : U \subseteq H \rightarrow H$ is Fréchet differentiable in x , then we denote by $(dT)_x$ its differential in x . Let us denote by $\partial\theta(x, \cdot)_x(y)$ the directional derivative of the function $\theta(x, \cdot) : H \rightarrow \mathbb{R}$ in $x \in D$, in direction $y \in H$, i.e.

$$\partial\theta(x, \cdot)_x(y) = \lim_{t \downarrow 0} \frac{\theta(x, x + ty) - \theta(x, x)}{t}.$$

We have the following result:

Theorem 4.9. *Let H be a real Hilbert space, $D \subseteq H$ open and convex and let $T : D \rightarrow H$ be an operator of class C^1 . Let $\theta : H \times H \rightarrow \mathbb{R}$ be a function with the property that $\theta(x, y) = \theta(y, x)$ and $\theta(x, x) = 0$ for all $x, y \in D$ and with the property that for all $x \in D$, the function $\theta(x, \cdot) : H \rightarrow \mathbb{R}$ has directional derivatives in $x \in D$ in every direction $y \in H$. If $\langle (dT)_x(y), y \rangle > \partial\theta(x, \cdot)_x(y)\|y\|$ for all $x \in D$ and all $y \in H$, then T is strict locally central θ -monotone on D . If in addition θ has the (m) property on D , then T is strictly θ -monotone on D .*

Proof. Let $x \in D$ be fixed. Since $\langle (dT)_x(y), y \rangle > \mathfrak{d}\theta(x, \cdot)_x(y)\|y\|$ for all $x \in D$ and all $y \in H$ we have

$$\left\langle \lim_{t \downarrow 0} \frac{T(x + ty) - Tx}{t}, y \right\rangle > \lim_{t \downarrow 0} \frac{\theta(x, x + ty) - \theta(x, x)}{t} \|y\|.$$

In other words, there exists $\epsilon > 0$ such that

$$\left\langle \frac{T(x + ty) - Tx}{t}, y \right\rangle > \frac{\theta(x, x + ty) - \theta(x, x)}{t} \|y\| \text{ for all } t \in (0, \epsilon),$$

or equivalently $\langle T(x + ty) - Tx, (x + ty) - x \rangle > (\theta(x, x + ty) - \theta(x, x))\|(x + ty) - x\| = \theta(x + ty, x)\|(x + ty) - x\|$ for all $t \in (0, \epsilon)$. This latter relation shows that for all $x \in D$ there exists an open neighborhood of x , say U_x , such that $\langle Tz - Tx, z - x \rangle > \theta(z, x)\|z - x\|$ for all $z \in U_x$, which means that T is strict locally central θ -monotone on D .

If, in addition, θ has the (m) property, according to Remark 4.7, T is strictly θ -monotone on D . ■

Remark 4.10. Obviously, under the assumptions of Theorem 4.9, if for all $x \in D$ and all $y \in H$ we have $\langle (dT)_x(y), y \rangle \geq \mathfrak{d}\theta(x, \cdot)_x(y)\|y\|$, then T is locally central θ -monotone on D . If, in addition, θ has the (m) property on D , then T is θ -monotone on D .

Remark 4.11. If $\theta(x, y) = C\|x - y\|$ for $C \in \mathbb{R}$, we have $\mathfrak{d}\theta(x, \cdot)_x(y) = C\|y\|$, thus for an operator $T : D \rightarrow H$ of class C^1 we conclude the following:

- (i) If $\langle (dT)_x(y), y \rangle \geq C\|y\|^2$ for all $y \in H$, where $C > 0$, then T is strongly monotone.
- (ii) If $\langle (dT)_x(y), y \rangle \geq 0$ for all $y \in H$, then T is Minty-Browder monotone.
- (iii) If $\langle (dT)_x(y), y \rangle \geq -C\|y\|^2$ for all $y \in H$, where $C > 0$, then T is C -relaxed monotone.

We conclude this section with several examples of θ -monotone operators. The first one provides a θ -monotone operator, which is not m -relaxed monotone (with $m \in (0, 1)$), premonotone or Minty-Browder monotone.

Example 4.12. Let $\theta : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $\theta((x, y), (u, v)) = -\sqrt{2}|(x - u) + (y - v)|$, and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (x - y, x - y)$. Then T is θ -monotone, but it is not Minty-Browder monotone, premonotone or C -relaxed monotone with $C \in (0, 1)$.

Indeed, it can be easily checked that the function θ has the (m) property on \mathbb{R}^2 , as well that $\mathfrak{d}\theta((x, y), \cdot)_{(x,y)}(u, v) = -\sqrt{2}|u + v|$.

Since $(dT)_{(x,y)}(u, v) = (u - v, u - v)$ it can be easily proved that

$$\langle (dT)_{(x,y)}(u, v), (u, v) \rangle \geq \mathfrak{d}\theta((x, y), \cdot)_{(x,y)}(u, v) \|(u, v)\|.$$

According to Remark 4.10 T is θ -monotone.

Since $\langle T(x, y), (x, y) \rangle = x^2 - y^2$ we obtain that the linear operator T is not positive semi-definite, so it is not monotone in Minty-Browder sense.

Suppose that T is premonotone. Then there exists a function $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ such that for all $(x, y), (u, v) \in \mathbb{R}^2$, one has $\langle T(x, y) - T(u, v), (x, y) - (u, v) \rangle \geq -\sigma(u, v) \|(x, y) - (u, v)\|$, therefore $(x - u)^2 - (y - v)^2 \geq -\sigma(u, v) \sqrt{(x - u)^2 + (y - v)^2}$ or equivalently $\sigma(u, v) \geq \frac{(x - u)^2 - (y - v)^2}{\sqrt{(x - u)^2 + (y - v)^2}}$. Let (u, v) be fixed and take $x = u$.

Then $\sigma(u, v) \geq \frac{(y - v)^2}{|y - v|}$ for all $y \in \mathbb{R}$. Taking the limit $y \rightarrow \infty$ we obtain $\sigma(u, v) \geq \infty$, a contradiction, consequently, T is not premonotone.

Suppose now that T is C -relaxed monotone, with $C \in (0, 1)$, i.e. there exists $m \in (0, 1)$ such that for all $(x, y), (u, v) \in \mathbb{R}^2$ one has

$$\langle T(x, y) - T(u, v), (x, y) - (u, v) \rangle \geq -C \|(x, y) - (u, v)\|^2. \text{ We have } (x - u)^2 - (y - v)^2 \geq -C((x - u)^2 + (y - v)^2) \text{ or, equivalently, } C \geq \frac{(x - u)^2 - (y - v)^2}{(x - u)^2 + (y - v)^2}.$$

Let (u, v) be fixed and take $y = v$. Then $C \geq \frac{(x - u)^2}{(x - u)^2} = 1$ for all $x \in \mathbb{R}$.

Contradiction, consequently T is not C -relaxed monotone.

Recall that an operator $T : D \subseteq X \rightarrow X^*$ is called *quasimonotone* if for all $x, y \in D$, $\langle Tx, y - x \rangle > 0$ implies $\langle Ty, y - x \rangle \geq 0$. The next example provides a θ -monotone operator which is not even quasimonotone.

Example 4.13. Let be $\theta : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $\theta((x, y), (u, v)) = -\sqrt{2}|(x - u) + 3(y - v)|$, and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (x + 2y, 2x + 3y)$. Then T is θ -monotone, but it is not even quasimonotone.

Indeed, it can be easily checked that the function θ has the (m) property on \mathbb{R}^2 , as well that $\mathfrak{d}\theta((x, y), \cdot)_{(x,y)}(u, v) = -\sqrt{2}|u + 3v|$. Since $(dT)_{(x,y)}(u, v) = (u + 2v, 2u + 3v)$, it can be easily proved that

$$\langle (dT)_{(x,y)}(u, v), (u, v) \rangle \geq \mathfrak{d}\theta((x, y), \cdot)_{(x,y)}(u, v) \|(u, v)\|.$$

According to Remark 4.10, T is θ -monotone.

On the other hand, let $(x, y) = \left(-\frac{1}{2}, \frac{1}{4}\right)$ and $(u, v) = \left(\frac{1}{4}, -\frac{1}{8}\right)$. Then $\langle T(x, y), (u, v) - (x, y) \rangle = \frac{3}{32} > 0$ and $\langle T(u, v), (u, v) - (x, y) \rangle = -\frac{3}{64} < 0$, which shows that T is not quasimonotone.

The next example provides a nonlinear θ -monotone operator.

Example 4.14. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (-y, x^2)$ and $\theta : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $\theta((x, y), (u, v)) = -\frac{|x+u-1|}{2} \sqrt{(x-u)^2 + (y-v)^2}$. Then T is θ -monotone but it is not monotone in Minty-Browder sense.

Indeed, since $\langle T(x, y) - T(u, v), (x, y) - (u, v) \rangle = (x+u-1)(x-u)(y-v)$, which obviously is not always nonnegative, we obtain that T it is not monotone in Minty-Browder sense. It can be easily checked that $(x+u-1)(x-u)(y-v) \geq -\frac{|x+u-1|}{2}((x-u)^2 + (y-v)^2)$, hence $\langle T(x, y) - T(u, v), (x, y) - (u, v) \rangle \geq \theta((x, y), (u, v))\|(x, y) - (u, v)\|$.

5. θ -CONVEX FUNCTIONS

In this section we introduce the concept of θ -convexity for real valued functions in Hilbert spaces. This concept generalizes some convexity notions known in literature, such as strong convexity and ϵ -convexity. We will show that this notion is strongly connected with the notion of θ -monotonicity, namely that a differentiable θ -convex function has as differential a 2θ -monotone operator with the same θ . Further, we will give an analytical condition upon θ that provides the θ -convexity of a differentiable real valued function. Everywhere in the sequel D denotes an open and convex subset of a real Hilbert space H , while the Fréchet differential of a function $f : D \rightarrow \mathbb{R}$ at $x \in D$ will be identified with $\nabla f(x)$.

Definition 5.1. Let $\theta : D \times D \rightarrow \mathbb{R}$ be a given function with the property that $\theta(x, y) = \theta(y, x)$ for all $x, y \in D$. One says that the function $f : D \rightarrow \mathbb{R}$ is θ -convex, if for all $x, y \in D$ and all $z \in (x, y)$ we have

$$(8) \quad \frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} + \theta(x, z) + \theta(z, y) \leq 0.$$

It can be easily observed that (8) is equivalent to $f((1-t)x + ty) \leq (1-t)f(x) + tf(y) - t(1-t)(\theta(x, (1-t)x + ty) + \theta((1-t)x + ty, y))\|x - y\|$, for all $t \in [0, 1]$ and all $x, y \in D$. Obviously, if $\theta(x, y) = \frac{c}{2}\|x - y\|$ for all $x, y \in D$ where $c \in \mathbb{R}_+ \setminus \{0\}$, we obtain the concept of strong convexity on D , while if $\theta(x, y) = 0$ for all $x, y \in D$, we obtain the concept of "classical" convexity on D .

One may ask what happens if in (8) we replace $\theta(x, z) + \theta(z, y)$ with $\theta(x, y)$. In this case a new notion of convexity defined by means of the function θ , the so called weak θ -convexity is obtained.

Definition 5.2. We say that the function $f : D \rightarrow \mathbb{R}$ is weak θ -convex if for all $x, y \in D$ and all $z \in (x, y)$ we have

$$(9) \quad \frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} + \theta(x, y) \leq 0.$$

It can be easily observed that (9) is equivalent to $f((1-t)x+ty) \leq (1-t)f(x) + tf(y) - t(1-t)\theta(x,y)\|x-y\|$, for all $t \in [0, 1]$ and all $x, y \in D$.

We say that the function $f : D \rightarrow \mathbb{R}$ is locally θ -convex, respectively, locally weak θ -convex, if for every $x_0 \in D$ there exists an open and convex neighborhood $U_{x_0} \subseteq D$ of x_0 such that the restriction of f on U_{x_0} , $f|_{U_{x_0}}$ is θ -convex, respectively weak θ -convex.

It can be easily observed, that in the case of $\theta(x, y) = C\|x - y\|$, where $C \in \mathbb{R}$, the notions of θ -convexity and of weak θ -convexity coincides. Even more, if $\theta(x, z) + \theta(z, y) = \theta(x, y)$ for all $x, y \in D$, $x \neq y$, $z \in (x, y)$, then the notions of θ -convexity and weak θ -convexity coincide. Obviously, if θ has the (m) property, then θ -convexity for a function implies its weak θ -convexity.

A local convexity notion, the so called *approximate convexity* (see for instance [6, 7]), is well known in literature. Recall that the function $f : D \rightarrow \mathbb{R}$ is called approximately convex, if for all $\epsilon > 0$ and all $x_0 \in D$ there exists $\delta > 0$, such that for every $x, y \in B(x_0, \delta)$ and $t \in (0, 1)$ one has

$$f((1-t)x+ty) \leq (1-t)f(x) + tf(y) + \epsilon t(1-t)\|x-y\|.$$

This notion is closely related to the notion of submonotonicity, i.e. a differentiable function is approximately convex if and only if its differential is submonotone.

It is easy to prove, that if θ is lower semicontinuous in (x, x) and $\theta(x, x) \geq 0$ for all $x \in D$, then a locally θ -convex, respectively, a locally weak θ -convex function is actually approximately convex. Indeed, according to the proof of Proposition 4.2 in Section 4, for all $\epsilon > 0$ there exists $\delta > 0$ such that $\theta(x, y) \geq -\frac{\epsilon}{2}$, respectively $\theta(x, y) \geq -\epsilon$ for all $x, y \in B(x_0, \delta) \subseteq U_{x_0}$, where U_{x_0} is an open neighborhood of x_0 where θ -convexity, respectively weak θ -convexity holds.

Since the function $\theta(x, y) = C\|x - y\|$, where $C \in \mathbb{R}$ is lower semicontinuous and $\theta(x, x) = 0$, any θ -convex function with $\theta(x, y) = C\|x - y\|$ is actually approximately convex.

If $\theta(x, y) = -C\|x - y\|^{\gamma-1}$, with $C > 0$ and $\gamma > 1$, then θ is lower semicontinuous and $\theta(x, x) = 0$. In this case a weak θ -convex function is called γ -*paraconvex*, (see [17]), i.e.

$$f((1-t)x+ty) \leq (1-t)f(x) + tf(y) + Ct(1-t)\|x-y\|^\gamma$$

for all $t \in [0, 1]$ and all $x, y \in D$.

It follows that a γ -paraconvex function is approximately convex.

The next result connects the θ -convexity property of a differentiable function with the 2θ -monotonicity property of its differential.

Proposition 5.3. *If $f : D \rightarrow \mathbb{R}$ is a differentiable θ -convex function, where $\theta(x, \cdot) : D \rightarrow \mathbb{R}$ is radially continuous and $\theta(x, x) = 0$ for all $x \in D$, then ∇f*

is 2θ -monotone, with the same θ . If $D = X$ and ∇f is hemicontinuous, then ∇f is maximal 2θ -monotone.

Proof. Let be $x, y \in D$, $x \neq y$. Since f is θ -convex, we have $f((1-t)x+ty) \leq (1-t)f(x) + tf(y) - t(1-t)(\theta(x, (1-t)x + ty) + \theta((1-t)x + ty, y))\|x - y\|$ for all $t \in [0, 1]$. This latter relation can be written as $\frac{f(x + t(y-x)) - f(x)}{t} \leq f(y) - f(x) - (1-t)(\theta(x, (1-t)x + ty) + \theta((1-t)x + ty, y))\|x - y\|$ for all $t \neq 0$, and taking the limit $t \downarrow 0$ we obtain that $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) - (\theta(x, x) + \theta(x, y))\|x - y\| = f(y) - f(x) - \theta(x, y)\|x - y\|$. In the same way we obtain that $\langle \nabla f(y), x - y \rangle \leq f(x) - f(y) - \theta(y, x)\|y - x\| = f(x) - f(y) - \theta(y, x)\|y - x\|$. By summing the two relations we get $\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 2\theta(y, x)\|y - x\|$.

If $D = X$ and ∇f is hemicontinuous, then the result follows from Proposition 3.9. \blacksquare

Remark 5.4. If $f : D \rightarrow \mathbb{R}$ is a differentiable θ -convex function, where $\theta(x, \cdot) : D \rightarrow \mathbb{R}$ is radially continuous and $\theta(x, x) = 0$ for all $x \in D$, one has, as follows from the proof of Proposition 5.3, that $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) - \theta(x, y)\|x - y\|$ for all $x, y \in D$. The converse statement also holds under milder assumptions.

Proposition 5.5. Let $f : D \rightarrow \mathbb{R}$ be a differentiable function. If for all $x, y \in D$ $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) - \theta(x, y)\|x - y\|$, then f is θ -convex.

Proof. Let $z = (1-t)x + ty = x + t(y-x)$, $t \in [0, 1]$. We have $f(x) \geq f(z) + \langle \nabla f(z), x - z \rangle + \theta(x, z)\|z - x\|$ and $f(y) \geq f(z) + \langle \nabla f(z), y - z \rangle + \theta(z, y)\|y - z\|$. Multiplying the first inequality by $1-t$ and the second one by t and summing we obtain $(1-t)f(x) + tf(y) \geq f(z) + \langle \nabla f(z), (1-t)x + ty - z \rangle + (1-t)\theta(x, z)\|z - x\| + t\theta(z, y)\|y - z\|$. But $(1-t)x + ty - z = 0$ and $\|z - x\| = t\|x - y\|$, $\|y - z\| = (1-t)\|x - y\|$, therefore we obtain $(1-t)f(x) + tf(y) \geq f((1-t)x + ty) + (1-t)t(\theta(x, z) + \theta(z, y))\|x - y\|$, which shows that f is θ -convex. \blacksquare

Next we will give a condition involving the function θ , such that the 2θ -monotonicity of the differential of a differentiable function provides the θ -convexity property of that function.

Theorem 5.6. If $f : D \rightarrow \mathbb{R}$ is a continuously differentiable function, the function $s : [0, 1] \rightarrow \mathbb{R}$, $s(t) = \theta(x, x+t(y-x))$ is integrable with $\int_0^1 s(t)dt \geq \frac{\theta(x, y)}{2}$ for all $x, y \in D$, $x \neq y$, and ∇f is 2θ -monotone, then f is θ -convex.

Proof. For $x, y \in D$, $x \neq y$, let be $g : [0, 1] \rightarrow \mathbb{R}$, $g(t) = f(x+t(y-x))$. Then $g'(t) = \langle \nabla f(x+t(y-x)), y-x \rangle$, $t \in (0, 1)$ and $g'(0) = \lim_{t \downarrow 0} g'(t) = \langle \nabla f(x), y-x \rangle$.

We get $g'(t) - g'(0) = \langle \nabla f(x + t(y - x)) - \nabla f(x), y - x \rangle = \frac{1}{t} \langle \nabla f(x + t(y - x)) - \nabla f(x), (x + t(y - x)) - x \rangle$ and according to the 2θ -monotonicity of ∇f we obtain $g'(t) - g'(0) \geq 2\theta(x, x + t(y - x))\|x - y\|$.

On the other hand $f(y) - f(x) - \langle \nabla f(x), y - x \rangle = g(1) - g(0) - g'(0) = \int_0^1 [g'(t) - g'(0)]dt$, therefore $f(y) - f(x) - \langle \nabla f(x), y - x \rangle \geq 2 \int_0^1 \theta(x, x + t(y - x))\|x - y\|dt \geq \theta(x, y)\|x - y\|$. The statement follows from Proposition 5.5. ■

If $\theta(x, y) = C\|x - y\|$, where $C \in \mathbb{R}$, we have that $s(t) = Ct\|y - x\|$ is integrable on $[0, 1]$ and $\int_0^1 s(t)dt = \frac{\theta(x, y)}{2} = \frac{C}{2}\|y - x\|$ for all $x, y \in D$, $x \neq y$, even more $\theta(x, \cdot) : D \rightarrow \mathbb{R}$ is radially continuous and $\theta(x, x) = 0$ for all $x \in D$. Thus according to Proposition 5.3 and Theorem 5.6 we conclude the following:

Proposition 5.7. *Let be $\theta(x, y) = C\|x - y\|$, with $C \in \mathbb{R}$. A differentiable function $f : D \rightarrow \mathbb{R}$ is θ -convex, if and only if ∇f is 2θ -monotone.*

Remark 5.8. If the function $\theta(x, \cdot)$ is concave and $\theta(x, x) = 0$ then $\int_0^1 \theta(x, x + t(y - x))dt \geq \frac{\theta(x, y)}{2}$ for all $x, y \in D$, $x \neq y$.

Indeed, since $\theta(x, \cdot)$ is concave and $\theta(x, x) = 0$ we have $\theta(x, x + t(y - x)) \geq (1 - t)\theta(x, x) + t\theta(x, y) = t\theta(x, y)$ for all $t \in [0, 1]$. Consequently, $\int_0^1 \theta(x, x + t(y - x))dt \geq \int_0^1 t\theta(x, y)dt = \frac{\theta(x, y)}{2}$.

Corollary 5.9. *Let $f : D \rightarrow \mathbb{R}$ be a differentiable function, $\theta(x, \cdot)$ concave and $\theta(x, x) = 0$ for all $x \in D$. The following assertions are true:*

- (a) *If f is locally θ -convex, then ∇f is 2θ -monotone.*
- (b) *If ∇f is locally 2θ -monotone, then f is θ -convex.*

Proof. a) Suppose that f is locally θ -convex. By Proposition 5.3, ∇f is locally 2θ -monotone and, according to Remark 4.4, the function θ has the (m) property. The result follows from Theorem 4.5.

b) Suppose that ∇f is locally 2θ -monotone. Since θ has the (m) property, from Theorem 4.5 we obtain that ∇f is 2θ -monotone. According to Remark 5.8, $\int_0^1 \theta(x, x + t(y - x))dt \geq \frac{\theta(x, y)}{2}$ for all $x, y \in D$, $x \neq y$. The result follows from Theorem 5.6. ■

Obviously, for a differentiable and weak θ -convex function Proposition 5.3 can be reformulated as follows:

Proposition 5.10. *If $f : D \rightarrow \mathbb{R}$ is a differentiable and weak θ -convex function, then ∇f is 2θ -monotone, with the same θ .*

The proof is identical to the proof of Proposition 5.3, therefore we omit it.

Remark 5.11. If in Proposition 5.5 θ has in addition the (m) property on D , it follows that the condition $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) - \theta(x, y)\|x - y\|$ for all $x, y \in D$, provides the weak θ -convexity of f , since in this case a θ -convex function is weak θ -convex. In the same manner we conclude that the assumptions of Theorem 5.6, in the case when θ has the (m) property on D , provides the weak θ -convexity of f .

In the case of γ -paraconvex functions, one has to define $\theta(x, y) = -C\|x - y\|^{\gamma-1}$, where $C > 0$ and $\gamma > 1$, hence θ has the (m) property on D if and only if $\gamma \geq 2$. In this case, from Theorem 5.6 and Remark 5.11 it follows that a function having its differential a γ -paramonotone operator is γ -paraconvex. Indeed, we have $\int_0^1 s(t)dt = \int_0^1 -C\|x - y\|^{\gamma-1}t^{\gamma-1}dt = \frac{-C\|x - y\|^{\gamma-1}}{\gamma} \geq \frac{-C\|x - y\|^{\gamma-1}}{2} = \frac{\theta(x, y)}{2}$.

If $\theta(x, y) = -\frac{\epsilon}{2}$, respectively $\theta(x, y) = -\epsilon$ for all $x, y \in D$, where $\epsilon > 0$ then a θ -convex, respectively, a weak θ -convex function is called ϵ -convex (see [10, 17]), i.e.

$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + \epsilon t(1-t)\|x - y\|$ for all $t \in [0, 1]$ and all $x, y \in D$.

This notion is closely related to the notion of ϵ -monotonicity, namely, a differentiable function f is ϵ -convex, if and only if ∇f is 2ϵ -monotone.

However, the only if part of this result does not follow nor from Theorem 5.6 neither from Remark 5.11, since in this case $\int_0^1 -\frac{\epsilon}{2}dt = -\frac{\epsilon}{2} < \frac{\theta(x, y)}{2}$ for all $x, y \in D$ and θ does not have the (m) property on D . The next theorem solves this problem, as we will see further.

Theorem 5.12. *If $f : D \rightarrow \mathbb{R}$ is a differentiable function, and the function θ has the property that $2\theta(u, v) \geq \theta(x, z) + \theta(z, y)$ for all $x, y \in D$, $x \neq y$, $z \in (x, y)$, $u \in (x, z)$, $v \in (z, y)$, and ∇f is 2θ -monotone, then f is θ -convex.*

Proof. Let be $x, y \in D$, $x \neq y$ and $z \in (x, y)$. According to the mean value theorem, there exists $u \in (x, z)$ such that $f(z) - f(x) = \langle \nabla f(u), z - x \rangle$ and there exists $v \in (z, y)$ such that $f(z) - f(y) = \langle \nabla f(v), z - y \rangle$. Using these relations we obtain

$$(10) \quad \frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} = \left\langle \nabla f(u), \frac{z - x}{\|z - x\|} \right\rangle + \left\langle \nabla f(v), \frac{z - y}{\|z - y\|} \right\rangle$$

Since $z = x + t(y - x)$ for some $t \in (0, 1)$, we get $\frac{z - x}{\|z - x\|} = \frac{y - x}{\|y - x\|}$ and $\frac{z - y}{\|z - y\|} = -\frac{y - x}{\|y - x\|}$, and since $u = x + s(y - x)$, $v = x + p(y - x)$ for some $s, p \in [0, 1]$, $p > s$ we obtain that $\frac{y - x}{\|y - x\|} = -\frac{u - v}{\|u - v\|}$.

Therefore (10) becomes

$$\frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} = \left\langle \nabla f(u), -\frac{u - v}{\|u - v\|} \right\rangle + \left\langle \nabla f(v), \frac{u - v}{\|u - v\|} \right\rangle.$$

Using the 2θ -monotonicity of ∇f from the assumptions we obtain

$$\frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} \leq -2\theta(u, v) \leq -\theta(x, z) - \theta(z, y). \quad \blacksquare$$

Remark 5.13. If $\theta(x, y) = -\frac{\epsilon}{2}$ we have that $2\theta(u, v) = \theta(x, z) + \theta(z, y)$, therefore, if a differentiable function f has its differential a 2ϵ -monotone operator, then f is ϵ -convex.

The next results connect the θ -convexity of a function with an analytical condition upon its Hessian.

Proposition 5.14. Let $f \in C^2(D)$ be a θ -convex function, where $\theta(x, \cdot) : D \rightarrow \mathbb{R}$ is radially continuous and $\theta(x, x) = 0$ for all $x \in D$. If in addition, the function $\theta(x, \cdot)$ has directional derivatives in every $y \in H$, then $\langle \nabla^2 f(x)y, y \rangle \geq 2\partial\theta(x, \cdot)_x(y)\|y\|$ for all $x \in D$ and all $y \in H$.

Proof. Since the conditions in Remark 5.4 are satisfied, we have $f(x + ty) - f(x) \geq t\langle \nabla f(x), y \rangle + t\theta(x, x + ty)\|y\|$, for all $x \in D$, all $y \in H$ and all $t > 0$ such that $x + ty \in D$. On the other hand, from Taylor's formula we obtain $f(x + ty) = f(x) + t\langle \nabla f(x), y \rangle + \frac{1}{2}t^2\langle \nabla^2 f(x)y, y \rangle + o(\|ty\|^2)$, where $\lim_{t \searrow 0} \frac{o(\|ty\|^2)}{t^2} = 0$.

These two relations lead to $\frac{1}{2}t^2\langle \nabla^2 f(x)y, y \rangle + o(\|ty\|^2) \geq t\theta(x, x + ty)\|y\|$, while dividing by t^2 we obtain $\langle \nabla^2 f(x)y, y \rangle + \frac{o(\|ty\|^2)}{t^2} \geq 2\frac{\theta(x, x + ty)\|y\|}{t} = 2\frac{(\theta(x, x + ty) - \theta(x, x))\|y\|}{t}$. Taking the limit $t \searrow 0$ we obtain

$$\langle \nabla^2 f(x)y, y \rangle \geq 2\partial\theta(x, \cdot)_x(y)\|y\|. \quad \blacksquare$$

The converse also holds under some supplementary assumptions.

Proposition 5.15. *Let $f \in C^2(D)$ and let the function $s : [0, 1] \rightarrow \mathbb{R}$, $s(t) = \theta(x, x + t(y - x))$ be integrable with $\int_0^1 s(t)dt \geq \frac{\theta(x, y)}{2}$ for all $x, y \in D$. If, in addition, θ has the (m) property, $\theta(x, x) = 0$ for all $x \in D$ and $\langle \nabla^2 f(x)y, y \rangle \geq 2\partial\theta(x, \cdot)_x(y)\|y\|$ for all $x \in D$ and all $y \in H$, then f is θ -convex.*

Proof. The statement follows from Theorem 4.9 and Theorem 5.6. ■

Corollary 5.16. *If $\theta(x, \cdot)$ is concave and $\theta(x, x) = 0$, then the condition $\langle \nabla^2 f(x)y, y \rangle \geq 2\partial\theta(x, \cdot)_x(y)\|y\|$ for all $x \in D$ and all $y \in H$, provides that f is θ -convex.*

Proof. According to Remark 4.4 and Remark 5.8 the assumptions of Proposition 5.15 are satisfied. ■

Recall that a function $f : D \rightarrow \mathbb{R}$ is called *quasiconvex* on D , if for all $x, y \in D$ and all $z \in [x, y]$, $f(z) \leq \max\{f(x), f(y)\}$. In what follows we will give an example of a θ -convex function which is not even quasiconvex.

Example 5.17. Let $\theta : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $\theta((x, y), (u, v)) = -\frac{\sqrt{2}}{2}|(x - u) + 3(y - v)|$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = \frac{x^2}{2} + 2xy + \frac{3y^2}{2}$. Then f is θ -convex but it is not even quasiconvex.

Since the function f is differentiable, its differential may be identified by the gradient operator $\nabla f(x, y) = (x + 2y, 2x + 3y)$, which is a 2θ -monotone operator as we have seen in the Example 4.13. It can be easily proved that the function $s : [0, 1] \rightarrow \mathbb{R}$, $s(t) = \theta((x, y), (x, y) + t((u, v) - (x, y)))$ satisfies $\int_0^1 s(t)dt = \frac{\theta((x, y), (u, v))}{2}$ for all $x, y \in D$, $x \neq y$, therefore, according to Theorem 5.6, the function f is θ -convex.

On the other hand $f(0, 0) = 0$, $f\left(\frac{1}{2}, -\frac{1}{4}\right) = -\frac{1}{32}$ and $f\left(-\frac{1}{4}, \frac{1}{8}\right) = -\frac{1}{128}$, and since

$$(0, 0) = \left(\frac{1}{2}, -\frac{1}{4}\right) + \frac{2}{3} \left(\left(-\frac{1}{4}, \frac{1}{8}\right) - \left(\frac{1}{2}, -\frac{1}{4}\right) \right)$$

we obtain $f(0, 0) > \max\left\{f\left(\frac{1}{2}, -\frac{1}{4}\right), f\left(-\frac{1}{4}, \frac{1}{8}\right)\right\}$, which shows that f is not quasiconvex.

6. APPLICATIONS TO SURJECTIVITY RESULTS

In what follows we will provide some surjectivity results involving θ -monotone operators in the case when $X = \mathbb{R}^n$.

Recall that an operator having $\|\cdot\| \times \|\cdot\|$ closed graph in $X \times X^*$ is called *outer semi-continuous*. The operator $T : \mathbb{R}^n \rightarrow 2^{X^*}$ is called *coercive*, if

$$\lim_{\|x\| \rightarrow \infty} \frac{\inf_{u \in Tx} \langle u, x \rangle}{\|x\|} = \infty.$$

We need the following theorem proved in [9]:

Theorem 6.1. *If $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is a locally bounded, convex-valued, coercive and outer semi-continuous operator, as well as $D(T) = \mathbb{R}^n$, then T is surjective.*

Obviously if the operator T is θ -monotone, then $T + \lambda I$ is θ -monotone (with the same θ), for all $\lambda > 0$, where I denotes the identity operator. Even more, $T + \lambda I$ is θ' monotone, with $\theta'(x, y) = \theta(x, y) + \lambda\|x - y\|$.

Theorem 6.2. *If the operator $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is θ -monotone, convex valued, outer semi-continuous and $D(T) = \mathbb{R}^n$, as well as the function $\theta(\cdot, y) : \mathbb{R}^n \rightarrow \mathbb{R}$ is lower semicontinuous for all $y \in \mathbb{R}^n$ and the function $\theta(\cdot, 0) : \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded below, then $T + \lambda I$ is surjective for all $\lambda > 0$.*

Proof. Obviously $T + \lambda I$ is θ -monotone for every $\lambda > 0$ and, according to Theorem 2.5, it is locally bounded in the interior of $D(T + \lambda I)$. Since $D(T) = \mathbb{R}^n$, we get that $D(T + \lambda I) = \mathbb{R}^n$, therefore $T + \lambda I$ is locally bounded on \mathbb{R}^n .

Easily can be checked, that $T + \lambda I$ is outer semi-continuous and convex valued. Remains to show the coercivity. Let be $v \in T(0)$. Then for all $x \in \mathbb{R}^n$, $\inf_{w \in (T + \lambda I)(x)} \langle w, x \rangle = \inf_{u \in Tx} \langle u + \lambda x, x \rangle = \inf_{u \in Tx} \langle u, x \rangle + \lambda\|x\|^2 = \inf_{u \in Tx} \langle u - v, x - 0 \rangle + \langle v, x \rangle + \lambda\|x\|^2 \geq \theta(x, 0)\|x\| - \|v\|\|x\| + \lambda\|x\|^2$. Dividing by $\|x\|$ and taking the limit $\|x\| \rightarrow \infty$ we obtain:

$$\lim_{\|x\| \rightarrow \infty} \frac{\inf_{u \in Tx} \langle u + \lambda x, x \rangle}{\|x\|} \geq \lim_{\|x\| \rightarrow \infty} (\lambda\|x\| + \theta(x, 0) - \|v\|) = \infty. \quad \blacksquare$$

The next Minty's type theorem ensures the surjectivity of $T + \lambda I$, when T is maximal θ -monotone.

Theorem 6.3. *Let $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a maximal θ -monotone operator with $D(T) = \mathbb{R}^n$. If $\theta(\cdot, y) : \mathbb{R}^n \rightarrow \mathbb{R}$ is lower semicontinuous for all $y \in \mathbb{R}^n$ and the function $\theta(\cdot, 0) : \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded below, then the operator $T + \lambda I$ is surjective for all $\lambda > 0$.*

Proof. As we have seen, in this case T is locally bounded on $\text{int } D(T) = \mathbb{R}^n$, as well as, according to Theorem 3.6 and Proposition 3.8, it is convex-valued and outer semi-continuous. In the proof of the precedent theorem we have shown, that in such case, $T + \lambda I$ is locally bounded, convex-valued, outer semi-continuous and coercive. Since $D(T + \lambda I) = \mathbb{R}^n$, according to Theorem 6.1, $T + \lambda I$ is surjective. ■

7. FINAL REMARKS AND COMMENTS

Since the concepts of θ -monotonicity and θ -convexity contain many monotonicity, respectively, convexity concepts as particular cases, the possibilities of further investigations are considerable.

For instance, it is natural to introduce a new subdifferential concept, the so-called θ -subdifferential.

Let X be a real Banach space and $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ a proper function. One says that $x^* \in X^*$ is a θ -subgradient of f in $x \in \text{dom}(f) = \{x \in X : f(x) < \infty\}$, if $\langle x^*, y - x \rangle \leq f(y) - f(x) - \theta(x, y)\|x - y\|$, $(\forall)y \in X$. The set

$$\partial_{\theta} f(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq f(y) - f(x) - \theta(x, y)\|x - y\|, (\forall)y \in X\}$$

is called the θ -subdifferential of f at $x \in \text{dom}(f)$.

According to the proof of Proposition 5.3, if $f : X \rightarrow \mathbb{R}$ is a differentiable, θ -convex function, where $\theta(x, \cdot) : D \rightarrow \mathbb{R}$ is radially continuous and $\theta(x, x) = 0$ for all $x \in \text{dom}(f)$, then $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) - \theta(x, y)\|x - y\|$ for all $y \in X$, which shows that in this case $\nabla f(x) \in \partial_{\theta} f(x)$. Moreover if ∇f is hemicontinuous then ∇f is maximal 2θ -monotone.

It is easy to check that $\partial_{\theta} f : X \rightarrow 2^{X^*}$ is a 2θ -monotone operator. Indeed let $x^* \in \partial_{\theta} f(x)$, $y^* \in \partial_{\theta} f(y)$. Then $\langle x^*, y - x \rangle \leq f(y) - f(x) - \theta(x, y)\|x - y\|$ and $\langle y^*, x - y \rangle \leq f(x) - f(y) - \theta(y, x)\|y - x\|$, which added give us $\langle y^* - x^*, y - x \rangle \geq 2\theta(x, y)\|x - y\|$.

It is worthwhile to investigate in the future the properties of this new subdifferential concept, due to the fact that the concept of cyclically θ -monotonicity may be introduced as well. One says that the operator $T : X \rightarrow 2^{X^*}$ is *cyclically θ -monotone*, if for all integer $n \geq 2$ and for arbitrary cycle $x_0, x_1, \dots, x_n = x_0$ and $u_i \in Tx_i$, $i = 1, 2, \dots, n$, we have

$$\sum_{i=1}^n \langle u_{i-1}, x_{i-1} - x_i \rangle \geq \sum_{i=1}^n \theta(x_{i-1}, x_i)\|x_{i-1} - x_i\|.$$

From here comes a new idea for future investigations, namely to establish conditions on θ such that the generalization of celebrated result of Rockafellar remain true (see [21]): a lower semicontinuous proper function is θ -convex, if and only if

its subdifferential a maximal cyclically θ -monotone operator. The works [6, 10, 23] are excellent starting points to develop these ideas.

ACKNOWLEDGMENTS

The author express his sincere thanks to Dr. R.I. Boş for his suggestions and help given for the improvement of this paper and is grateful also to two anonymous referees for their helpful comments and suggestions, which led to improvement of the originally submitted version of this work.

The author wishes to thank for the financial support provided from programs co-financed by The SECTORAL OPERATIONAL PROGRAMME HUMAN RESOURCES DEVELOPMENT, Contract POSDRU 6/1.5/S/3 -“Doctoral studies: through science towards society”.

REFERENCES

1. J. M. Borwein, Maximality of sums of two maximal monotone operators in general Banach space, *Proc. Am. Math. Soc.*, **135**(12) (2007), 3917-3924.
2. F. E. Browder, Multi-valued monotone nonlinear mappings, *Trans. AMS*, **118** (1965), 338-551.
3. F. E. Browder, Nonlinear maximal monotone mappings in Banach spaces, *Math. Ann.*, **175** (1968), 81-113, Springer, 2008.
4. F. E. Browder, The fixed point theory of multi-valued mappings in topological vector spaces, *Math. Ann.*, **177** (1968), 283-301.
5. R. S. Burachik and A. N. Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, Springer Optimization and Its Applications, Springer, US, 2008.
6. A. Daniilidis and P. Georgiev, Approximate convexity and submonotonicity, *J. Math. Anal. Appl.*, **291** (2004), 292-301.
7. P. Georgiev, Submonotone mappings in Banach spaces and applications, *Set-Valued Analysis*, **5** (1997), 1-35.
8. N. Hadjisavas and S. Schaible, Generalized monotone maps, in: *Handbook of Generalized Convexity and Generalized Monotonicity*, N. Hadjisavas, S. Komlósi and S. Schaible, eds.; Springer, Series Nonconvex Optimization and Its Applications, USA, 2005, pp. 389-420.
9. A. Iusem, G. Kassay and W. Sosa, An existence result for equilibrium problems with some surjectivity consequences, *Journal of Convex Analysis*, **16**(3&4) (2009), 807-826.
10. A. Jofré, D. T. Luc and M. Théra, ϵ -Subdifferential and ϵ -Monotonicity, *Nonlinear Analysis*, **33** (1998), 71-90.
11. A. Jourani, Subdifferentiability and Subdifferential monotonicity of γ parconvex functions, *Control Cibernet*, **25** (1996), 721-737.

12. G. Kassay and J. Kolumbán, Multivalued parametric variational inequalities with α -pseudomonotone maps, *Journal of Optimization Theory and Applications*, **107(1)** (2000), 35-50.
13. G. Kassay and C. Pinteá, On preimages of a class of generalized monotone operators, *Nonlinear Analysis Series A: Theory, Methods & Applications*, **73(11)** (2010), 3537-3545.
14. G. J. Minty, Monotone (nonlinear) operators in Hilbert spaces, *Duke Math. J.*, **29** (1962), 341-346.
15. G. J. Minty, On some aspects of theory of monotone operators, in: *Theory and Applications of Monotone Operators*, Odersi, Gubbio, 1969, pp. 67-82.
16. B. S. Mordukhovich, *Variational Analysis and Generalized Differentiations. I. Basic Theory, II. Applications*, Springer, Series Fundamental Principles of Mathematical Sciences, Vol. 330, p. 601; Vol. 331, p. 632, 2006.
17. D. T. Luc, H. V. Ngai and M. Théra, On ϵ -convexity and ϵ -monotonicity, in *Calculus of Variations and Differential Equations*, A. Ioffe, S. Reich and I. Shafir, (eds.), Research Notes in Mathematics Series, Chapman & Hall, 1999, pp. 82-100.
18. H. V. Ngai and J. P. Penot, In Asplund spaces, approximately convex functions and regular functions are generically differentiable, *Taiwanese J. Math.*, **12(6)** (2008), 1477-1492.
19. R. G. Otero and A. Iusem, *Regularity Results for Semimonotone Operators*, http://www.preprint.impa.br/Shadows/SERIE_A/2010/672.html
20. R. John, Uses of Generalized Convexity and Generalized Monotonicity in Economics, in: *Handbook of Generalized Convexity and Generalized Monotonicity*, N. Hadjisavvas, S. Komlósi and S. Schaible, eds., Springer, Series Nonconvex Optimization and Its Applications, USA, 2005, pp. 619-666.
21. R. T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific Journal of Mathematics*, **33(1)** (1970), 209-216.
22. S. Rolewicz, On $\alpha(\cdot)$ -monotone multifunctions and differentiability of γ -paraconvex functions, *Studia Math.*, **133** (1999), 29-37.
23. S. Rolewicz, Φ -convex functions defined on metric spaces, *Journal of Mathematical Sciences*, **115(5)** (2003), 2631-2652.
24. R. U. Verma, A-monotonicity and its role in nonlinear variational inclusions, *Journal of Optimization Theory and Applications*, **129(3)** (2006), 457-467.

Szilárd László
Faculty of Mathematics and Computer Sciences
Babeş-Bolyai University
400084
M. Kogalniceanu 1, Cluj-Napoca
Romania
E-mail: laszloszisi@yahoo.com