TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 2, pp. 635-649, April 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

LEVITIN-POLYAK WELL-POSEDNESS FOR VECTOR QUASI-EQUILIBRIUM PROBLEMS WITH FUNCTIONAL CONSTRAINTS

Jian-Wen Peng, Yan Wang and Soon-Yi Wu*

Abstract. In this paper, four types of Levitin-Polyak well-posedness of vector quasi-equilibrium problems with a functional constraint, as well as an abstract set constraint are investigated. Some criteria and characterizations for these types of Levitin-Polyak well-posedness with or without gap functions of vector quasi-equilibrium problems are obtained. The results in this paper generalize and extend some known results in the literature.

1. INTRODUCTION

It is well known that the well-posedness is very important for both optimization theory and numerical methods of optimization problems, which guarantees that, for approximating solution sequences, there is a subsequence which converges to a solution. The study of well-posedness originates from Tykhonov [1] in dealing with unconstrained optimization problems. Levitin and Polyak [2] extended the notion to constrained (scalar) optimization, allowing minimizing sequences $\{x_n\}$ to be outside of the feasible set X_0 and requiring $d(x_n, X_0)$ (the distance from x_n to X_0) to tend to zero. The Levitin and Polyak well-posedness is generalized in [3, 4] for problems with explicit constraint (i.e., functional constraint) $g(x) \in K$, where g is a continuous map between two metric spaces and K is a closed set. For minimizing sequences $\{x_n\}$, instead of $d(x_n, X_0)$, here the distance $d(g(x_n), K)$ is required to tend to zero. This generalization is appropriate for penalty-type methods

Received December 20, 2009, accepted December 10, 2010.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: 47J05, 47J25, 47H09.
Key words and phrases: Vector quasi-equilibrium problems, Levitin-Polyak well-posedness, Approximating solution sequence, Functional constraint, Gap function.
This research was supported by the National Natural Science Foundation of China (Grants No. 11171863 and Grants No. 10831009), the Natural Science Foundation of Chongqing (Grant No. CSTC, 2009BB8240), the Research Project of Chongqing Normal University (Grant 08XLZ05).
*Corresponding author.

(e.g., penalty function methods, augmented Lagrangian methods) with iteration processes terminating when $d(g(x_n), K)$ is small enough (but $d(x_n, X_0)$ may be large). Recently, the study of generalized Levitin-Polyak well-posedness was extended to nonconvex vector optimization problem problems with abstract and functional constraints (see [5]), variational inequality problems with an abstract set constraint and a functional constraint (see [6]), generalized inequality problems with an abstract set constraint and a functional constraint [7], generalized vector inequality problems with an abstract set constraint and a functional constraint [8], equilibrium problems with an abstract set constraint and a functional constraint [9]. Most recently, Li and Li [10] introduced and researched two types of Levitin-Polyak well-posedness of vector equilibrium problems with variable domination structures. Huang, Long and Zhao [11] introduced and researched the Levitin-Polyak well-posedness of vector quasi-equilibrium problems. Li, Li and Zhang [12] introduced and researched the Levitin-Polyak well-posedness for two types of generalized vector quasi-equilibrium problems. However, there is no study on the (generalized) Levitin-Polyak well-posedness for vector quasi-equilibrium problems with explicit constraint $g(x) \in K$.

Motivated and inspired by the above works, in this paper, we introduce four types of Levitin-Polyak well-posedness of vector quasi-equilibrium problems with functional constraints, as well as an abstract set constraint and investigate criteria and characterizations for these types of Levitin-Polyak well-posedness with or without a gap function for vector quasi-equilibrium problems. The results in this paper generalize and extend some known results in [6, 9, 10].

2. Preliminaries

Let (X, d_X), (Z, d_Z) and Y be locally convex Hausdorff topological vector spaces, where d_X (d_Z) is the metric which compatible with the topology of X (Z). Throughout this paper, we suppose that $K \subset Z$ and $X_1 \subset X$ are nonempty and closed sets, $C : X \to 2^Y$ is a set-valued mapping such that for any $x \in X$, $C(x)$ is a pointed, closed and convex cone in Z with nonempty interior $intC(x)$, $e : X \to Y$ is a continuous vector-valued mapping and satisfies that for any $x \in X$, $e(x) \in intC(x)$, $f : X \times X_1 \to Y$ and $g : X_1 \to Z$ are two vector-valued mappings, $S : X_1 \to 2^{X_1}$ is a strict set-valued map (i.e. $S(x) \neq \varnothing, \forall x \in X_1$), and $X_0 = \{x \in X_1 : g(x) \in K\}$. Let $X_2 = \{x \in X_1 : x \in S(x)\}$. We consider the following vector quasi-equilibrium problem with functional constraints, as well as an abstract set constraint: finding a point $x^* \in X_0$ such that $x^* \in S(x^*)$ and

$$f(x^*, y) \notin -intC(x^*), \forall y \in S(x^*). \text{(VQEP)}$$

We denote by Ω the set of solutions of (VQEP).

Let (P, d) be a metric space, $P_1 \subseteq P$ and $x \in P$. We denote by $d(x, P_1) = \inf\{d(x, p) : p \in P_1\}$ the distance function from the point $x \in P$ to the set P_1.

Throughout this paper, we always assume that $X_0 \neq \varnothing$ and g is continuous on X_1.

Definition 2.1. (i) A sequence $\{x_n\} \subset X_1$ is called a type I Levitin-Polyak (in short LP) approximating solution sequence if there exists a sequence $\{\epsilon_n\} \subseteq \mathbf{R}_+ = \{r \in \mathbf{R} : r \geq 0\}$ with $\epsilon_n \to 0$ such that

$$d(x_n, X_0) \leq \epsilon_n, \tag{2.1}$$

$$x_n \in S(x_n), \tag{2.2}$$

and

$$f(x_n, y) + \epsilon_n e(x_n) \notin -intC(x_n), \forall y \in S(x_n). \tag{2.3}$$

(ii) A sequence $\{x_n\} \subset X_1$ is called a type II LP approximating solution sequence if there exists a sequence $\{\epsilon_n\} \subseteq \mathbf{R}_+$ with $\epsilon_n \to 0$ such that (2.1), (2.2) and (2.3) hold, and for any $n \in N$ there exists $y_n \in S(x_n)$ such that

$$f(x_n, y_n) - \epsilon_n e(x_n) \in -C(x_n). \tag{2.4}$$

(iii) A sequence $\{x_n\} \subset X_1$ is called a generalized type I LP approximating solution sequence if there exists a sequence $\{\epsilon_n\} \subseteq \mathbf{R}_+$ with $\epsilon_n \to 0$ such that

$$d(g(x_n), K) \leq \epsilon_n, \tag{2.5}$$

and (2.2) and (2.3) hold.

(iv) A sequence $\{x_n\} \subset X_1$ is called a generalized type II LP approximating solution sequence if there exists a sequence $\{\epsilon_n\} \subseteq \mathbf{R}_+$ with $\epsilon_n \to 0$ such that (2.2), (2.3) and (2.5) hold, and for any $n \in N$ there exists $y_n \in S(x_n)$ such that (2.4) holds.

Definition 2.2. (VQEP) is said to be type I (resp. type II, generalized type I, generalized type II) LP well-posed if the solution set Ω of (VQEP) is nonempty and for every type I (resp. type II, generalized type I, generalized type II) LP approximating solution sequence $\{x_n\}$ for (VQEP), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in \Omega$ such that $x_{n_j} \to \bar{x}$.

Remark 2.1. (i) It is clear that any (generalized) type II LP approximating solution sequence of (VQEP) is a (generalized) type I LP approximating solution sequence of (VQEP). Thus the (generalized) type I LP well-posedness of (VQEP) implies the (generalized) type II LP well-posedness of (VQEP).

(ii) Each type of LP well-posedness for (VQEP) implies that the solution set is nonempty and compact.

(iii) Suppose that g is uniformly continuous on the set

$$S(\delta_0) = \{x \in X_1 : d(x, X_0) \leq \delta_0\}, \tag{2.6}$$

for some $\delta_0 \geq 0$. Then, generalized type I (resp. generalized type II) LP well-posedness of (VQEP) implies type I (resp. type II) LP well-posedness of (VQEP).

(iv) If $S(x) = X_0$ for all $x \in X_1$, then the type I (resp. type II) LP well-posedness of (VQEP) in Definition 2.2 reduces to type I (resp. type II) LP well-posedness of vector equilibrium problem introduced by Li and Li [10].

(v) If $Y = \mathbf{R}$ and $C(x) = \mathbf{R}_+$ for all $x \in X$, and $S(x) = X_0$ for all $x \in X_1$, then the type I (resp. type II, generalized type I, generalized type II) LP well-posedness of (VQEP) defined in Definition 2.2 reduces to the type I (resp. type II, generalized type I, generalized type II) LP well-posedness of the scalar equilibrium problem with an abstract set constraint and a functional constraint introduced by Long, Huang and Teo [9]. Moreover, if X^* is the topological dual space of X, $F : X_1 \to X^*$ is a mapping, $\langle F(x), z\rangle$ denotes the value of the functional $F(x)$ at z, and $f(x, y) =< F(x), y - x >$ for all $x, y \in X_1$, then the type I (resp. type II, generalized type I, generalized type II) LP well-posedness of (VQEP) defined in Definition 2.2 reduces to the type I (resp. type II, generalized type I, generalized type II) LP well-posedness for the variational inequality with abstract and functional constraints introduced by Huang, Yang and Zhu [6].

3. Criteria and Characterizations for LP Well-posedness of (VQEP) without Involving Gap functions of (VQEP)

In this subsection, we give some criteria and characterizations for the (generalized) LP well-posedness of (VQEP) without using any gap functions of (VQEP).

Now we consider the Kuratowski measure of noncompactness for a nonempty subset A of X (see [13]) defined by

$$\alpha(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i, \text{ for every } A_i, diamA_i < \epsilon\},$$

where $diamA_i$ is the diameter of A_i defined by $diamA_i = \sup\{d(x_1, x_2) : x_1, x_2 \in A_i\}$. Given two nonempty subsets A and B of X, the excess of set A and B is defined by $e(A, B) = \sup\{d(a, B) : a \in A\}$, and the Hausdorff distance between A and B is defined by $H(A, B) = \max\{e(A, B), e(B, A)\}$.

For any $\epsilon > 0$, four types of approximating solution sets for (VQEP) are defined, respectively, by

$T_1(\epsilon) := \{x \in X_1 : x \in S(x)$ and $d(x, X_0) \leq \epsilon$ and $f(x, y) + \epsilon e(x) \notin -intC(x), \forall y \in S(x)\}$,

$T_2(\epsilon) := \{x \in X_1 : x \in S(x)$ and $d(g(x), K) \leq \epsilon$ and $f(x, y) + \epsilon e(x) \notin -intC(x), \forall y \in S(x)\}$,

$T_3(\epsilon) := \{x \in X_1 : x \in S(x)$ and $d(x, X_0) \leq \epsilon$ and $f(x, y) + \epsilon e(x) \notin -intC(x), \forall y \in S(x)$ and $f(x, y) - \epsilon e(x) \in -C(x)$, for some $y \in S(x)\}$, and

$T_4(\epsilon) := \{x \in X_1 : x \in S(x)$ and $d(g(x), K) \leq \epsilon$ and $f(x,y) + \epsilon e(x) \notin -intC(x), \forall y \in S(x)$ and $f(x,y) - \epsilon e(x) \in -C(x)$, for some $y \in S(x)\}$.

Theorem 3.1. *Let X be complete.*

(i) (*VQEP*) *is type I LP well-posed if and only if the solution set Ω is nonempty, compact and* $e(T_1(\epsilon), \Omega) \to 0$ *as* $\epsilon \to 0$.

(ii) (*VQEP*) *is generalized type I LP well-posed if and only if the solution set Ω is nonempty, compact and*

$$e(T_2(\epsilon), \Omega) \to 0 \text{ as } \epsilon \to 0. \tag{3.1}$$

(iii) (*VQEP*) *is type II LP well-posed if and only if the solution set Ω is nonempty, compact and* $e(T_3(\epsilon), \Omega) \to 0$ *as* $\epsilon \to 0$.

(iv) (*VQEP*) *is generalized type II LP well-posed if and only if the solution set Ω is nonempty, compact and* $e(T_4(\epsilon), \Omega) \to 0$ *as* $\epsilon \to 0$.

Proof. The proof of (i), (iii) and (iv) are similar with that of (ii) and they are omitted here. Let (VQEP) be generalized type I LP well-posed. Then Ω is nonempty and compact. Now we show that (3.1) holds. Suppose to the contrary that there exist $l > 0$, $\epsilon_n > 0$ with $\epsilon_n \to 0$ and $z_n \in T_2(\epsilon_n)$ such that

$$d(z_n, \Omega) \geq l. \tag{3.2}$$

Since $\{z_n\} \subset T_2(\epsilon_n)$ we know that $\{z_n\}$ is generalized type I LP approximating solution for (VQEP). By the generalized type I LP well-posedness of (VQEP), there exist a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ converging to some element of Ω. This contradicts (3.2). Hence (3.1) holds.

Conversely, suppose that Ω is nonempty, compact and (3.1) holds. Let $\{x_n\}$ be a generalized type I LP approximating solution for (VQEP). Then there exists a sequence $\{\epsilon_n\}$ with $\{\epsilon_n\} \subseteq \mathbf{R}^1_+$ and $\epsilon_n \to 0$ such that (2.2), (2.3) and (2.5) hold. Thus, $\{x_n\} \subset T_2(\epsilon_n)$. It follows from (3.1) that there exists a sequence $\{z_n\} \subseteq \Omega$ such that

$$d(x_n, z_n) = d(x_n, \Omega) \leq e(T_2(\epsilon_n), \Omega) \to 0.$$

Since Ω is compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converging to $x_0 \in \Omega$. And so the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to x_0. Therefore (VQEP) is generalized type I LP well-posed. This completes the proof.■

Theorem 3.2. *Let X be complete. Assume that*

(i) *the vector-valued function f is continuous on $X_1 \times X_1$;*

(ii) *the mapping $W : X \to 2^Y$ defined by $W(x) = Y \setminus -intC(x)$ is closed;*

(iii) *S be lower semi-continuous and closed on X_1.*

Then (*VQEP*) *is generalized type I LP well-posed if and only if*

$$T_2(\epsilon) \neq \varnothing, \forall \epsilon > 0 \text{ and } \lim_{\epsilon \to 0} \alpha(T_2(\epsilon)) = 0. \tag{3.3}$$

Proof. First we show that for every $\epsilon > 0$, $T_2(\epsilon)$ is closed. In fact, let $\{x_n\} \subset T_2(\epsilon)$ and $x_n \to \bar{x}$. Then (2.2) holds,

$$d(g(x_n), K) \leq \epsilon, \tag{3.5}$$

and

$$f(x_n, y) + \epsilon e(x_n) \notin -intC(x_n), \forall y \in S(x_n). \tag{3.6}$$

From (2.2) and the closedness of S, we get $\bar{x} \in S(\bar{x})$. From (3.5), we obtain $d(g(\bar{x}), K) \leq \epsilon$.

Since S is lower semi-continuous, for any $v \in S(\bar{x})$, we can find $v_n \in S(x_n)$ with $v_n \to v$ such that

$$f(x_n, v_n) + \epsilon e(x_n) \in W(x_n). \tag{3.7}$$

By assumption (i), (ii) and (3.7), we have $f(\bar{x}, v) + \epsilon e(\bar{x}) \notin -intC(\bar{x}), \forall v \in S(\bar{x})$. Hence $\bar{x} \in T_2(\epsilon)$.

Second, we show that

$$\Omega = \cap_{\epsilon>0} T_2(\epsilon). \tag{3.8}$$

It is obvious that $\Omega \subset \cap_{\epsilon>0} T_2(\epsilon)$. Now suppose that $\epsilon_n > 0$ with $\epsilon_n \to 0$ and $x^* \in \cap_{n=1}^{\infty} T_2(\epsilon_n)$. Then, we have $x^* \in S(x^*)$,

$$d(g(x^*), K) \leq \epsilon_n, \forall n \in \mathbf{N}, \tag{3.9}$$

and

$$f(x^*, y) + \epsilon_n e(x^*) \notin -intC(x^*), \forall y \in S(x^*). \tag{3.10}$$

Since K is closed, g is continuous and (3.9) holds, we have $x^* \in X_0$. It follows from (3.10) and closedness of $W(x^*)$ that $f(x^*, y) \in W(x^*), \forall y \in S(x^*)$. i.e., $x^* \in \Omega$. Hence (3.8) hold.

Now we assume that (3.3) holds. Clearly, $T_2(.)$ is increasing with $\epsilon > 0$. By the Kuratowski theorem (see [14]), we have

$$H(T_2(\epsilon), \Omega) \to 0, \text{ as } \epsilon \to 0. \tag{3.11}$$

Let $\{x_n\}$ be any generalized type I LP approximating solution sequence for (VQEP). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that (2.2), (2.3) and (2.5) hold. Thus, $x_n \in T_2(\epsilon_n)$. It follows from (3.11) that $d(x_n, \Omega) \to 0$. So $\exists u_n \in \Omega$, such that

$$d(x_n, u_n) \to 0. \tag{3.12}$$

Since Ω is compact, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and a solution $x^* \in \Omega$ satisfying

$$u_{n_j} \to x^*. \tag{3.13}$$

From (3.12) and (3.13), we get $d(x_{n_j}, x^*) \to 0$.

Conversely, let (VQEP) be generalized type I LP well-posed. Observe that for every $\epsilon > 0$,

$$H(T_2(\epsilon), \Omega) = max\{e(T_2(\epsilon), \Omega), e(\Omega, T_2(\epsilon))\} = e(T_2(\epsilon), \Omega).$$

Hence,

$$\begin{aligned} \alpha(T_2(\epsilon)) &\leq 2H(T_2(\epsilon), \Omega) + \alpha(\Omega) \\ &= 2e(T_2(\epsilon), \Omega), \end{aligned} \tag{3.14}$$

where $\alpha(\Omega) = 0$ since Ω is compact. From Theorem 3.1(ii), we know that $e(T_2(\epsilon), \Omega) \to 0$ as $\epsilon \to 0$. It follows from (3.14) that (3.3) holds. This completes the proof. ■

By similar argument with the proof of Theorem 3.2, we can prove the following result:

Theorem 3.3. Let X be complete. Assume that

(i) the vector-valued function f is continuous on $X_1 \times X_1$;

(ii) the mapping $W : X \to 2^Y$ defined by $W(x) = Y \setminus -intC(x)$ is closed;

(iii) S be lower semi-continuous and closed on X_1.

Then (VQEP) is type I LP well-posed if and only if

$$T_1(\epsilon) \neq \varnothing, \forall \epsilon > 0 \text{ and } \lim_{\epsilon \to 0} \alpha(T_1(\epsilon)) = 0.$$

4. Criteria and Characterizations Invovling the Gap Functions of the (VQEP)

In this subsection, we give some criteria and characterizations for the four types of LP well-posedness of (VQEP) involving the gap functions of (VQEP).

Chen, Yang and Yu [15] introduced a nonlinear scalarization function $\xi_e : X \times Z \to \mathbf{R}$ defined by:

$$\xi_e(x, y) = \inf\{\lambda \in \mathbf{R} : y \in \lambda e(x) - C(x)\}.$$

Definition 4.1. A mapping $\phi : X_1 \to \mathbf{R} \cup \{+\infty\}$ is said to be a gap function on X_0 for (VQEP) if

(i) $\phi(x) \geq 0, \forall x \in X_0 \cap X_2$;

(ii) $\phi(x^*) = 0$ and $x^* \in X_0 \cap X_2$ iff $x^* \in \Omega$.

We introduce a function as follows:

$$\phi(x) = \sup_{y \in S(x)} \{-\xi_e(x, f(x,y))\}, \forall x \in X_1. \tag{4.1}$$

By Proposition 4 in [12], we can easily obtain the following Proposition 4.1:

Proposition 4.1. *If for any $x \in X_0 \cap X_2$, $f(x,x) \in -\partial C(x)$, where $\partial C(x)$ is the topological boundary of $C(x)$. Then the mapping ϕ defined by (4.1) is a gap function of (VQEP).*

Proposition 4.2. *Let function ϕ defined by (4.1). Assume that*

(i) the vector-valued function f is continuous on $X \times X_1$;

(ii) the mapping $W : X \to 2^Y$ defined by $W(x) = Y \setminus -intC(x)$ is upper-semi-continuous;

(iii) S is lower semi-continuous on X_1.

Then ϕ is lower semi-continuous from X_1 to $\mathbf{R} \cup \{+\infty\}$. Further assume that the solution set Ω of (VQEP) is nonempty, then $\mathrm{Dom}(\phi) \neq \emptyset$.

Proof. First, it is obvious that $\phi(x) > -\infty, \forall x \in X_1$. Otherwise, suppose that there exists $x_0 \in X_1$ such that $\phi(x_0) = -\infty$. Then

$$\xi_e(x_0, f(x_0, y)) \geq +\infty, \forall y \in S(x_0).$$

which is impossible, since $\xi_e(x_0, \cdot)$ is a finite function on X.

Second, we show that ϕ is lower semi-continuous on X_1. Indeed, $\xi_e(.,.)$ is upper semi-continuous by the condition (i), (ii) and Theorem 2.1 in [15]. It follows from Proposition 19 in Section 1 of Chapter 3 [16] that ϕ defined by (4.1) is lower semi-continuous on X_1. Furthermore, if $\Omega \neq \emptyset$, by Proposition 4.1, we see that $\mathrm{Dom}(f) \neq \emptyset$. This completes the proof. ■

In order to relate the various LP well-posedness of (VQEP) with that of constrained minimization problems studied in [7], we recall the various LP well-posedness of the following general constrained program [7]:

$$(P) \qquad \begin{aligned} &\min \phi(x) \\ &s.t.\ x \in X_1' \\ &g(x) \in K. \end{aligned}$$

where $X_1' \subseteq X_1$ is nonempty and closed set and $\phi : X_1 \to \mathbf{R} \cup \{\infty\}$ is proper and lower semicontinuous. The feasible set of (P) is $X_0' = \{x \in X_1' : g(x) \in K\}$. The optimal set and optimal value of (P) are denoted by $\bar{\Omega}$ and $\bar{v}$, respectively. Note that if $\mathrm{Dom}(\phi) \cap X_0' \neq \emptyset$, then $\bar{v} < +\infty$, where $\mathrm{Dom}(\phi) = \{x \in X_1 : \phi(x) < +\infty\}$.

Definition 4.1. [7] (i) A sequence $\{x_n\} \subset X_1'$ is called a type I LP minimizing sequence for (P) if

$$\limsup_{n\to+\infty} \phi(x_n) \le \bar{v}, \tag{4.2}$$

and

$$d(x_n, X_0') \to 0. \tag{4.3}$$

(ii)$\{x_n\} \subset X_1'$ is called a type II LP minimizing sequence for (P) if

$$\lim_{n\to\infty} \phi(x_n) = \bar{v}, \tag{4.4}$$

and (4.3) hold.

(iii) $\{x_n\} \subset X_1'$ is called a generalized type I LP minimizing sequence for (P) if

$$d(g(x_n), K) \to 0, \tag{4.5}$$

and (4.2) hold.

(iv) $\{x_n\} \subset X_1'$ is called a generalized type II LP minimizing sequence for (P) if (4.4) and (4.5) hold.

Definition 4.2. (P) is said to be type I (resp. type II, generalized type I, generalized type II) LP well-posed if $\bar{\Omega} \neq \varnothing$, and for any type I (resp. type II, generalized type I, generalized type II) LP minimizing sequence $\{x_n\}$ for (P), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in \bar{\Omega}$ such that $x_{n_j} \to \bar{x}$.

In the rest of this paper, we set $X_1^{'}$ in (P) equal to $X_1 \cap X_2$. Note that if S is closed on X_1, then $X_1^{'}$ is closed. The following lemma reveals some relationship between LP approximating solution sequence of (VQEP) and LP minimizing sequence of (P).

Lemma 4.1. *Let the function ϕ defined by (4.1).*

(i) $\{x_n\} \subset X_1$ is a sequence such that there exist $\{\epsilon_n\} \subseteq \mathbf{R}_+$ with $\epsilon_n \to 0$ satisfying (2.2) and (2.3) if and only if $\{x_n\} \subset X_1'$ and (4.2) holds with $\bar{v} = 0$.

(ii) $\{x_n\} \subset X_1$ is a sequence such that there exist $\{\epsilon_n\} \subseteq \mathbf{R}_+$ with $\epsilon_n \to 0$ satisfying (2.2) and (2.3) and for any $n \in \mathbf{N}$, there exists $y_n \in S(x_n)$ satisfying (2.4) if and only if $\{x_n\} \subset X_1'$ and (4.4) holds with $\bar{v} = 0$.

Proof. (i) Let $\{x_n\} \subset X_1$ be any sequence, if there exists $\{\epsilon_n\} \subseteq \mathbf{R}_+$ with $\epsilon_n \to 0$ satisfying (2.2) and (2.3), then we can easily verify that $\{x_n\} \subset X_1'$ and $\phi(x_n) \le \epsilon_n$. It follows that (4.2) holds with $\bar{v} = 0$.

For the converse, let $\{x_n\} \subset X_1'$ and (4.2) hold. We can see that $\{x_n\} \subset X_1$ and (2.2) hold. Furthermore, by (4.2), we have that there exist $\{\epsilon_n\} \subseteq \mathbf{R}_+$ with $\epsilon_n \to 0$ such that $\phi(x_n) \le \epsilon_n$. That is,

$$\sup_{y\in S(x_n)} \{-\xi_e(x_n, f(x_n, y))\} \leq \epsilon_n.$$

Hence,

$$\xi_e(x_n, f(x_n, y)) \geq -\epsilon_n, \forall y \in S(x_n).$$

It follows from Proposition 2.3 (iii) in [15] that (2.3) holds.

(ii) Let $\{x_n\} \subset X_1$ be any sequence, we can verify that $\liminf_{n\to+\infty} \phi(x_n) \geq 0$ hold if and only if there exists $\{\alpha_n\} \subseteq \mathbf{R}_+$ with $\alpha_n \to 0$ such that for any $n \in \mathbf{N}$ there exists $y_n \in S(x_n)$ satisfying

$$f(x_n, y_n) - \alpha_n e(x_n) \in -C(x_n).$$

From proof of (i), we know that

$$\limsup_{n\to+\infty} \phi(x_n) \leq 0$$

and $\{x_n\} \subset X_1'$ hold if and only if $\{x_n\} \subseteq X_1$, such that there exist $\{\beta_n\} \subseteq \mathbf{R}_+$ with $\beta_n \to 0$ satisfy (2.2) and (2.3). Finally, we let $\epsilon_n = \max\{\alpha_n, \beta_n\}$ and the conclusion follows. This completes the proof. ■

Theorem 4.1. *Assume that* $\Omega \neq \emptyset$. *Then*

(i) (*VQEP*) *is generalized type I* (*resp. generalized type II*) *LP well-posed if and only if* (*P*) *is generalized type I* (*resp. generalized type II*) *LP well-posed with* $\phi(x)$ *defined by* (*4.1*).

(ii) *If* (*VQEP*) *is type I* (*resp. type II*) *LP well-posed, then* (*P*) *is type I* (*resp. type II*) *LP well-posed with* $\phi(x)$ *defined by* (*4.1*).

Proof. Let $\phi(x)$ be defined by (4.1). Since $\Omega \neq \emptyset$, it is easily checked that $\bar{x} \in \Omega$ is a solution of (VQEP) if and only if $\bar{x}$ is an optimal solution of (P) with $\bar{v} = \phi(\bar{x}) = 0$.

(i) Similar to the proof of Lemma 4.1, it is also routine to check that a sequence $\{x_n\}$ is a generalized type I (resp., generalized type II) LP approximating solution sequence of (VQEP) if and only if it is a generalized type I (resp., generalized type II) LP approximating solution sequence of (P). So, (VQEP) is generalized type I (resp. generalized type II) LP well-posed if and only if (P) is generalized type I (resp. generalized type II) LP well-posed with $\phi(x)$ defined by (4.1).

(ii) Since ${X_0}' \subseteq X_0$, $d(x, X_0) \leq d(x, {X_0}')$ for any x. This fact together with Lemma 4.1 implies that a type I (resp. type II) LP minimizing sequence of (P) is a type I (resp. type II) LP approximating solution sequence of (VQEP). So the type I (resp. type II) LP well-posedness of (VQEP) implies that the type I (resp. type II) LP well-posedness of (P) with $\phi(x)$ defined by (4.1).This completes the proof. ■

Next section, we give some criteria and characterizations for types of LP well-posedness of (VQEP).

Now consider a real-valued function $c = c(t, s, r)$ defined for $t, s, r \geq 0$ sufficiently small, such that

$$c(t, s, r) \geq 0, \ \forall t, s, r, \ c(0, 0, 0) = 0, \tag{4.6}$$

$$s_n \to 0, t_n \geq 0, r_n = 0, c(t_n, s_n, r_n) \to 0 \text{ implythat } t_n \to 0. \tag{4.7}$$

Theorem 4.2. *If* (*VQEP*) *is type II LP well-posed, the set-valued map* S *is closed-valued, then there exists a function* c *satisfying* (*4.6*) *and* (*4.7*) *such that*

$$|\phi(x)| \geq c(d(x, \Omega), d(x, X_0), d(x, S(x))), \forall x \in X_1, \tag{4.8}$$

where $\phi(x)$ *is defined by* (*4.1*)*. Conversely, suppose that* Ω *is nonempty compact, and* (*4.8*) *hold for some* c *satisfying* (*4.6*) *and* (*4.7*)*. Then* (*VQEP*) *is type II LP well-posed.*

Proof. Let $c(t, s, r)$ be the real-valued function defined by

$$c(t, s, r) = \inf\{|\phi(x)| : x \in X_1, d(x, \Omega) = t, d(x, X_0) = s, d(x, S(x)) = r\}. \tag{4.9}$$

Since $\Omega \neq \varnothing$, it is obvious that $c(0, 0, 0) = 0$. Moreover, if $s_n \to 0, t_n \geq 0, r_n = 0$ and $c(t_n, s_n, r_n) \to 0$, then there exists a sequence $\{x_n\} \subset X_1$ with

$$d(x_n, \Omega) = t_n, \tag{4.10}$$

$$d(x_n, X_0) = s_n \to 0, \tag{4.11}$$

$$d(x_n, S(x_n)) = r_n = 0, \tag{4.12}$$

such that

$$|\phi(x_n)| \to 0. \tag{4.13}$$

Since S is closed-valued, $x_n \in S(x_n)$ for any n. This combined with (4.11), (4.13) and Lemma 4.1 imply that $\{x_n\}$ is a type II approximating solution sequence of (VQEP). By proposition 3.1, we have that $t_n \to 0$.

Conversely, let $\{x_n\}$ be a type II approximating solution sequence of (VQEP). Then by (4.9) we have $|\phi(x_n)| \geq c(d(x_n, \Omega), d(x_n, X_0), d(x_n, S(x_n)))$. Let $t_n = d(x_n, \Omega)$, $s_n = d(x_n, X_0)$, $r_n = d(x_n, S(x_n))$. Then $s_n \to 0$ and $r_n = 0$ $\forall n \in \mathbf{N}$. Moreover, by Lemma 4.1, we have $|\phi(x_n)| \to 0$. Then $c(t_n, s_n, r_n) \to 0$. These facts together with the properties of the function c implies that $t_n \to 0$. By proposition 3.1, we see that (VQEP) is type II well-posed. This completes the proof. ■

Theorem 4.3. *If* (*VQEP*) *is generalized type II LP well-posed, the set-valued map* S *is closed-valued, then there exists a function* c *satisfying* (*4.6*) *and* (*4.7*) *such that*

$$|\phi(x)| \geq c(d(x,\Omega), d(g(x),K), d(x,S(x))), \forall x \in X_1, \tag{4.14}$$

where $\phi(x)$ *is defined by* (*4.1*). *Conversely, suppose that* Ω *is nonempty compact, and* (*4.14*) *hold for some* c *satisfying* (*4.6*) *and* (*4.7*). *Then* (*VQEP*) *is generalized type II LP well-posed.*

Proof. The proof is almost the same as Theorem 4.2. The only difference lies in the proof of the first part of Theorem 4.2. Here we define

$$c(t,s,r) = \inf\{|\phi(x)| : x \in X_1, d(x,\Omega) = t, d(g(x),K) = s, d(x,S(x)) = r\}.$$

This completes the proof. ∎

Definition 4.2. [6]. (i) Let Z be a topological space and let $Z_1 \subset Z$ be a nonempty subset. Suppose that $G : Z \to \mathbf{R} \cup \{+\infty\}$ is an extend real-valued function. Then the function G is said to be level-compact on Z_1 if for any $s \in \mathbf{R}$ the subset $\{z \in Z_1 : G(z) \leq s\}$ is compact.

(ii) Let Z be a finite dimensional normed space and $Z_1 \subset Z$ be nonempty. A function $h : Z \to \mathbf{R} \cup \{+\infty\}$ is said to be level-bounded on Z_1 if Z_1 is bounded or

$$\lim_{z \in Z_1, ||z|| \to +\infty} h(z) = +\infty.$$

The following proposition presents some sufficient conditions for type I LP well-posedness of (VQEP).

Proposition 4.3. *Suppose that the solution set* Ω *is nonempty, for any* $x \in X_0 \cap X_2$, $f(x,x) \in -\partial C(x)$, *the vector-valued function* f *is continuous on* $X \times X_1$, *the mapping* $W : X \to 2^Y$ *defined by* $W(x) = Y \setminus -intC(x)$ *is upper-semi-continuous and the set-valued map* S *is lower semi-continuous and closed on* X_1. *Further assume that one of the following conditions holds.*

(*i*) *there exists* $0 < \delta_1 < \delta_0$ *such that* $X_1(\delta_1)$ *is compact where*

$$X_1(\delta_1) = \{x \in X_1 \cap X_2 : d(x, X_0) \leq \delta_1\}; \tag{4.15}$$

(*ii*) *the function* ϕ *defined by* (*4.1*) *is level-compact on* $X_1 \cap X_2$;

(*iii*) X *is a finite-dimensional normed space and*

$$\lim_{x \in X_1 \cap X_2, ||x|| \to +\infty} \max\{\phi(x), d(x, X_0)\} = +\infty; \tag{4.16}$$

(iv) *there exists* $0 < \delta_1 < \delta_0$ *such that* ϕ *is level-compact on* $X_1(\delta_1)$ *defined by* (4.15)*;*

Then $(VQEP)$ *is type I LP well-posed.*

Proof. First we show that each one of (i), (ii) and (iii) implies (iv). Clearly, either of (i) and (ii) implies (iv). Now we show that condition (iii) implies condition (iv). We notes that the set $X_1 \cap X_2$ is closed by the closedness of S on X_1. Then, we need only to show that for any $t \in \mathbf{R}$ the set $A = \{x \in X_1(\delta_1) : \phi(x) \leq t\}$ is bounded since X is finite-dimensional space and the function ϕ defined by (4.1) is lower semi-continuous on $X_1 \cap X_2$ and thus, A is closed. Suppose to the contrary, there exists $t \in \mathbf{R}$ and $\{x'_n\} \subset X_1(\delta_1)$ such that $||x'_n|| \to +\infty$ and $\phi(x'_n) \leq t$. From $\{x'_n\} \subset X_1(\delta_1)$ we have $d(x'_n, X_0) \leq \delta_1$. Thus, $\max\{\phi(x'_n), d(x'_n, X_0)\} \leq max\{t, \delta_1\}$, which contradicts condition (4.16).

Therefore, we only need to prove that if condition (iv) hold, then (VQEP) is type I LP well-posed. Let $\{x_n\}$ be a type I LP approximating solution sequence of (VQEP). Then there exists $\{\epsilon_n\} \subset \mathbf{R}_+$ with $\epsilon_n > 0$ such that (2.1), (2.2) and (2.3) hold. From (2.1) and (2.2), we can assume without loss of generality that $\{x_n\} \subset X_1(\delta_1)$. By Lemma 4.1, we can assume without loss of generality that $\{x_n\} \subseteq \{x \in X_1(\delta_1) : \phi(x) \leq 1\}$, where ϕ is defined by (4.1). By the level-compact of ϕ on $X_1(\delta_1)$, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in X_1(\delta_1)$ such that $x_{n_j} \to \bar{x}$. Taking limit in (2.1) we have $\bar{x} \in X_0$. Since S is closed and (2.2) holds, we also have $\bar{x} \in S(\bar{x})$. That is,

$$\bar{x} \in X_0 \cap X_2 = X'_0. \tag{4.17}$$

Furthermore, by Proposition 4.2 and Lemma 4.1, we have

$$\phi(\bar{x}) \leq \liminf_{j \to +\infty} \phi(x_{n_j}) \leq \limsup_{j \to +\infty} \phi(x_{n_j}) \leq 0.$$

This fact combined with (4.17) and Proposition 4.1 implies that $\phi(\bar{x}) = 0$ and $\bar{x} \in \Omega$. This completes the proof. ■

Similar to Proposition 4.3, we can prove the following result:

Proposition 4.4. *Suppose that the solution set* Ω *is nonempty, for any* $x \in X_0 \cap X_2$, $f(x, x) \in -\partial C(x)$, *the vector-valued function* f *is continuous on* $X \times X_1$, *the mapping* $W : X \to 2^Y$ *defined by* $W(x) = Y \setminus -intC(x)$ *is upper-semi-continuous and the set-valued map* S *is lower semi-continuous and closed on* X_1. *Further assume that one of the following conditions holds.*

(i) *there exists* $0 < \delta_1 < \delta_0$ *such that* $X_2(\delta_1)$ *is compact where*

$$X_2(\delta_1) = \{x \in X_1 \cap X_2 : d(g(x), K) \leq \delta_1\}; \tag{4.18}$$

(ii) *the function* ϕ *defined by (4.1) is level-compact on* $X_1 \cap X_2$*;*
(iii) X *is a finite-dimensional normed space and*

$$\lim_{x \in X_1 \cap X_2, ||x|| \to +\infty} \max\{\phi(x), d(g(x), K)\} = +\infty;$$

(iv) *there exists* $0 < \delta_1 < \delta_0$ *such that* ϕ *is level-compact on* $X_2(\delta_1)$ *defined by* (*4.18*)*;*
Then (*VQEP*) *is generalized type I LP well-posed.*

Remark 4.1. If X is a finite dimensional space, then the "level-compactness" condition in Propositions 4.3 and 4.4 can be replaced by the "level-boundedness" condition.

Remark 4.2. (i) Theorems 3.3, Propositions 4.1, 4.2, Theorems 4.1 and 4.2 respectively, generalize and extend Theorem 3.1, Propositions 4.1, 4.2, Theorems 4.1 and 4.2 in [10] from vector equilibrium problems case to the vector quasi-equilibrium problems case. And so Theorems 3.2 and 4.3, respectively, generalize and extend Theorems 3.1 and 4.2 in [10] in several ways.

(ii) Theorems 3.1(i), 3.1(2), 3.3, 3.2, Propositions 4.3 and 4.4, respectively, generalize and extend Theorems 3.1, 3.5, 3.4, 3.6, propositions 4.2 and 4.3 in [9] from scalar equilibrium problems case to the vector quasi-equilibrium problems case.

(iii) It is easy to see that the results in this paper generalize and extend the main results in [6] in several aspects.

REFERENCES

1. A. N. Tykhonov, On the stability of the functional optimization problem, *USSRJ. Comput. Math. Math. Phys.*, **6** (1966), 28-33.

2. E. S. Levitin and B. T. Polyak, Convergence of minimizing sequences in conditional extremum problem, *Soviet Mathematics Doklady*, **7** (1966), 764-767.

3. A. S. Konsulova and J. P. Revalski, Constrained convex optimization problemswell-posedness and stability, *Numer. Funct. Anal. Optim.*, **15** (1994), 889-907.

4. X. X. Huang and X. Q. Yang, Generalized Levitin-Polyak well-posedness in constrained optimization, *SIAM J. Optim.*, **17** (2006), 243-258.

5. X. X. Huang and X. Q. Yang, Levitin-Polyak well-posedness of constrained vector optimization problems, *J. Glob. Optim.*, **37** (2007), 287-304.

6. X. X. Huang, X. Q. Yang and D. L. Zhu, Levitin-Polyak well-posedness of variational inequalities problems with functional constraints, *J. Glob. Optim.*, **44** (2009), 159-174.

7. X. X. Huang and X. Q. Yang, Levitin-Polyak well-posedness in generalized variational inequalities problems with functional constraints, *J. Ind. Manag. Optim.*, **3** (2007), 671-684.

8. Z. Xu, D. L. Zhu and X. X. Huang, Levitin-Polyak well-posedness in generalized vector variational inequality problem with functional constraints, *Math. Meth. Oper. Res.*, **67** (2008), 505-524.

9. X. J. Long, N. J. Huang and K. L. Teo, *Levitin-Polyak well-posedness for Equilibrium Problems with functional Constraints*, Journal of Inequalities and Applications, Vol. 2008, Article ID 657329, 14 pages.

10. S. J. Li and M. H. Li, Levitin-Polyak well-posedness of Vector Equilibrium Problems, *Math. Meth. Oper. Res.*, **69** (2008), 125-140.

11. N. J. Huang, X. J. Long and C. W. Zhao, Well-posedness for vector quasi-equilibrium problems with applications, *J. Ind. Manag. Optim.*, **5** (2009), 341-349.

12. M. H. Li, S. J. Li and W. Y. Zhang, Levitin-Polyak well-posedness of generalized vector quasi-equilibrium problems, *J. Ind. Manag. Optim.*, **5** (2009), 683-696.

13. M. Furi and A. Vignoli, About well-posed optimization problems for functions in metric spaces, *J. Optim. Theory Appl.*, **5** (1970), 225-229.

14. C. Kuratowski, *Topologie*, *Panstwove Wydanictwo Naukowe*, Warszawa, Poland, 1952.

15. G. Y. Chen, X. Q. Yang and H. Yu, A nonlinear scalarization function and generalized quasi-vector equilibrium problems, *J. Glob. Optim.*, **32** (2005), 451-466.

16. J. P. Aubin and I. Ekeland, *Applied Nonlinear Analisis*, John Wiley & Sons, 1984.

Jian-Wen Peng and Yan Wang
School of Mathematics
Chongqing Normal University
Chongqing 400047
P. R. China

Soon-Yi Wu
Department of Mathematics
National Cheng Kung University
Tainan 701, Taiwan
E-mail: soonyi@mail.ncku.edu.tw