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MULTIPLICITY RESULTS FOR A NEUMANN BOUNDARY VALUE PROBLEM INVOLVING THE $P(X)$-LAPLACIAN

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Abstract. In this paper we are interested in the multiplicity of weak solutions to the following Neumann problem involving the $p(x)$-Laplacian operator

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda\alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

We establish the existence of at least three solutions to this problem by using, as main tool, a recent variational principle due to Ricceri.

1. INTRODUCTION

This paper is concerned with the existence of weak solutions to the following Neumann problem

$$(\mathrm{N}_\lambda) \qquad \begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda\alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary, $\lambda \in \mathbb{R}$, ν is the outward unit normal to $\partial\Omega$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous nonconstant functions, $\alpha, \beta \in L^1(\Omega)$ are nonnegative nonconstant functions and $p \in L^\infty(\Omega)$ is such that

$$N < p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \le p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < +\infty.$$

The operator $\Delta_{p(x)}$ defined by $\Delta_{p(x)}u := div\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is known as $p(x)$-Laplacian and represents a generalization of the classical p-Laplacian operator, obtained when p is a positive constant. As a matter of fact, the $p(x)$-Laplacian has

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more complicated nonlinearities than the p-Laplacian. For example, it is not homogeneous and thus, some techniques which can be applied in the case of the p-Laplacian, such as the Lagrange Multiplier Theorem, will fail in this new situation. For this and other reasons, elliptic equations involving the operator $\Delta_{p(x)}$ are not trivial generalizations of similar problems studied in the constant case.
The study of differential equations with $p(x)$-growth conditions is an interesting and attractive topic and has been the object of considerable attention in recent years. The reason for such an interest relies on the fact that they model phenomena arising from various fields; we cite, for instance, the motion of electrorheological fluids, which are characterized by their ability to drastically change their mechanical properties under the influence of an exterior electromagnetic field, the thermo-convective flows of non-Newtonian fluids and the image processing. In this last context, the variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise.
The starting point of our approach to problem (N_λ) has been [9], in which the author considers the following ordinary Neumann problem

$$(1) \qquad \begin{cases} -u'' + u = \lambda\alpha(t)f(u) + \beta(t)g(u) & \text{in } [0,1] \\ u'(0) = u'(1) = 0, \end{cases}$$

where $\lambda \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R},\ \alpha, \beta : [0,1] \to [0, +\infty[$ are four continuous nonconstant functions. Thanks to a new multiplicity result established in the same [9], Ricceri has proved that the problem above admits at least three nonzero solutions. His work can be considered, somehow, an improvement of previous results of [6] and [10] which ensured, for $\beta \equiv 0$ in (1), the existence, respectively, of one solution for all $\lambda > 0$ and two solutions for all $\lambda > 0$ sufficiently large.
The aim of this paper is to extend problem (1) to the $p(x)$-Laplacian case, namely to problem (N_λ). We point out that there are several other existence results for $p(x)$-Laplacian equations like (N_λ) with Neumann boundary conditions; some of them can be found, for instance, in [1, 3, 7].

By using the variational methods, a technical lemma (Lemma 2.1) and the above mentioned Ricceri's three-critical-points theorem (Theorem 1 of [9]), we get the main result Theorem 3.1. This paper is organized as follows. In Section 2 we establish the variational setting to our problem by giving some background facts concerning the generalized Lebesgue and Sobolev spaces and we recall Ricceri's result. In Section 3 we present our main result and its proof with some remarks and applications.

2. Preliminaries

Since problem (N_λ) involves the operator $\Delta_{p(x)}$, the Lebesgue and Sobolev spaces with variable exponents are the most suitable contexts in which this problem can be studied. Here we state some basic results on the theory of these spaces, useful for our purpose. For more details we refer the reader to [4, 5, 8] and the references therein.

Throughout this paper, Ω denotes a nonempty bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ \text{ measurable} : \ \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}$$

with the norm

$$|u|_{p(x)} = \inf\left\{ \sigma > 0 : \ \int_\Omega \left| \frac{u(x)}{\sigma} \right|^{p(x)} dx \le 1 \right\} ,$$

and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

endowed with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)} .$$

$L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

By Theorem 2.2 of [4], due to the assumption $p^- > N$, it follows that there exists a compact embedding of $W^{1,p(x)}(\Omega)$ into $C^0(\overline{\Omega})$, namely, there exists $c > 0$ such that

$$\sup_{x\in\overline{\Omega}} |u(x)| \le c\, \|u\|$$

for all $u \in W^{1,p(x)}(\Omega)$.

We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution to problem (N$_\lambda$) if

$$\int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) dx \ + \int_\Omega |u(x)|^{p(x)-2} u(x) v(x) dx \, -$$
$$- \, \lambda \int_\Omega \alpha(x) f(u(x)) v(x) dx \ - \int_\Omega \beta(x) g(u(x)) v(x) dx = 0$$

for all $v \in W^{1,p(x)}(\Omega)$. Now, we introduce the functionals involved in our problem and state some properties that we need in the sequel. First, we define

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} \right) dx$$

for all $u \in W^{1,p(x)}(\Omega)$. Similar arguments as those used in [8] ensure that Φ is a sequentially weakly lower semicontinuous C^1 functional in $W^{1,p(x)}(\Omega)$ whose derivative is given by

$$\Phi'(u)(v) = \int_\Omega \left(|\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla_v(x) + |u(x)|^{p(x)-2} u(x) v(x) \right) dx$$

for all $u, v \in W^{1,p(x)}(\Omega)$. Moreover, straightforward computations show that Φ satisfies the following inequalities:

Proposition 2.1. *Let* $u \in W^{1,p(x)}(\Omega)$*; then*

(i) $\|u\| < 1 \implies \frac{1}{p^+} \|u\|^{p^+} \leq \Phi(u) \leq \frac{1}{p^-} \|u\|^{p^-}$;

(ii) $\|u\| > 1 \implies \frac{1}{p^+} \|u\|^{p^-} \leq \Phi(u) \leq \frac{1}{p^-} \|u\|^{p^+}$.

Thanks to (ii) of the above proposition, we can immediately infer that Φ is coercive. Furthermore, it is not difficult to prove that the operator $\Phi' : W^{1,p(x)}(\Omega) \to \left(W^{1,p(x)}(\Omega)\right)^*$ is coercive, hemicontinuous and uniformly monotone, which implies, via Theorem 26.A(d) of [11], that Φ' is invertible with continuous inverse (see, for instance, [1]).

If $\gamma : \Omega \to [0, +\infty[$ is a nonconstant function in $L^1(\Omega)$ and $h : \mathbb{R} \to \mathbb{R}$ is a continuous nonconstant function, we denote by $J_{\gamma,h}$ the functional defined on $W^{1,p(x)}(\Omega)$ by putting

$$J_{\gamma,h}(u) = \int_\Omega \gamma(x) h(u(x)) dx$$

for all $u \in W^{1,p(x)}(\Omega)$. The following lemma involving the functional $J_{\gamma,h}$ represents a useful ingredient for the proof of our main result.

Lemma 2.1. *Let* $\gamma : \Omega \to [0, +\infty[$ *be a nonzero function in* $L^1(\Omega)$ *and* $h : \mathbb{R} \to \mathbb{R}$ *a continuous nonzero function. Then, one has*

$$\limsup_{u \to 0} \frac{J_{\gamma,h}(u)}{\|u\|^{p^+}} \leq c^{p^+} \|\gamma\|_{L^1(\Omega)} \max\left\{0, \limsup_{\xi \to 0} \frac{h(\xi)}{|\xi|^{p^+}}\right\} \tag{2}$$

and

$$\limsup_{\|u\| \to +\infty} \frac{J_{\gamma,h}(u)}{\|u\|^{p^-}} \leq c^{p^-} \|\gamma\|_{L^1(\Omega)} \max\left\{0, \limsup_{|\xi| \to +\infty} \frac{h(\xi)}{|\xi|^{p^-}}\right\}. \tag{3}$$

Proof. Fixed $\eta > \max\left\{0, \limsup_{\xi\to 0} \frac{h(\xi)}{|\xi|^{p^+}}\right\}$, there exists some $\delta > 0$ such that

$$h(\xi) \le \eta|\xi|^{p^+}$$

for all $\xi \in [-\delta, \delta]$. For $u \in W^{1,p(x)}(\Omega)$ with $\|u\| \le \dfrac{\delta}{c}$, we have $\sup_{x\in\overline{\Omega}} |u(x)| \le \delta$ and so

$$h(u(x)) \le \eta|u(x)|^{p^+}.$$

Thus, multiplying by γ (recall that $\gamma \ge 0$), integrating and taking the embedding of $W^{1,p(x)}(\Omega)$ in $C^0(\overline{\Omega})$ into account, we get

$$J_{\gamma,h}(u) \le \eta \left(\sup_{x\in\overline{\Omega}} |u(x)|\right)^{p^+} \|\gamma\|_{L^1(\Omega)} \le c^{p^+} \eta \|\gamma\|_{L^1(\Omega)} \|u\|^{p^+}.$$

From this it follows that

$$\limsup_{u\to 0} \frac{J_{\gamma,h}(u)}{\|u\|^{p^+}} \le c^{p^+} \eta \|\gamma\|_{L^1(\Omega)}$$

and so (2), by the arbitrariness of η.
Now, fixed $\theta > \max\left\{0, \limsup_{|\xi|\to+\infty} \frac{h(\xi)}{|\xi|^{p^-}}\right\}$, for some $\omega > 0$ we have

$$h(\xi) \le \theta|\xi|^{p^-}$$

for all $\xi \in \mathbb{R} \setminus [-\omega, \omega]$. Thanks to the embedding of $W^{1,p(x)}(\Omega)$ in $C^0(\overline{\Omega})$, for each $u \in W^{1,p(x)}(\Omega) \setminus \{0\}$, we have

$$\frac{J_{\gamma,h}(u)}{\|u\|^{p^-}} = \frac{\displaystyle\int_{u^{-1}([-\omega,\omega])} \gamma(x)h(u(x))dx}{\|u\|^{p^-}} + \frac{\displaystyle\int_{u^{-1}(\mathbb{R}\setminus[-\omega,\omega])} \gamma(x)h(u(x))dx}{\|u\|^{p^-}} \le$$

$$\le \frac{\|\gamma\|_{L^1(\Omega)} \sup_{[-\omega,\omega]} h}{\|u\|^{p^-}} + c^{p^-}\theta \|\gamma\|_{L^1(\Omega)}.$$

Hence

$$\limsup_{\|u\|\to+\infty} \frac{J_{\gamma,h}(u)}{\|u\|^{p^-}} \le c^{p^-}\theta \|\gamma\|_{L^1(\Omega)}$$

and (3) follows by the arbitrariness of θ. ■

If $\gamma : \Omega \to [0, +\infty[$ is a nonconstant function in $L^1(\Omega)$ and $h : \mathbb{R} \to \mathbb{R}$ is a continuous nonconstant function, using the same notations as above, consider the functional $J_{\gamma,H}$, where H is defined by

$$H(\xi) = \int_0^\xi h(t)dt$$

for all $\xi \in \mathbb{R}$. By arguing as in [8], the functional $J_{\gamma,H}$ turns out to be in $C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ and its derivative is given by

$$J'_{\gamma,H}(u)(v) = \int_\Omega \gamma(x)h(u(x))v(x)dx$$

for all $u, v \in W^{1,p(x)}(\Omega)$. Moreover, the compact embedding of $W^{1,p(x)}(\Omega)$ in $C^0(\overline{\Omega})$ implies that $J'_{\gamma,H} : W^{1,p(x)}(\Omega) \to \left(W^{1,p(x)}(\Omega)\right)^*$ is compact.
From the expressions of Φ' and $J'_{\gamma,H}$ we can deduce that $u \in W^{1,p(x)}(\Omega)$ is a weak solution to (N_λ) if and only if u is a critical point of the functional

$$\Phi - \lambda J_{\alpha,F} - J_{\beta,G}$$

which represents, therefore, the energy functional related to problem (N_λ).
As already said in the introduction, our approach in facing problem (N_λ) is based on the very recent multiplicity result established in [9], that we recall below for the reader's convenience.

Theorem 2.1. (Theorem 1 of [9]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a coercive and sequentially weakly lower semicontinuous C^1 functional whose derivative admits a continuous inverse on X^*; $\Psi_1, \Psi_2 : X \to \mathbb{R}$ two C^1 functionals with compact derivative. Assume that there exist two points $u_0, v_0 \in X$ with the following properties:*

(i) *u_0 is a strict local minimum of Φ and $\Phi(u_0) = \Psi_1(u_0) = \Psi_2(u_0) = 0$;*

(ii) *$\Phi(v_0) \le \Psi_1(v_0)$ and $\Psi_2(v_0) > 0$.*

Moreover, suppose that, for some $\rho \in \mathbb{R}$, one has either

$$\begin{aligned} &\sup_{\lambda>0} \inf_{u\in X} (\lambda(\Phi(u) - \Psi_1(u) - \rho) - \Psi_2(u)) \\ &< \inf_{u\in X} \sup_{\lambda>0} (\lambda(\Phi(u) - \Psi_1(u) - \rho) - \Psi_2(u)) \end{aligned} \tag{4}$$

or

$$\begin{aligned} &\sup_{\lambda>0} \inf_{u\in X} (\Phi(u) - \Psi_1(u) - \lambda(\rho + \Psi_2(u))) \\ &< \inf_{u\in X} \sup_{\lambda>0} (\Phi(u) - \Psi_1(u) - \lambda(\rho + \Psi_2(u))). \end{aligned} \tag{5}$$

Finally, assume that

$$\max\left\{\limsup_{\|u\|\to+\infty}\frac{\Psi_1(u)}{\Phi(u)},\limsup_{u\to u_0}\frac{\Psi_1(u)}{\Phi(u)}\right\}<1 \tag{6}$$

and

$$\max\left\{\limsup_{\|u\|\to+\infty}\frac{\Psi_2(u)}{\Phi(u)},\limsup_{u\to u_0}\frac{\Psi_2(u)}{\Phi(u)}\right\}\le 0. \tag{7}$$

Under such hypotheses, there exists $\lambda^*>0$ such that the equation

$$\Phi'(u)=\Psi_1'(u)+\lambda^*\Psi_2'(u)$$

has at least four solutions in X, u_0 being one of them.

Remark 2.1. It is important to remark that, in view of Theorem 1 of [2], condition (4) is equivalent to the existence of $u_1, v_1 \in X$ satisfying

$$\Phi(u_1)-\Psi_1(u_1)<\rho<\Phi(v_1)-\Psi_1(v_1)$$

and

$$\frac{\sup\limits_{(\Phi-\Psi_1)^{-1}(]-\infty,\rho])}\Psi_2-\Psi_2(u_1)}{\rho-\Phi(u_1)+\Psi_1(u_1)}<\frac{\sup\limits_{(\Phi-\Psi_1)^{-1}(]-\infty,\rho])}\Psi_2-\Psi_2(v_1)}{\rho-\Phi(v_1)+\Psi_1(v_1)}.$$

Likewise, condition (5) is equivalent to the existence of $u_1, v_1 \in X$ satisfying

$$\Psi_2(v_1)<\rho<\Psi_2(u_1)$$

and

$$\frac{\Phi(u_1)-\Psi_1(u_1)-\inf\limits_{\Psi_2^{-1}([\rho,+\infty[)}(\Phi-\Psi_1)}{\Psi_2(u_1)-\rho}<\frac{\Phi(v_1)-\Psi_1(v_1)-\inf\limits_{\Psi_2^{-1}([\rho,+\infty[)}(\Phi-\Psi_1)}{\Psi_2(v_1)-\rho}.$$

3. Results

Our main result is given by the following theorem.

Theorem 3.1. *Let $f, g:\mathbb{R}\to\mathbb{R}$ be two continuous nonconstant functions and let $\alpha,\beta:\Omega\to[0,+\infty[$ be two nonconstant functions in $L^1(\Omega)$. Assume that*

$$(\mathrm{a}_1)\quad \max\left\{\limsup_{|\xi|\to+\infty}\frac{\int_0^\xi f(t)dt}{|\xi|^{p^-}},\ \limsup_{\xi\to 0}\frac{\int_0^\xi f(t)dt}{|\xi|^{p^+}}\right\}\le 0,$$

$$(\mathrm{a}_2)\ \sup_{\xi\in\mathbf{R}}\int_0^{\xi} g(t)dt<+\infty,\quad \limsup_{\xi\to 0}\frac{\int_0^{\xi} g(t)dt}{|\xi|^{p^+}}<\frac{1}{c^{p^+}p^+\,\|\beta\|_{L^1(\Omega)}}.$$

Finally, suppose that there exist $\sigma > c\,(p^+)^{\frac{1}{p^-}}\max\left\{1,\left(\|\beta\|_{L^1(\Omega)}\sup_{\mathbf{R}} G\right)^{\frac{1}{p^-}}\right\}$ *and* $\xi_1\in\mathbb{R}$ *such that*

$$(\mathrm{a}_3)\ 0<\int_0^{\xi_1} f(t)dt=\sup_{|\xi|\le\sigma}\int_0^{\xi} f(t)dt<\sup_{\xi\in\mathbf{R}}\int_0^{\xi} f(t)dt,$$

$$(\mathrm{a}_4)\ \max\left\{|\xi_1|^{p^-},|\xi_1|^{p^+}\right\}\le\frac{p^-\,\|\beta\|_{L^1(\Omega)}}{|\Omega|}\int_0^{\xi_1} g(t)dt.$$

Under such hypotheses, there exists $\lambda^*>0$ *such that the problem*

$$\begin{cases} -\Delta_{p(x)}u+|u|^{p(x)-2}u=\lambda^*\alpha(x)f(u)+\beta(x)g(u) & \text{in } \Omega\\ \dfrac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega\end{cases}$$

has at least three nonzero solutions.

Proof. In order to apply Theorem 2.1, take $X=W^{1,p(x)}(\Omega)$ and $\Phi,\ \Psi_1,\ \Psi_2$ equal, respectively, to $\Phi,\ J_{\beta,G},\ J_{\alpha,F}$ defined in Section 2. Now, take $u_0=0$ and $v_0=\xi_1$; of course, (i) is evident. Moreover, thanks to (a_3) and (a_4), one has

$$\begin{aligned}\Phi(v_0)&\le\frac{1}{p^-}\int_\Omega|\xi_1|^{p(x)}dx\le\frac{|\Omega|}{p^-}\max\left\{|\xi_1|^{p^-},|\xi_1|^{p^+}\right\}\\ &\le\|\beta\|_{L^1(\Omega)}\int_0^{\xi_1} g(t)dt=J_{\beta,G}(v_0)\end{aligned}$$

and

$$J_{\alpha,F}(v_0)=\|\alpha\|_{L^1(\Omega)}\int_0^{\xi_1} f(t)dt>0,$$

which prove (ii). Now, if $\|u\|>1$, due to Proposition 2.1 we have

$$\frac{J_{\beta,G}(u)}{\Phi(u)}\le\frac{p^+J_{\beta,G}(u)}{\|u\|^{p^-}}$$

and thanks to Lemma 2.1 we get

$$\limsup_{\|u\|\to+\infty}\frac{J_{\beta,G}(u)}{\Phi(u)}\le p^+c^{p^-}\,\|\beta\|_{L^1(\Omega)}\max\left\{0,\limsup_{|\xi|\to+\infty}\frac{\int_0^{\xi} g(t)dt}{|\xi|^{p^-}}\right\}\le 0.$$

If $\|u\|<1$, due to Proposition 2.1 we have

$$\frac{J_{\beta,G}(u)}{\Phi(u)} \le \frac{p^+ J_{\beta,G}(u)}{\|u\|^{p^+}}$$

and thanks to Lemma 2.1 and (a_2) we get

$$\limsup_{u\to 0} \frac{J_{\beta,G}(u)}{\Phi(u)} \le p^+ c^{p^+} \|\beta\|_{L^1(\Omega)} \max\left\{0, \limsup_{\xi\to 0} \frac{\int_0^\xi g(t)dt}{|\xi|^{p^+}}\right\} < 1.$$

So (6) is fulfilled. Likewise, from Lemma 2.1 and (a_1), we get

$$\limsup_{\|u\|\to+\infty} \frac{J_{\alpha,F}(u)}{\Phi(u)} \le p^+ c^{p^-} \|\alpha\|_{L^1(\Omega)} \max\left\{0, \limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(t)dt}{|\xi|^{p^-}}\right\} \le 0$$

and

$$\limsup_{u\to 0} \frac{J_{\alpha,F}(u)}{\Phi(u)} \le p^+ c^{p^+} \|\alpha\|_{L^1(\Omega)} \max\left\{0, \limsup_{\xi\to 0} \frac{\int_0^\xi f(t)dt}{|\xi|^{p^+}}\right\} \le 0,$$

which satisfy (7). Finally, let us check that (4) holds.
Being $\sigma > c(p^+)^{\frac{1}{p^-}}$, we have that

$$1 - \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G < \frac{\sigma^{p^-}}{c^{p^-} p^+} - \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G$$

and so it is possible to choose $\rho \in \mathbb{R}$ such that

$$\max\left\{0, 1 - \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G\right\} < \rho < \frac{\sigma^{p^-}}{c^{p^-} p^+} - \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G.$$

Let $u \in X$ such that $\Phi(u) - J_{\beta,G}(u) \le \rho$. If $\|u\| \le 1$, by the definition of σ and the choice of ρ we get

$$\frac{1}{p^+} \|u\|^{p^+} \le \Phi(u) \le \rho + \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G$$

and then, being $\rho + \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G > 1$, we obtain

$$\|u\| \le \left(p^+ \left(\rho + \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G\right)\right)^{\frac{1}{p^+}} \le \left(p^+ \left(\rho + \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G\right)\right)^{\frac{1}{p^-}} \le \frac{\sigma}{c}.$$

On the other hand, if $\|u\| > 1$, we get

$$\frac{1}{p^+} \|u\|^{p^-} \le \Phi(u) \le \rho + \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G$$

and then

$$\|u\| \le \left(p^+ \left(\rho + \|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G\right)\right)^{\frac{1}{p^-}} \le \frac{\sigma}{c}.$$

Thus, owing to the embedding of $W^{1,p(x)}(\Omega)$ in $C^0(\overline{\Omega})$ we have the inclusion

$$\{u \in X : \Phi(u) - J_{\beta,G}(u) \leq \rho\} \subseteq \left\{ u \in X : \sup_{x\in\overline{\Omega}} |u(x)| \leq \sigma \right\}. \tag{8}$$

Now, in order to fulfill the equivalent formulation of (4) recalled in Remark 2.1, choose $u_1 = v_0$ and take as v_1 any constant d such that $F(d) > \sup\limits_{[-\sigma,\sigma]} F$. Such a d does exist by (a_3). Thanks to (a_4) we have

$$\Phi(u_1) - J_{\beta,G}(u_1) \leq \frac{|\Omega|}{p^-} \max\left\{ |\xi_1|^{p^-}, |\xi_1|^{p^+} \right\} - \|\beta\|_{L^1(\Omega)} \int_0^{\xi_1} g(t)dt \leq 0 < \rho.$$

Moreover, $\Phi(v_1) - J_{\beta,G}(v_1)$ has to be necessarily strictly greater than ρ, otherwise, by (8) we would have $|d| \leq \sigma$ and $F(d) \leq \sup\limits_{[-\sigma,\sigma]} F$, a contradiction. Then, due to (a_3) and to the choice of d, we readily obtain that

$$\sup_{(\Phi - J_{\beta,G})^{-1}(]-\infty,\rho])} J_{\alpha,F} \leq J_{\alpha,F}(u_1)$$

and

$$\sup_{(\Phi - J_{\beta,G})^{-1}(]-\infty,\rho])} J_{\alpha,F} \leq J_{\alpha,F}(v_1).$$

Thus, the following inequalities hold

$$\frac{\sup\limits_{(\Phi - J_{\beta,G})^{-1}(]-\infty,\rho])} J_{\alpha,F} - J_{\alpha,F}(u_1)}{\rho - \Phi(u_1) + J_{\beta,G}(u_1)} < 0 < \frac{\sup\limits_{(\Phi - J_{\beta,G})^{-1}(]-\infty,\rho])} J_{\alpha,F} - J_{\alpha,F}(v_1)}{\rho - \Phi(v_1) + J_{\beta,G}(v_1)}$$

and, each assumption of Theorem 2.1 being satisfied, Our problem admits at least three weak solutions. ■

Remark 3.1. It is worth pointing out that, if assumption (a_3) of Theorem 3.1 is replaced by the existence of

$$\sigma > c\left(p^+\right)^{\frac{1}{p^-}} \max\left\{ 1, \left(\|\beta\|_{L^1(\Omega)} \sup_{\mathbb{R}} G \right)^{\frac{1}{p^-}} \right\}$$

and $\xi_1, \xi_2 \in \mathbb{R}$, with $\xi_1\xi_2 > 0$, such that

$$0 < \int_0^{\xi_1} f(t)dt = \sup_{|\xi|\leq\sigma} \int_0^{\xi} f(t)dt < \int_0^{\xi_2} f(t)dt, \tag{9}$$

under the additional assumption

$$|\Omega| \geq \frac{p^-}{c^{p^-} p^+}, \tag{10}$$

we can ensure that the three nonzero solutions of the thesis of Theorem 3.1 are nonnegative (respectively nonpositive) provided $\xi_1 > 0$ (respectively $\xi_2 < 0$). To see this, it suffices to apply Theorem 3.1 to the functions $f_0, g_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_0(\xi) = \begin{cases} f(\xi) & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0, \end{cases}$$

$$g_0(\xi) = \begin{cases} g(\xi) & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0, \end{cases}$$

when $\xi_1 > 0$ or by

$$f_0(\xi) = \begin{cases} f(\xi) & \text{if } \xi \leq 0 \\ 0 & \text{if } \xi > 0, \end{cases}$$

$$g_0(\xi) = \begin{cases} g(\xi) & \text{if } \xi \leq 0 \\ 0 & \text{if } \xi > 0, \end{cases}$$

when $\xi_1 < 0$. In fact, conditions (9) and (a_2) ensure that $f(0) = 0$ and $g(0) = 0$, namely, that f_0 and g_0 are continuous functions. Furthermore, condition (10) guarantees that $|\xi_1| \leq \sigma$; in fact by (a_4) one has

$$|\xi_1|^{p^-} \leq \frac{p^- \|\beta\|_{L^1(\Omega)} \sup_{\mathbf{R}} G}{|\Omega|} \leq \frac{p^- \sigma^{p^-}}{c^{p^-} p^+ |\Omega|} \leq \sigma^{p^-}.$$

Hence, the function f_0 satisfies assumption (a_3).

From Theorem 3.1, applied with $f = g$ and $\alpha = \beta$, via Remark 3.1, we get:

Corollary 3.1. *Let* $|\Omega| > \dfrac{p^-}{c^{p^-} p^+}$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

$$(\mathrm{b}_1)\ \sup_{\xi \in \mathbf{R}} \int_0^\xi f(t)dt < +\infty, \quad \limsup_{\xi \to 0} \frac{\int_0^\xi f(t)dt}{|\xi|^{p^+}} \leq 0.$$

Moreover, suppose that there exist $\sigma > 0$ *and* $\xi_1, \xi_2 \in \mathbb{R}$, *with* $\xi_1 \xi_2 > 0$, *such that*

$$(\mathrm{b}_2)\ 0 < \int_0^{\xi_1} f(t)dt = \sup_{|\xi| \leq \sigma} \int_0^\xi f(t)dt < \int_0^{\xi_2} f(t)dt$$

and

$$(\mathrm{b}_3)\quad \frac{|\Omega| \max\left\{|\xi_1|^{p^-}, |\xi_1|^{p^+}\right\}}{p^- \int_0^{\xi_1} f(t)dt} < \frac{\sigma^{p^-}}{c^{p^-} p^+ \sup_{\xi\in\mathbf{R}} \int_0^{\xi} f(t)dt}.$$

Under such hypotheses, for every nonconstant function $\alpha : \Omega \to [0, +\infty[$ *in* $L^1(\Omega)$ *satisfying*

$$(\mathrm{b}_4)\quad \max\left\{\frac{1}{\sup_{\mathbf{R}} F}, \frac{|\Omega| \max\left\{|\xi_1|^{p^-}, |\xi_1|^{p^+}\right\}}{p^- \displaystyle\int_0^{\xi_1} f(t)dt}\right\} \le \|\alpha\|_{L^1(\Omega)} < \frac{\sigma^{p^-}}{c^{p^-} p^+ \displaystyle\sup_{\xi\in\mathbf{R}} \int_0^{\xi} f(t)dt},$$

there exists $\widehat{\lambda} > 1$ *such that the problem*

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \widehat{\lambda}\alpha(x)f(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three nonzero solutions which are nonnegative or nonpositive according to whether $\xi_1 > 0$ *or* $\xi_1 < 0$*.*

Another consequence of Theorem 3.1 is as follows:

Proposition 3.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function and* a, b, c, σ *four positive constants, with* $a < b < \sigma < c$*, such that*

$$f(\xi) \ge 0$$

for all $\xi \in]-\infty, -c] \cup [-\sigma, 0] \cup [a, b]$*, while*

$$f(\xi) \le 0$$

for all $\xi \in [-c, -\sigma] \cup [0, a] \cup [b, +\infty[$ *and*

$$0 < \int_0^b f(t)dt < \int_0^{-c} f(t)dt.$$

Moreover, let $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function and let* $\beta : \Omega \to [0, +\infty[$ *be a function in* $L^1(\Omega)$ *such that*

$$\int_0^b g(t)dt > 0, \quad \sup_{\xi\in\mathbf{R}} \int_0^{\xi} g(t)dt < +\infty, \quad \limsup_{\xi\to 0} \frac{\int_0^{\xi} g(t)dt}{|\xi|^{p^+}} < \frac{1}{c^{p^+} p^+ \|\beta\|_{L^1(\Omega)}}$$

and

$$\max\left\{\frac{1}{\sup_{\mathbf{R}} G}, \frac{|\Omega| \max\left\{b^{p^-}, b^{p^+}\right\}}{p^- \displaystyle\int_0^b g(t)dt}\right\} \le \|\beta\|_{L^1(\Omega)} < \frac{\sigma^{p^-}}{c^{p^-} p^+ \displaystyle\sup_{\xi\in\mathbf{R}} \int_0^{\xi} g(t)dt}.$$

Under such hypotheses, for each nonzero function $\alpha : \Omega \to [0, +\infty[$ *in* $L^1(\Omega)$, *there exists* $\lambda^* > 0$ *such that the problem*

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda^*\alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three nonzero solutions.

Proof. The assumptions on the sign of f readily imply that

$$\int_0^b f(t)dt = \sup_{|\xi|\le\sigma}\int_0^\xi f(t)dt < \sup_{\xi\in\mathbf{R}}\int_0^\xi f(t)dt = \int_0^{-c} f(t)dt.$$

The same assumptions also imply that $\displaystyle\int_0^\xi f(t)dt \le 0$ for all $\xi \in [-\sigma, a]$, and so (a_1) holds. Consequently, if we take $\xi_1 = b$, the assumptions of Theorem 3.1 are satisfied and the conclusion follows. ■

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