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(2,1)-TOTAL NUMBER OF JOINS OF PATHS AND CYCLES

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Abstract. The $(2,1)$-total number $\lambda_2^t(G)$ of a graph G is the width of the smallest range of integers that suffices to label the vertices and edges of G such that no two adjacent vertices or two adjacent edges have the same label and the difference between the label of a vertex and its incident edges is at least 2. In this paper, we characterize completely the $(2,1)$-total number of the join of two paths and the join of two cycles.

1. INTRODUCTION

Motivated by the Frequency Channel Assignment problem, Griggs and Yeh [7] introduced the $L(2,1)$-labelling of graphs. This notion was subsequently generalized to the $L(p,q)$-labelling problem of graphs. Let p and q be two nonnegative integers. An $L(p,q)$-*labelling* of a graph G is a function f from its vertex set $V(G)$ to the set $\{0,1,\ldots,k\}$ for some positive integer k such that $|f(x)-f(y)|\geq p$ if x and y are adjacent, and $|f(x)-f(y)|\geq q$ if x and y are at distance 2. The $L(p,q)$-*labelling number* $\lambda_{p,q}(G)$ of G is the smallest k such that G has an $L(p,q)$-labelling f with $\max\{f(v)\mid v\in V(G)\}=k$.

The $L(p,q)$-labelling of graphs have been studied rather extensively in recent years [3, 4, 13, 15, 16, 17, 18]. Whittlesey, Georges and Mauro investigated the $L(2,1)$-labelling of incidence graphs [21]. The *incidence graph* of a graph G is the graph obtained from G by replacing each edge by a path of length 2. The $L(2,1)$-labelling of the incidence graph of G is equivalent to an assignment of integers to each element of $V(G)\cup E(G)$ such that adjacent vertices have different labels, adjacent edges have different labels, and incident vertex and edge have labels that differ by at least 2. Such a labelling is called a $(2,1)$-total labelling of G, which was introduced by Havet and Yu and generalized to the $(d,1)$-total labelling [8].

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Let $d \geq 1$ be an integer. A k-$(d,1)$-*total labelling* of a graph G is a function f from $V(G) \cup E(G)$ to the set $\{0, 1, \ldots, k\}$ such that $f(u) \neq f(v)$ if u and v are two adjacent vertices, $f(e) \neq f(e')$ if e and e' are two adjacent edges, and $|f(u) - f(e)| \geq d$ if vertex u is incident to edge e. The $(d,1)$-*total number*, denoted by $\lambda_d^t(G)$, is the least k such that G has a k-$(d,1)$-total labelling.

When $d = 1$, the $(1,1)$-total labelling is the well-known total coloring of a graph, which has been extensively studied [2, 10, 12, 19].

Let $\Delta(G)$ (or simply Δ) denote the maximum degree of a graph G. Havet and Yu [8] proposed the following conjecture.

$(d,1)$-**Total Labelling Conjecture.** $\lambda_d^t(G) \leq \min\{\Delta + 2d - 1, 2\Delta + d - 1\}$.

In [8], it was shown that for any graph G, (i) $\lambda_d^t(G) \leq 2\Delta + d - 1$; (ii) $\lambda_d^t(G) \leq 2\Delta - 2\log(\Delta + 2) + 2\log(16d - 8) + d - 1$, (iii) $\lambda_2^t(G) \leq 2\Delta$; and (iv) $\lambda_2^t(G) \leq 2\Delta - 1$ if $\Delta \geq 5$ is odd. The $(d,1)$-total labelling for a few special graphs has been studied, e.g., complete graphs [8], complete bipartite graphs [11], planar graphs [1], outerplanar graphs [5], trees [9, 20], products of graphs [6], graphs with a given maximum average degree [14], etc.

The *join* $G \vee H$ of two vertex-disjoint graphs G and H is the graph obtained by joining each vertex of G to each vertex of H. If $C_m = u_1u_2 \ldots u_mu_1$ and $C_n = v_1v_2 \ldots v_nv_1$, with $n, m \geq 3$, are vertex-disjoint cycles, then

$V(C_m \vee C_n) = V(C_m) \cup V(C_n)$,

$E(C_m \vee C_n) = E(C_m) \cup E(C_n) \cup \{u_iv_j : i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$.

If $P_m = u_1u_2 \ldots u_m$ and $P_n = v_1v_2 \ldots v_n$, $n, m \geq 1$, are vertex-disjoint paths, then

$V(P_m \vee P_n) = V(P_m) \cup V(P_n)$,

$E(P_m \vee P_n) = E(P_m) \cup E(P_n) \cup \{u_iv_j : i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$.

In this paper, we will characterize completely the $(2,1)$-total number of the join of two paths and the join of two cycles.

2. Join of Cycles

The following two lemmas appeared in [8]:

Lemma 1. *Let G be a graph. Then*

(1) $\lambda_2^t(G) \geq \Delta + 1$.

(2) *For any $(\Delta+1)$-$(2,1)$-total labelling f of G using the labels $0, 1, \ldots, \Delta+1$, every vertex of maximum degree of G is assigned 0 or $\Delta + 1$.*

(3) *If H is a subgraph of G, then $\lambda_2^t(H) \leq \lambda_2^t(G)$.*

Lemma 2. *Let $n \geq 3$ be any integer. Then*

$$\lambda_2^t(K_n) = \begin{cases} n+1, & \text{if } n = 6,8 \text{ or } n \text{ is odd;} \\ n+2, & \text{otherwise.} \end{cases}$$

Let G_Δ denote the subgraph induced by all vertices of maximum degree in G. Chen and Wang [5] proved the following result:

Lemma 3. *If $\Delta(G_\Delta) \geq \Delta - 1$, then $\lambda_2^t(G) \geq \Delta + 2$.*

Lemma 4. *If G_Δ is not bipartite, then $\lambda_2^t(G) \geq \Delta + 2$.*

Proof. By Lemma 1, we may assume to the contrary that $\lambda_2^t(G) = \Delta + 1$. Let f be a $(\Delta+1)$-$(2,1)$-total labelling of G using $0, 1, \ldots, \Delta+1$. Thus, every vertex v of maximum degree of G has $f(v) = 0$ or $f(v) = \Delta + 1$. This implies that f is a 2-coloring restricted on G_Δ, hence G_Δ is bipartite, contradicting the assumption on G_Δ. ∎

Given a k-$(2,1)$-total labelling f of the graph G using the label set $B = \{0, 1, \ldots, k\}$, let σ_i and β_i denote the number of vertices and edges having the label i, respectively. Moreover, $\{x_1, x_2, \ldots, x_s\} \to (b_1, b_2, \ldots, b_l)$ denotes that the sequences of vertices or edges $x_1, x_2, \ldots, x_s$ are alternately labelled with repeated uses of the sequences of labels $b_1, b_2, \ldots, b_l$. For example, $\{v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5\} \to (1, 2, 3, 4)$ means that all elements in the subset $\{v_1, v_3, v_5\}$ are labelled with 1, $\{e_1, e_3\}$ with 2, $\{v_2, v_4\}$ with 3, and $\{e_2, e_4\}$ with 4, respectively. For a subset $S \subseteq V(G) \cup E(G)$ and a label $i \in B$, let $f(S) = i$ denote that all the elements in S are assigned label i, i.e., $f(x) = i$ for each $x \in S$. In particular, we simply write to indicate $f(x) = i$ for each $x \in \{a, b, \ldots, c\}$.

Theorem 5. *Let n, m be integers with $n \geq m \geq 3$. Then*

$$\lambda_2^t(C_m \vee C_n) = \begin{cases} n+3 & \text{if either } n \geq m+2 \text{ and } m \text{ is even,} \\ & \text{or } n = m+1 \text{ and } m \equiv 2, 4 \text{ (mod 12);} \\ n+4 & \text{otherwise.} \end{cases}$$

Proof. Let $G = C_m \vee C_n$ and write $\Delta = \Delta(G)$. Since $n \geq m \geq 3$, we see that $\Delta = n + 2$ by definition. We assume that all indices are taken modulo m for u_i and modulo n for v_j in the following argument. The proof is split into two cases.

Case 1. m is even.

Subcase 1.1. $n \geq m + 3$.

By Lemma 1(1), $\lambda_2^t(G) \geq \Delta + 1 = n + 3$. It thus suffices to establish an $(n+3)$-$(2,1)$-total labelling f of G using the labels $0, 1, \ldots, n+3$:

$\{u_1, u_1u_2, u_2, u_2u_3, \ldots, u_{m-1}u_m, u_m, u_mu_1\} \to (0, 2, n+3, 3)$,
$f(v_1) = f(v_3) = 1$, $f(v_2) = 2$,
$f(v_j) = j - 2$ for $4 \leq j \leq n - m + 2$, $f(v_j) = j$ for $n - m + 3 \leq j \leq n$,
$f(v_jv_{j+1}) = m + 3 + j$ for $j = 1, 2, 3$,
$f(v_jv_{j+1}) = j + 1$ for $4 \leq j \leq n - m + 1$,
$f(v_{n-m+2}v_{n-m+3}) = 0$.
Let $n - m + 3 \leq j \leq n$.

If n is odd, we set $f(v_jv_{j+1}) = 2$ when j is even, and $f(v_jv_{j+1}) = 3$ when j is odd.

If n is even, we set $f(v_jv_{j+1}) = 3$ when j is even, and $f(v_jv_{j+1}) = 2$ when j is odd.

For all i, j with $i + j \geq 3$, if $i + j + 1 \leq n + 3$, we set $f(u_iv_j) = i + j + 1$; otherwise, $f(u_iv_j) = p + 3$, where $i + j + 1 \equiv p$ (mod $(n + 3)$) and $p \geq 1$.

We relabel u_mv_{n-m+1} with 0, u_mv_{n-m+2} with 1 and u_1v_1 with $n + 3$. For $i = 2, 4, \ldots, m - 2$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled 1, and the edge u_iv_{j+1} with $f(u_iv_{j+1}) = n + 3$ is relabelled 0.

For example, a 14-$(2, 1)$-total labelling of $C_8 \vee C_{11}$ is given in Table 1.

Table 1: A 14-(2,1)-total labelling of $C_8 \vee C_{11}$.

3

			12	13	14	5	0	2	3	2	3	2	
		$C_8 \backslash C_{11}$	1	2	1	2	3	6	7	8	9	10	11
	2	0	14	4	5	6	7	8	9	10	11	12	13
	3	14	4	5	6	7	8	9	10	11	12	1	0
	2	0	5	6	7	8	9	10	11	12	13	14	4
3	3	14	6	7	8	9	10	11	12	1	0	4	5
	2	0	7	8	9	10	11	12	13	14	4	5	6
	3	14	8	9	10	11	12	1	0	4	5	6	7
	2	0	9	10	11	12	13	14	4	5	6	7	8
		14	10	11	12	0	1	4	5	6	7	8	9

In Table 1, the label 3 in the first row is assigned to the edge $v_{11}v_1$. The sequence of labels $12, 13, 14, \ldots, 3, 2$ in the second row are assigned to edges $v_1v_2, v_2v_3, v_3v_4, \ldots, v_9v_{10}, v_{10}v_{11}$, respectively. The sequence of labels $1, 2, 1, \ldots, 10, 11$ in the third row are assigned to vertices $v_1, v_2, v_3, \ldots, v_{10}, v_{11}$, respectively. The label 3 in the first column is assigned to the edge u_8u_1. The sequence of labels $2, 3, 2, \ldots, 3, 2$ in the second column are assigned to edges $u_1u_2, u_2u_3, u_3u_4, \ldots, u_6u_7, u_7u_8$, respectively. The sequence of labels $0, 14, 0, \ldots, 0, 14$ in the third column are assigned to vertices $u_1, u_2, u_3, \ldots, u_7, u_8$, respectively. Other labels in the table are assigned to edges u_iv_j for $i = 1, 2, \ldots, 8$ and $j = 1, 2, \ldots, 11$.

Subcase 1.2. $n = m + 2$.

Since $\lambda_2^t(G) \geq \triangle + 1 = n + 3$ by Lemma 1, it suffices to give an $(n + 3)$-$(2, 1)$-total labelling f of G using the labels $0, 1, \ldots, n + 3$:

$f(u_i) = 0$ if $i \geq 1$ is odd, $f(u_i) = n + 3$ if $i \geq 2$ is even.

$f(u_iu_{i+1}) = i + 1$ for $i = 1, 2, \ldots, m - 1$, $f(u_mu_1) = m + 1$.

$f(v_1v_2) = n + 2$, $f(v_nv_1) = n + 1$.

$f(v_j) = j$ if $1 \leq j \leq m - 2$, $f(v_{m-1}) = m$, $f(v_j) = j - 1$ if $m \leq j \leq n$.

For all $i, j \geq 1$, if $i + j + 1 \leq n + 3$, we set $f(u_iv_j) = i + j + 1$; otherwise, $f(u_iv_j) = p + 1$, where $i + j + 1 \equiv p$ (mod $(n + 3)$) and $p \geq 1$.

For $i = 3, 7, 11, \ldots$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled $n + 3$, and u_iv_j with $f(u_iv_j) = n + 3$ is relabelled $n + 2$.

Afterwards, we consider two subcases:

(a) If $m = 0$ (mod 4), we set $\{v_2v_3, v_3v_4, \ldots, v_{n-1}v_n\} \rightarrow (0, n + 2, 1, n + 3)$.

For $i = 2, 6, 10, \ldots$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled 0, and u_iv_j with $f(u_iv_j) = n + 3$ is relabelled 1.

For $i = 4, 8, 12, \ldots$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled 1, and u_iv_j with $f(u_iv_j) = n + 3$ is relabelled 0.

Finally, we relabel u_1v_3 with n+3, u_1u_j with $j-2$ for all $j = 7, 11, 15, \ldots, n-3$.

(b) If $m = 2$ (mod 4), we set $\{v_2v_3, v_3v_4, \ldots, v_{n-1}v_n\} \rightarrow (0, n + 3, 1, n + 2)$.

For $i = 4, 8, 12, \ldots$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled 0, and u_iv_j with $f(u_iv_j) = n + 3$ is relabelled 1.

For $i = 2, 6, 10, \ldots$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled 1, and u_iv_j with $f(u_iv_j) = n + 3$ is relabelled 0.

Finally, we relabel u_1v_1 with n+3, u_1v_j with $j-2$ for all $j = 5, 9, 13, \ldots, n$–3.

Subcase 1.3. $n = m + 1$.

Subcase 1.3.1. $m \not\equiv 2, 4$ (mod 12).

First, we give an $(n + 4)$-$(2, 1)$-total labelling f of G using $0, 1, \ldots, n + 4$:

$f(u_1) = 0$, $f(u_1u_2) = n + 3$, $f(u_2) = 1$, $f(u_2u_3) = n + 4$,

$\{u_3, u_3u_4, \ldots, u_m, u_mu_1\} \rightarrow (0, 3, 1, 4)$.

$f(v_1) = n + 4$, $f(v_1v_2) = n + 2$, $f(v_2) = 1$, $f(v_2v_3) = n + 3$, $f(v_3) = 2$,

$\{v_3v_4, v_4, \ldots, v_n, v_nv_1\} \rightarrow (0, 3, 1, 4)$.

For all $i, j \geq 1$, if $i + j \leq n + 4$, we set $f(u_iv_j) = i + j$; otherwise, $f(u_iv_j) = p + 4$, where $i + j \equiv p$ (mod $(n + 4)$) and $p \geq 1$.

For $i = 5, 7, 9, \ldots$, the edge u_iv_j with $f(u_iv_j) = 5$ is relabelled 2. Moreover, we relabel u_1v_3 with $n + 4$.

To show that $\lambda_2^t(G) \geq n + 4 = m + 5$, we suppose to the contrary that $\lambda_2^t(G) \leq n + 3 = m + 4$. Let f be an $(m + 4)$-$(2, 1)$-total labelling using $B = \{0, 1, \ldots, m + 4\}$. We may, by Lemma 1(2), assume that $f(u_i) = 0$ if i is odd, and $f(u_i) = m + 4$ if i is even. This implies that $\sigma_0 = \sigma_{m+4} = \frac{m}{2}$. Since $|V(G)| = 2m+1$ and $|E(G)| = m(m+1)+m+m+1 = m^2+3m+1$, we have

$$\sum_{i=0}^{m+4} \sigma_i = 2m + 1, \tag{1}$$

and

$$\sum_{i=0}^{m+4} \beta_i = m^2 + 3m + 1. \tag{2}$$

From (1), we conclude that $\sigma_1 + \sigma_2 + \cdots + \sigma_{m+3} = m + 1$. Let $S_i = \sigma_{i-1} + \sigma_i + \sigma_{i+1}$ for each $i \in B$, where $\sigma_{-1} = \sigma_{m+5} = 0$. Thus,

$$\beta_i \leq \lfloor \frac{2m + 1 - S_i}{2} \rfloor \leq m + \frac{1}{2} - \frac{1}{2}S_i. \tag{3}$$

Further,

$$\begin{aligned}
\sum_{i=0}^{m+4} \beta_i &\leq (m + 5)(m + \frac{1}{2}) - \frac{1}{2} \sum_{i=0}^{m+4} S_i \\
&= (m + 5)(m + \frac{1}{2}) - \frac{1}{2}[2\sigma_0 + 3(\sigma_1 + \sigma_2 + \cdots + \sigma_{m+3}) + 2\sigma_{m+4}] \\
&= m^2 + \frac{11}{2}m + \frac{5}{2} - \frac{1}{2}(2m + 3m + 3) \\
&= m^2 + 3m + 1.
\end{aligned}$$

By (2) and (3), $\sum_{i=0}^{m+4} \beta_i = m^2 + 3m + 1$ if and only if $\beta_i = \frac{2m+1-S_i}{2}$ for all $i \in B$. So, all S_i's must be odd. Since m is even, to finish the proof, we have two possibilities as follows:

(i) Assume that $m \equiv 0 \pmod 4$. In this case, $\sigma_0 = \sigma_{m+4} = \frac{m}{2}$ is even. Since $S_0 = \sigma_0 + \sigma_1$ is odd, it follows that $\sigma_1 = S_0 - \sigma_0$ is odd. Since $S_1 = \sigma_0 + \sigma_1 + \sigma_2$ is odd, it follows that $\sigma_2 = S_1 - \sigma_0 - \sigma_1$ is even. Since $S_2 = \sigma_1 + \sigma_2 + \sigma_3$ is odd, it follows that $\sigma_3 = S_2 - \sigma_1 - \sigma_2$ is even. Continuing this process, we derive that $\sigma_1, \sigma_4, \sigma_7, \ldots, \sigma_m, \sigma_{m+3}$ are odd, and $\sigma_0, \sigma_2, \sigma_3, \sigma_5, \sigma_6,\ \sigma_{m+1}, \sigma_{m+2}, \sigma_{m+4}$ are even. This implies that $m + 5 \equiv 0 \pmod 3$, so $m = 3k_1 + 1$ for some integer $k_1 \geq 1$. Note that $m \equiv 0 \pmod 4$, i.e., $m = 4k_2$ for some integer $k_2 \geq 2$.

Combining these two facts, we obtain that $m \equiv 4 \pmod{12}$, which contradicts the assumption.

(ii) Assume that $m \equiv 2 \pmod 4$. We note that $\sigma_0 = \sigma_{m+4} = \frac{m}{2}$ is odd. Since $S_0 = \sigma_0 + \sigma_1$ is odd, similar to discussion for (i), σ_i is odd for precisely $i = 0, 3, 6, 9, \ldots, m+1, m+4$, where $m+4$ divides 3. This implies that $m \equiv 2$ (mod 3) and (by assumption) $m \equiv 2$ (mod 4), so we have a contradiction that $m \equiv 2$ (mod 12).

Subcase 1.3.2. $m \equiv 2 \pmod{12}$.

It suffices to give an $(n+3)$-$(2,1)$-total labelling f of G using $0, 1, \ldots, n+3$:

$f(u_i) = 0$ if $i \geq 1$ is odd, $f(u_i) = n+3$ if $i \geq 2$ is even.

$f(u_i u_{i+1}) = i+1$ for $i = 1, 2, \ldots, m-1$, $f(u_m u_1) = m+1$.

$f(v_j) = j-1$ if $j \equiv 0 \pmod 3$; otherwise, we set $f(v_j) = j+1$.

$f(v_1v_2) = 0$, $f(v_2v_3) = n+2$, $f(v_nv_1) = n+3$,

$\{v_3v_4, v_4v_5, \ldots, v_{n-1}v_n\} \to (0, n+3, 1, n+2)$.

For all $i, j \geq 1$, if $i+j+1 \leq n+3$, we set $f(u_iv_j) = i+j+1$; otherwise, $f(u_iv_j) = p+1$, where $i+j+1 \equiv p \pmod{(n+3)}$ and $p \geq 1$.

For $i \equiv 1 \pmod 4$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled $n+3$, and u_iv_j with $f(u_iv_j) = n+3$ is relabelled $n+2$.

For $i \equiv 2 \pmod 4$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled 0, and u_iv_j with $f(u_iv_j) = n+3$ is relabelled 1.

For $i \equiv 0 \pmod 4$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled 1, and u_iv_j with $f(u_iv_j) = n+3$ is relabelled 0.

Finally, we relabel u_1v_1 with $n+2$, u_1v_2 with $n+3$, u_1v_n with $n+1$, and u_1v_j with $j-1$ for all $j \not\equiv 0 \pmod 3$ and $j \geq 4$. Relabel u_mv_n with 0, u_mv_2 with 1, and u_mv_j with $j+1$ for all $j \equiv 0 \pmod 3$ and $3 \leq j < n$.

Subcase 1.3.3. $m \equiv 4 \pmod{12}$.

It suffices to give an $(n+3)$-$(2,1)$-total labelling f of G using $0, 1, \ldots, n+3$:

$f(u_i) = 0$ if $i \geq 1$ is odd, $f(u_i) = n+3$ if $i \geq 2$ is even.

$f(u_1u_2) = 3$, $f(u_iu_{i+1}) = i$ for $i = 2, 3, \ldots, m-1$, $f(u_mu_1) = m$.

$f(v_1) = 1$, $f(v_2) = n+2$, $f(v_j) = j-2$ if $j \equiv 2 \pmod 3$ and $j \geq 3$; otherwise, we set $f(v_j) = j$.

$f(v_1v_2) = 3$, $\{v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1\} \to (0, n+3, 1, n+2)$.

For all $i, j \geq 1$, if $i+j \leq n+3$, we set $f(u_iv_j) = i+j$; otherwise, $f(u_iv_j) = p+1$, where $i+j \equiv p \pmod{(n+3)}$ and $p \geq 1$.

For $i \equiv 1 \pmod 4$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled $n+3$, and u_iv_j with $f(u_iv_j) = n+3$ is relabelled $n+2$.

For $i \equiv 2 \pmod 4$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled 0, and

u_iv_j with $f(u_iv_j) = n + 3$ is relabelled 1.

For $i \equiv 0 \pmod 4$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled 1, and u_iv_j with $f(u_iv_j) = n + 3$ is relabelled 0.

Then, we relabel u_1v_1 with n, u_1v_2 with $n+3$, u_1v_3 with $n+2$, and u_1v_j with $j-2$ for all $j \not\equiv 2 \pmod 3$ and $j \geq 4$. Relabel u_2v_1 with 1, u_mv_1 with 2, u_mv_2 with $n+1$, and u_mv_j with j for all $j \equiv 2 \pmod 3$ and $j \geq 5$. Finally, we need to exchange the obtained labels of u_iv_1 and u_iv_2 for all $i = 1, 2, \ldots, m$.

Subcase 1.4. $n = m$.

This means that G is an $(n+2)$-regular graph. By Lemma 3, $\lambda_2^t(G) \geq \Delta+2 = n + 4$. It thus suffices to give an $(n + 4)$-$(2, 1)$-total labelling f of G using the labels $0, 1, \ldots, n + 4$:

$\{u_1, u_1u_2, u_2, u_2u_3, \ldots, u_{n-1}u_n, u_n, u_nu_1\} \to (1, 3, 0, 4)$.

$\{v_1, v_1v_2, v_2, v_2v_3, \ldots, v_{n-1}v_n, v_n, v_nv_1\} \to (3, 1, 4, 0)$.

For all $i, j = 1, 2, \ldots, n$, if $i + j + 1 \leq n + 4$, we set $f(u_iv_j) = i + j + 1$; otherwise, $f(u_iv_j) = p + 4$, where $i + j + 1 \equiv p \pmod{(n + 4)}$ and $p \geq 1$.

We relabel u_1v_1 with $n + 3$, both u_1v_2 and u_2v_1 with $n + 4$, and u_iv_j with 2 if j is even and $f(u_iv_j) = 5$.

Case 2. m is odd.

Subcase 2.1. $n \geq m + 1$.

Since C_m is an odd cycle and $G_\Delta = C_m$, Lemma 4 shows that $\lambda_2^t(G) \geq \Delta+2 = n+4$. It suffices to establish an $(n+4)$-$(2, 1)$-total labelling f of G using $0, 1, \ldots, n + 4$:

$f(u_1) = n + 4$, $f(u_1u_2) = 2$, $\{u_2, u_2u_3, \ldots, u_m, u_mu_1\} \to (0, 3, 1, 4)$.

$f(v_1) = m + 4$, $f(v_2) = 3$, $f(v_3) = 2$, $f(v_2v_3) = m + 5$.

For all $i, j \geq 1$, if $i + j + 1 \leq n + 4$, we set $f(u_iv_j) = i + j + 1$; otherwise, $f(u_iv_j) = p + 4$, where $i + j + 1 \equiv p \pmod{(n + 4)}$ and $p \geq 1$. Afterwards, when $i \geq 4$ is even, the edge u_iv_j with $f(u_iv_j) = 5$ is relabelled 2.

If n is odd, we set $f(v_1v_2) = 1$ and relabel u_1v_2 with 0, and

$\{v_3v_4, v_4, v_4v_5, \ldots, v_{n-1}v_n, v_n, v_nv_1\} \to (0, 3, 1, 4)$.

If n is even, we set $f(v_1v_2) = 0$ and relabel u_1v_2 with 1, and

$\{v_3v_4, v_4, v_4v_5, \ldots, v_{n-1}v_n, v_n, v_nv_1\} \to (0, 4, 1, 3)$.

Subcase 2.2. $n = m$.

Since $C_3 \vee C_3$ is just K_6, $\lambda_2^t(K_6) = 7$ by Lemma 2. Thus, we only need to consider the case for $n = m \geq 5$. It is obvious that $\lambda_2^t(G) \geq \Delta + 2 = n + 4$ by Lemma 4. It suffices to give an $(n+4)$-$(2, 1)$-total labelling f of G using the labels $0, 1, \ldots, n + 4$:

$f(u_n) = n + 1$, $f(v_n) = n$, $f(u_1u_2) = n + 2$, $f(v_1v_2) = 2$,
$f(u_i) = 1$ if $1 \le i \le n - 2$ is odd, $f(u_i) = 0$ if $2 \le i \le n - 1$ is even,
$f(v_j) = n + 3$ if $1 \le j \le n - 2$ is odd, $f(v_j) = n+4$ if $2 \le j \le n-1$ is even,
$\{u_2u_3, u_3u_4, \ldots, u_{n-1}u_n, u_nu_1\} \to (n + 3, n + 4)$,
$\{v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1\} \to (0, 1)$,
$f(u_1v_n) = n + 3$, $f(u_2v_n) = n + 4$, $f(u_{n-1}v_n) = n + 2$, $f(u_nv_n) = 2$,
$f(u_iv_n) = i$ for $3 \le i \le n - 2$.

For odd j, if $i+j \le n+1$, we set $f(u_iv_j) = i+j$; otherwise, $f(u_iv_j) = p+1$, where $i + j \equiv p \pmod{(n + 1)}$, $p \ge 1$ and $j \le n - 2$.

For even j, if $i+j \le n+2$, we set $f(u_iv_j) = i+j$; otherwise, $f(u_iv_j) = p+2$, where $i + j \equiv p \pmod{(n + 2)}$, $p \ge 1$ and $j \le n - 1$.

Finally, we relabel u_1v_1 with $n + 1$, u_nv_1 with 0, u_nv_2 with 1.

This completes the proof. ∎

3. Join of Paths

In this section, we give a complete classification for the join of two paths according to their $(2, 1)$-total numbers. More precisely, we obtain the following result:

Theorem 6. *Let n, m be integers with $n \ge m \ge 1$. Then*

$$\lambda_2^t(P_m \vee P_n) = \begin{cases} n+1 & \text{if } m = 1 \text{ and } n \ge 4; \\ n+2 & \text{if } m = 1 \text{ and } 1 \le n \le 3, \text{ or } m = 2 \text{ and } n \ge 4; \\ n+3 & \text{if } m = 2 \text{ and } n = 3, \text{ or } m \ge 3 \text{ and } n \ge m+1; \\ n+4 & \text{if } m = n \ge 2. \end{cases}$$

Proof. We write simply $G = P_m \vee P_n$ and $\Delta = \Delta(G)$. In the following proof, all indices are taken modulo m for u_i and modulo n for v_j. We consider several cases, depending on the values of m and n.

Case 1. $m = 1$.

In this case, G is a fan with $\Delta = n$. If $n = 1$, then it is easy to check that $G = K_2$ and $\lambda_2^t(G) = 3 = n+2$. If $n = 2$, then $G = K_3$ and $\lambda_2^t(G) = 4 = n+2$. If $n = 3$, then G is the graph obtained by removing an edge of K_4. It is not difficult to verify that $\lambda_2^t(G) = 5 = n + 2$.

Assume that $n \ge 4$. On the one hand, $\lambda_2^t(G) \ge \Delta + 1 = n + 1$ by Lemma 1(1). On the other hand, an $(n + 1)$-$(2, 1)$-total labelling f of G using the labels $0, 1, \ldots, n + 1$ is constructed as follows:

$f(u_1, v_1v_2, v_4v_5) = 0$, $f(v_3) = 1$, $f(v_2, v_4, u_1v_1) = 2$, $f(u_1v_3) = 3$,

$f(v_5, v_3v_4, u_1v_2) = 4$, $f(v_1, v_2v_3) = 5$, $\{v_5v_6, v_6, \ldots, v_{n-1}v_n, v_n\} \to (1, 3, 5)$,
$f(u_1v_j) = j + 1$ for $j = 4, 5, \ldots, n$.

Case 2. $m = 2$.

If $n = 2$, then G is K_4 and $\lambda_2^t(G) = 6 = n + 4$ by Lemma 2.

If $n = 4$, to show that $\lambda_2^t(G) = 6 = n + 2$, it suffices to give a 6-$(2, 1)$-total labelling f of G using the labels $0, 1, \ldots, 6$ as follows:

$f(u_1, v_1v_2, u_2v_4) = 0$, $f(v_3, u_2v_1) = 1$, $f(v_2, u_1u_2) = 2$, $f(v_4, u_1v_1, u_2v_3) = 3$,
$f(u_2v_2, u_1v_3) = 4$, $f(v_1, u_1v_4, v_2v_3) = 5$, $f(u_2, u_1v_2, v_3v_4) = 6$.

Assume that $n = 3$. Since G contains a 3-cycle consisting of three vertices, u_1, u_2, v_2, of maximum degree, we have $\lambda_2^t(G) \geq \Delta + 2 = 6 = n + 3$ by Lemma 4. Since $P_2 \vee P_3$ is a subgraph of $P_2 \vee P_4$, $\lambda_2^t(P_2 \vee P_3) \leq \lambda_2^t(P_2 \vee P_4) = 6 = n + 3$ by Lemma 1(3) and the previous proof. Thus, $\lambda_2^t(G) = 6 = n + 3$.

Assume that $n \geq 5$. Since $\lambda_2^t(G) \geq \Delta + 1 = n + 2$, it suffices to give an $(n + 2)$-$(2, 1)$-total labelling f of G using the labels $0, 1, \ldots, n + 2$:

$f(u_1, v_{n-3}v_{n-2}, u_2v_{n-1}) = 0$, $f(v_{n-1}v_n, u_2v_{n-2}) = 1$.
$f(u_1u_2) = 2$, $f(u_2v_n) = 3$, $f(u_2, v_{n-2}v_{n-3}) = n + 2$.
$f(v_j) = j$ for $j = 1, 2, \ldots, n$.
$f(u_1v_j) = j + 2$ for $j = 1, 2, \ldots, n$.
$f(u_2v_j) = j + 3$ for $j = 1, 2, \ldots, n - 3$.
$f(v_jv_{j+1}) = j + 5$ for $j = 1, 2, \ldots, n - 4$.

Case 3. $m \geq 3$.

Subcase 3.1. $n = m = 3$.

Our goal is to show that $\lambda_2^t(G) = n + 4 = 7$. Since $G \subseteq K_6$ and $\lambda_2^t(G) \leq \lambda_2^t(K_6) = 7$ by Lemmas 1(3) and 2, it suffices to prove that $\lambda_2^t(G) \geq 7$. Assume to the contrary that G has a 6-$(2, 1)$-total labelling f using the label set $B = \{0, 1, \ldots, 6\}$. Since G has 6 vertices and 13 edges, we derive

$$\sum_{i=0}^{6} \sigma_i = 6, \tag{4}$$

and

$$\sum_{i=0}^{6} \beta_i = 13. \tag{5}$$

Since u_2 and v_2 are vertices of maximum degree, $\{f(u_2), f(v_2)\} = \{0, 6\}$ by Lemma 1(2), say $f(u_2) = 0$ and $f(v_2) = 6$. Hence, $f(x) \notin \{0, 6\}$ for all

$x \in V(G) \setminus \{u_2, v_2\}$. This implies that $\sigma_0 = \sigma_6 = 1$. Since only u_1 and u_3, or v_1 and v_3, may have the same label, it follows that $\sigma_i \le 2$ for all $1 \le i \le 5$.

Claim 1. *For each $i \in B$, $\beta_i \le \lfloor \frac{6-\sigma_{i-1}-\sigma_i-\sigma_{i+1}}{2} \rfloor$, where $\sigma_{-1} = \sigma_7 = 0$.*

Claim 1 implies that $\beta_i \le 3$ for all $i \in B$. Furthermore, since $\sigma_0 = \sigma_6 = 1$, we have $\beta_i \le 2$ for $i = 0, 1, 5, 6$. We consider two cases as follows:

Case (i). *There is some $k \in B$ such that $\beta_k = 3$.*

We notice that $k \in \{2, 3, 4\}$. By symmetry, we consider two subcases:

• $\beta_2 = 3$. Then $\sigma_1 = \sigma_2 = \sigma_3 = 0$, and $f(u_1) = f(u_3) = i_1$ and $f(v_1) = f(v_3) = i_2$ with $\{i_1, i_2\} = \{4, 5\}$. It is easy to see that $\beta_5 = 0$, $\beta_4, \beta_6 \le 1$, $\beta_3 \le 2$ by Claim 1. Thus,

$$\sum_{i=0}^{6} \beta_i \le 2 + 2 + 3 + 2 + 1 + 0 + 1 = 11,$$

which contradicts (5).

• $\beta_3 = 3$. We note that $\sigma_2 = \sigma_3 = \sigma_4 = 0$, and $f(u_1) = f(u_3) = i_1$ and $f(v_1) = f(v_3) = i_2$ with $\{i_1, i_2\} = \{1, 5\}$. It follows that $\beta_0, \beta_1, \beta_5, \beta_6 \le 1$, $\beta_2, \beta_4 \le 2$ and hence

$$\sum_{i=0}^{6} \beta_i \le 3 + 2 \times 2 + 4 \times 1 = 11,$$

again contradicting (5).

Case (ii). *For all $i \in B$, $\beta_i \le 2$.*

If $\sigma_i \le 1$ for all $i \in B$, then there must exist two distinct labels $p, q \in \{1, 2, \ldots, 5\}$ such that $\sigma_{p-1} = \sigma_p = \sigma_{p+1} = 1$ and $\sigma_{q-1} = \sigma_q = \sigma_{q+1} = 1$, which implies that $\beta_p = \beta_q = 1$ by Claim 1 and therefore

$$\sum_{i=0}^{6} \beta_i \le 2 \times 1 + 5 \times 2 = 12,$$

which contradicts (5).

Suppose that $\sigma_{i_0} = 2$ for some $i_0 \in B$. It is immediate to derive that $i_0 \in \{1, 2, \ldots, 5\}$. By symmetry, it suffices to handle the case for $i_0 \in \{1, 2, 3\}$.

If $i_0 = 1$, then $\sigma_0 + \sigma_1 = 1 + 2 = 3$ and $\beta_0 = \beta_1 = 1$ by Claim 1. Consequently, $\sum_{i=0}^{6} \beta_i \le 2 \times 1 + 5 \times 2 = 12$.

Assume that $i_0 = 2$. Since $\sigma_0 = 1$ and $\sigma_2 = 2$, $\beta_1 \le \lfloor (6 - 1 - 2)/2 \rfloor = 1$. If $\sigma_1 \ge 1$ or $\sigma_3 \ge 1$, then $\sigma_1 + \sigma_2 + \sigma_3 \ge 3$ to make that $\beta_2 = 1$ and $\sum_{i=0}^{6} \beta_i \le 12$. If $\sigma_1 = \sigma_3 = 0$, then $\sigma_4 + \sigma_5 + \sigma_6 = 6 - 1 - 2 = 3$, and hence $\beta_5 = 1$ and $\sum_{i=0}^{6} \beta_i \le 12$.

Assume that $i_0 = 3$. If $\sigma_2 \geq 1$ or $\sigma_4 \geq 1$, then $\beta_3 = 1$ and at least one of β_2 and β_4 is equal to 1, thus $\sum_{i=0}^{6} \beta_i \leq 12$. If $\sigma_2 = \sigma_4 = 0$, then $\sigma_1 + \sigma_5 = 2$. If $\sigma_1 = \sigma_5 = 1$, then $\beta_2 = \beta_4 = 1$ and $\sum_{i=0}^{6} \beta_i \leq 12$. If $\sigma_1 = 2$ or $\sigma_5 = 2$, then we may assume that $\sigma_1 = 2$ (up to symmetry). Since this is the case that $i_0 = 1$, we can obtain that $\sum_{i=0}^{6} \beta_i \leq 12$.

Since each assumption yields the contradiction $\sum_{i=0}^{6} \beta_i \leq 12$, Subcase 3.1 is concluded.

Subcase 3.2. $n = m \geq 4$.

Since G contains a 3-cycle consisting of three vertices of maximum degree, $\lambda_2^t(G) \geq \Delta + 2 = n + 4$ by Lemma 4. Since $P_n \vee P_n$ is a subgraph of $C_n \vee C_n$, we derive $\lambda_2^t(G) \leq n + 4$ by Lemma 1, Subcases 1.3 and 2.2 in Theorem 5. Consequently, $\lambda_2^t(G) = n + 4$.

Subcase 3.3. $n = m + 1$.

It is obvious that $\lambda_2^t(G) \geq \Delta + 1 = n + 3$ by Lemma 1. It suffices to establish an $(n + 3)$-$(2, 1)$-total labelling f of G using the labels $0, 1, \ldots, n + 3$:

$f(u_i) = 0$ if $i \geq 1$ is odd, $f(u_i) = n + 3$ if $i \geq 2$ is even.

$f(v_1) = n + 2$, $f(v_j) = j - 1$ for $j = 2, 3, \ldots, n$.

$f(u_1u_2) = 3$, $f(u_iu_{i+1}) = i$ for $i = 2, 3, \ldots, m - 1$.

For all $i, j \geq 1$, if $i + j \leq n + 3$, we set $f(u_iv_j) = i + j$; otherwise, $f(u_iv_j) = p + 1$, where $i + j \equiv p \pmod{(n + 3)}$ and $p \geq 1$.

$f(v_1v_2) = 3$, $f(v_2v_3) = n + 2$.

If m is odd, then $\{v_3v_4, v_4v_5, \ldots, v_{n-1}v_n\} \to (0, n + 2, 1, n + 3)$.
If m is even, then $\{v_3v_4, v_4v_5, \ldots, v_{n-1}v_n\} \to (n + 3, 0, n + 2, 1)$.
To relabel some edges, we need to consider two cases as follows:

(a) If $m \equiv 0$ or 3 (mod 4), we relabel u_1v_2 with $n+3$, u_2v_1 with 0, u_2v_n with 1.

For $i = 4, 8, 12, \ldots$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled 0, and u_iv_{j+1} with $f(u_iv_{j+1}) = n + 3$ is relabelled 1.

For $i = 6, 10, 14, \ldots$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled 1, and u_iv_{j+1} with $f(u_iv_{j+1}) = n + 3$ is relabelled 0.

For $i = 5, 9, 13, \ldots$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled $n + 3$, and u_iv_{j+1} with $f(u_iv_{j+1}) = n + 3$ is relabelled $n + 2$.

(b) If $m \equiv 1$ or 2 (mod 4), we relabel u_1v_2 with $n + 3$, u_2v_1 with 1, u_2v_n with 0.

For $i = 4, 8, 12, \ldots$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled 1, and u_iv_{j+1} with $f(u_iv_{j+1}) = n+3$ is relabelled 0.

For $i = 6, 10, 14, \ldots$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled 0, and u_iv_{j+1} with $f(u_iv_{j+1}) = n+3$ is relabelled 1.

For $i = 3, 7, \ldots$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled $n+3$, and u_iv_{j+1} with $f(u_iv_{j+1}) = n+3$ is relabelled $n+2$.

Subcase 3.4. $n = m+2$.

By Lemma 1(1), $\lambda_2^t(P_m \vee P_n) \geq \Delta + 1 = n+3$.

If m is even, the result follows from Subcase 1.2 in Theorem 5.

If m is odd, we only need to give an $(n+3)$-$(2,1)$-total labelling f of G using the labels $0, 1, \ldots, n+3$:

$f(u_i) = 0$ if $i \geq 1$ is odd, $f(u_i) = n+3$ if $i \geq 2$ is even.

$f(v_1) = n+1$, $f(v_j) = j-1$ for $j = 2, 3, \ldots, n$.

$f(u_1u_2) = 3$, $f(u_iu_{i+1}) = i$ for $i = 2, 3, \ldots, m-1$.

$f(v_1v_2) = 3$, $f(v_2v_3) = n+2$, $f(v_3v_4) = n+3$.

If $m \equiv 1$ (mod 4), then $\{v_4v_5, v_5v_6, \ldots, v_{n-1}v_n\} \rightarrow (1, n+2, 0, n+3)$.

If $m \equiv 3$ (mod 4), then $\{v_4v_5, v_5v_6, \ldots, v_{n-1}v_n\} \rightarrow (0, n+2, 1, n+3)$.

For all $i, j \geq 1$, if $i + j \leq n+3$, we set $f(u_iv_j) = i+j$; otherwise, $f(u_iv_j) = p+1$, where $i+j \equiv p$ (mod $(n+3)$) and $p \geq 1$.

If $m \equiv 1$ (mod 4), for $i = 3, 7, 11, \ldots$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled $n+3$, and u_iv_{j+1} with $f(u_iv_{j+1}) = n+3$ is relabelled $n+2$.

If $m \equiv$3 (mod 4), for $i = 5, 9, 13, \ldots$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled $n+3$, and u_iv_{j+1} with $f(u_iv_{j+1}) = n+3$ is relabelled $n+2$.

We relabel u_1v_2 with $n+3$, u_2v_1 with 0, u_2v_n with 1.

For $i = 4, 8, 12, \ldots$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled 0, and u_iv_{j+1} with $f(u_iv_{j+1}) = n+3$ is relabelled 1.

For $i = 6, 10, 14, \ldots$, the edge u_iv_j with $f(u_iv_j) = n+2$ is relabelled 1, and u_iv_{j+1} with $f(u_iv_{j+1}) = n+3$ is relabelled 0.

Subcase 3.5. $n \geq m+3$.

By Lemma 1(1), $\lambda_2^t(P_m \vee P_n) \geq \Delta + 1 = n+3$. It suffices to give an $(n+3)$-$(2,1)$-total labelling f of G using $0, 1, \ldots, n+3$:

$f(v_1v_2) = m+4$, $f(v_2v_3) = m+5$, $f(v_3v_4) = m+6$.

$\{u_1, u_1u_2, u_2, u_2u_3, \ldots, u_{m-1}u_m, u_m\} \rightarrow (0, 2, n+3, 3)$.

$f(v_j) = j$ for $j = 1, 2, \ldots, n$, $\{v_4v_5, v_5v_6, \ldots, v_{n-1}v_n\} \rightarrow (2, 3)$.

For all $i, j \geq 1$, if $i+j+1 \leq n+3$, we set $f(u_iv_j) = i+j+1$; otherwise, $f(u_iv_j) = p+3$, where $i+j+1 \equiv p$ (mod $(n+3)$) and $p \geq 1$.

For $i = 2, 4, 6, \ldots$, the edge u_iv_j with $f(u_iv_j) = n + 2$ is relabelled 0, and u_iv_{j+1} with $f(u_iv_{j+1}) = n + 3$ is relabelled 1. ■

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