TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 2, pp. 579-604, April 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

CONTINUOUS-TIME QUADRATIC PROGRAMMING PROBLEMS: APPROXIMATE SOLUTIONS AND ERROR ESTIMATION

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Abstract. A class of continuous-time quadratic programming problems (CQP) is discussed in this paper. We propose a discrete approximation procedure to find numerical solutions of (CQP). The provided computational procedure can yield feasible numerical solutions and bounds on the error introduced by the numerical approximation. It can also be shown that the searched sequence of approximate solution functions weakly-star converges to an optimal solution of (CQP). Besides, some numerical examples are provided to implement our proposed method and to show the quality of the proposed error bound.

1. INTRODUCTION

Let $L^\infty([0,T],\mathbb{R}^p)$ be the space of all essentially bounded measurable functions from $[0,T]$ into $\mathbb{R}^p$, where $[0,T]$ is a time space and $T>0$ is fixed. Let $C([0,T],\mathbb{R}^p)$ be the space of all continuous functions from $[0,T]$ into $\mathbb{R}^p$. The object of our study here is the following optimization problem, which is called the *continuous-time quadratic programming problem* (CQP):

$$\begin{aligned}
\text{(CQP)}\quad \text{maximize}\quad & \int_0^T \left\{1/2\,\mathbf{x}(t)^\top D(t)\,\mathbf{x}(t)+\mathbf{f}(t)^\top\mathbf{x}(t)\right\}dt\\
\text{subject to}\quad & B\mathbf{x}(t)-\int_0^t K\mathbf{x}(s)ds\le \mathbf{g}(t)\ \text{for}\ t\in[0,T],\\
& \mathbf{x}(\cdot)\in L^\infty([0,T],\mathbb{R}^q_+),
\end{aligned}$$

where B and K are $p\times q$ matrices, $D(t)=[\,d_{ij}(t)\,]_{q\times q}$ is a symmetric negative semi-definite matrix in which $d_{ij}(t)\in C([0,T],\mathbb{R})$, $\mathbf{f}(\cdot)\in C([0,T],\mathbb{R}^q)$,

Received March 10, 2010, accepted December 1, 2010.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: 90C20, 90C48.
Key words and phrases: Continuous-time quadratic programming problems, Continuous-time linear programming problems.
This work was supported by NSC-100-2115-M-037-001 and NSC-100-2115-M-238-001.
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$\mathbf{g}(\cdot) \in C([0,T], \mathbb{R}^p_+)$, $\mathbb{R}^p_+ := \{(x_1, \cdots, x_p)^\top : x_i \geq 0 \text{ for } i = 1, \cdots, p\}$ and the superscript " $^\top$ " denotes the transpose operation.

In the special case when the given $D(t)$ is the zero matrix, the (CQP) reduces to what is known as the *continuous-time linear programming problem* (CLP). (CLP) was first introduced by Bellman [5] under the name "bottleneck problem". The theory of continuous-time linear programming problem has received considerable attention for a long time. Tyndall [42] established the duality theory of (CLP). The result was later extended by Levison [20], Tyndall [43], Hanson [17, 18], Hanson and Mond [19], Gogia and Kanpur [13] and Grinold [14, 15, 16]. Besides, Meidan and Perold [21], Papageorgiou [24] and Schechter [34] have also obtained some interesting results of various classes of (CLP). On the other hand, Anderson et al. [1, 2, 3], Fleischer and Sethuraman [10], Pullan [25, 26, 27, 28, 29] and Wang et al. [44] investigated a subclass of continuous-time linear programming problem, which is called the separated continuous-time linear programming problem and can be used to model the job-shop scheduling problems. Recently, Weiss [45] proposed a simplex-like algorithm to solve the separated continuous-time linear programming problem. Besides, Wen et al. [46, 49] developed numerical methods to solve the non-separated continuous-time linear programming problem.

The optimization problem in which the objective function appears as a ratio of two real-valued function is known as a fractional programming problem. Due to its significance appearing in the information theory, stochastic programming and decomposition algorithms for large linear systems, the various theoretical and computational issues have received particular attention in the last decades. For more details on this topic, we may refer to Stancu-Minasian [40] and Schaible et al. [11, 35, 36, 37]. On the other hand, Zalmai [50, 51, 52, 53] investigated the continuous-time fractional programming problems. Moreover, Stancu-Minasian and Tigan [41] studied the stochastic continuous-time linear fractional programming problem. Recently, Wen et al. [47] used the Charnes and Cooper's transformation to develop a numerical algorithm for solving a class of continuous-time linear fractional programming problems. Meanwhile, Wen and Wu [48] proposed a Dinkelbach-type algorithm to solve the same class of problems.

The other nonlinear type of continuous-time optimization problems was also studied by Farr and Hanson [8, 9], Grinold [14, 15], Hason [17, 18], Hanson and Mond [19], Reiland [30, 31], Reiland and Hanson [32] and Singh [38]. The nonsmooth continuous-time optimization problems was studied by Rojas-Medar et al. [33] and Singh and Farr [39]. The nonsmooth continuous-time multiobjective programming problems was also studied by Nobakhtian and Pouryayevali [22, 23].

Although the nonlinear type of continuous-time optimization problems has been investigated as mentioned above, the computational study is very scanty in the literature. For solving the problem (CQP), Andreani et al. [4] developed a numerical method which is an extension of the purely linear cases studied by Buie and Abrham [6] and Pullan [25] to linear-quadratic problems. However, the provided method in [4] has several drawbacks. For instance, the searched numerical solutions may not be feasible; one can not know how accurate the searched solution is, and there is no easily checked termination criterion. Motivated by the above disadvantages, we shall provide a computational procedure which yields feasible numerical solutions and bounds on the error introduced by the numerical approximation. Besides, the proposed procedure can also generate an approximate solution with pre-defined error bound. These show the usefulness of the proposed method.

This paper is organized as follows. In Section 2, we introduce the dual problem of (CQP) and review the duality properties developed by Gogia and Kanpur [13]. In Section 3, we formulate the discretization problem of the continuous-time quadratic programming problem and also derive their relationships that will be used for designing the practical algorithm. In Section 4, based on the solutions obtained from the discretization problem, we can construct the feasible solutions of the continuous-time quadratic programming problem depending on the step sizes of discretization, which will also be termed as natural solutions. In Section 5, we show that the sequence of the natural solutions weakly-star converges to the optimal solutions of the problem (CQP). In the final Section 6, the computational procedure is proposed and the numerical examples are provided to demonstrate the usefulness of this practical algorithm.

2. Duality Properties of (CQP)

For convenience, we write $F(\text{P})$ and $V(\text{P})$ to denote the feasible set and optimal objective value of an optimization problem (P), respectively. According to Gogia and Kanpur [13], the dual problem of (CQP) can be formulated as follows:

$$\begin{aligned}
\text{(DCQP)}\quad \text{minimize}\quad & \int_0^T \{-1/2\,\mathbf{u}(t)^\top D(t)\mathbf{u}(t) + \mathbf{g}(t)^\top \mathbf{w}(t)\}dt \\
\text{subject to}\quad & B^\top \mathbf{w}(t) - \int_t^T K^\top \mathbf{w}(s)ds \geq D(t)\mathbf{u}(t) + \mathbf{f}(t) \text{ for } t \in [0,T], \\
& \mathbf{w}(\cdot) \in L^\infty([0,T],\mathbb{R}_+^p) \text{ and } \mathbf{u}(\cdot) \in L^\infty([0,T],\mathbb{R}^q).
\end{aligned}$$

Gogia and Kanpur [13] has shown the duality properties under some suitable assumptions, which are given below.

Theorem 2.1. (Weak Duality). *Let* $\mathbf{x}^{(0)}(t)$ *and* $(\mathbf{u}^{(0)}(t), \mathbf{w}^{(0)}(t))$ *be feasible for (CQP) and (DCQP), respectively, then*

$$\begin{aligned}
& \int_0^T \{1/2\,\mathbf{x}^{(0)}(t)^\top D(t)\mathbf{x}^{(0)}(t) + \mathbf{f}(t)^\top \mathbf{x}^{(0)}(t)\}dt \\
\leq & \int_0^T \{-1/2\,\mathbf{u}^{(0)}(t)^\top D(t)\mathbf{u}^{(0)}(t) + \mathbf{g}(t)^\top \mathbf{w}^{(0)}(t)\}dt
\end{aligned}$$

and hence $V(CQP) \leq V(DCQP)$.

Theorem 2.2. (Strong Duality). *Suppose the following conditions are satisfied:*

$(H1)$ $\{\mathbf{x} \in \mathbb{R}^q : B\mathbf{x} \leq \mathbf{0}, \mathbf{x} \geq \mathbf{0}\} = \{\mathbf{0}\}$*;*

$(H2)$ B*,* K *and* $\mathbf{g}(t)$ *have nonnegative components for all* $t \in [0, T]$.

If there exists an optimal solution $\widetilde{\mathbf{x}}(t)$ *of primal problem (CQP), then there exists an optimal solution* $(\widetilde{\mathbf{u}}(t), \widetilde{\mathbf{w}}(t))$ *of dual problem (DCQP) such that* $\widetilde{\mathbf{u}}(t) = \widetilde{\mathbf{x}}(t)$ *and*

$$\int_0^T \{1/2\,\widetilde{\mathbf{x}}(t)^\top D(t)\widetilde{\mathbf{x}}(t) + \mathbf{f}(t)^\top \widetilde{\mathbf{x}}(t)\}dt = \int_0^T \{-1/2\,\widetilde{\mathbf{u}}(t)^\top D(t)\widetilde{\mathbf{u}}(t) + \mathbf{g}(t)^\top \widetilde{\mathbf{w}}(t)\}dt.$$

That is, if (CQP) is solvable then $V(CQP) = V(DCQP)$.

Let $B = [B_{ij}]_{p\times q}$ and $K = [K_{ij}]_{p\times q}$. For the remainder of this paper, we assume that the considered problem (CQP) satisfies the following conditions:

$(A1)$ $\mathbf{g}(t) \geq \mathbf{0}$ for all $t \in [0, T]$, $B_{ij} \geq 0$ and $K_{ij} \geq 0$ for all $i = 1, 2, \cdots, p$, and $j = 1, 2, \cdots, q$.

$(A2)$ $\sum_{i=1}^p B_{ij} > 0$ for all $j = 1, 2, \cdots, q$.

Now, we have the following observations.

- It is not difficult to see that the above conditions (A1) and (A2) are equivalent to the conditions (H1) and (H2) in Theorem 2.2.
- Under assumptions (A1) and (A2), we can see that both the problems (CQP) and (DCQP) are feasible by the forthcoming Lemma 4.1 and Lemma 4.2.
- Since the primal and dual problems are feasible as shown above, we have

$$-\infty < V(\text{CQP}) \leq V(\text{DCQP}) < \infty. \tag{1}$$

- By the forthcoming Theorem 5.1, we can see that the problem (CQP) is solvable, and hence the dual problem (DCQP) is also solvable with $V(\text{CQP}) = V(\text{DCQP})$ by Theorem 2.2.

3. Discretization of Continuous-time Quadratic Programming Problems

Given any $n \in \mathbb{N}$, let $\mathcal{P}_n = \{0, \frac{1}{n}T, \frac{2}{n}T, \cdots, \frac{n-1}{n}T, T\}$ be a partition on $[0, T]$ which divides the time interval $[0, T]$ into n subintervals with equal length $\frac{T}{n}$. Define

$$D^{(n,l)} = \left[d_{ij}^{(n,l)}\right]_{q\times q}, \text{ where } d_{ij}^{(n,l)} = min\left\{d_{ij}(t) : t \in \left[\frac{l-1}{n}T, \frac{l}{n}T\right]\right\}. \tag{2}$$

For $l = 1, 2, \cdots, n$, we let

(3) $$\mathbf{b}_l^{(n)} := (b_{1l}^{(n)}, b_{2l}^{(n)}, \cdots, b_{pl}^{(n)})^\top \in \mathbb{R}^p$$

and

(4) $$\mathbf{c}_l^{(n)} := (c_{1l}^{(n)}, c_{2l}^{(n)}, \cdots, c_{ql}^{(n)})^\top \in \mathbb{R}^q,$$

where

(5) $$b_{il}^{(n)} = min\{g_i(x) : x \in [\frac{l-1}{n}T, \frac{l}{n}T]\}$$

and

(6) $$c_{jl}^{(n)} = min\{f_j(x) : x \in [\frac{l-1}{n}T, \frac{l}{n}T]\}.$$

We define step functions $\mathbf{f}^{(n)} : [0, T] \mapsto \mathbb{R}^q$ and $\mathbf{g}^{(n)} : [0, T] \mapsto \mathbb{R}^p$ as follows:

$$\mathbf{f}^{(n)}(t) = (f_1^{(n)}(t), f_2^{(n)}(t), \cdots, f_q^{(n)}(t))^\top$$

and

$$\mathbf{g}^{(n)}(t) = (g_1^{(n)}(t), g_2^{(n)}(t), \cdots, g_p^{(n)}(t))^\top,$$

where for $1 \le i \le p$ and $1 \le j \le q$,

(7) $$f_j^{(n)}(t) = \begin{cases} c_{jl}^{(n)}, & \text{if } t \in [\frac{l-1}{n}T, \frac{l}{n}T) \text{ for some } 1 \le l \le n, \\ c_{jn}^{(n)}, & \text{if } t = T, \end{cases}$$

and

(8) $$g_i^{(n)}(t) = \begin{cases} b_{il}^{(n)}, & \text{if } t \in [\frac{l-1}{n}T, \frac{l}{n}T) \text{ for some } 1 \le l \le n, \\ b_{in}^{(n)}, & \text{if } t = T. \end{cases}$$

We also define the function $D^{(n)} : [0, T] \mapsto \mathbb{R}^{q \times q}$ by

$$D^{(n)}(t) = \left[d_{ij}^{(n)}(t) \right]_{q \times q}, \text{ where}$$

(9) $$d_{ij}^{(n)}(t) = \begin{cases} d_{ij}^{(n,l)}, & \text{if } t \in [\frac{l-1}{n}T, \frac{l}{n}T) \text{ for some } 1 \le l \le n, \\ d_{ij}^{(n,n)}, & \text{if } t = T \end{cases}$$

and $d_{ij}^{(n,l)}$ is defined in (2). The discretization of primal problem (CQP) is formulated as the following quadratic programming problem

$$
\begin{aligned}
&(\mathrm{P}_n) \quad \text{maximize} \quad && \frac{T}{n}\sum_{l=1}^{n}\{1/2\,\mathbf{x}_l^\top D^{(n,l)}\,\mathbf{x}_l + (\mathbf{c}_l^{(n)})^\top \mathbf{x}_l\} \\
&\text{(10)} \qquad \text{subject to} && B\mathbf{x}_l - \frac{T}{n}K\sum_{r=1}^{l-1}\mathbf{x}_r \le \mathbf{b}_l^{(n)} \text{ for } l = 1,\cdots,n \\
& && \mathbf{x}_l \in \mathbb{R}_+^q \text{ for } l = 1,\cdots,n,
\end{aligned}
$$

where the "empty sum" $\sum_1^0$ is regarded as the zero vector. According to [7], the dual problem (D_n) of (P_n) is defined as follows:

$$
\begin{aligned}
(\mathrm{D}_n) \quad \text{minimize} \quad & \frac{T}{n}\sum_{l=1}^{n}\{-1/2\,\mathbf{u}^\top D^{(n,l)}\mathbf{u} + (\mathbf{b}_l)^\top \mathbf{w}_l\} \\
\text{subject to} \quad & B^\top \mathbf{w}_l - \frac{T}{n}K^\top \sum_{r=l+1}^{n}\mathbf{w}_r \ge D^{(n,l)}\mathbf{u}_l + \mathbf{c}_l^{(n)} \text{ for } l = 1,2,\cdots,n \\
& \mathbf{w}_l \in \mathbb{R}_+^p \text{ for } l = 1,\cdots,n \text{ and} \\
& \mathbf{u}_l \in \mathbb{R}^q \text{ for } l = 1,\cdots,n,
\end{aligned}
$$

where "empty sum" $\sum_{n+1}^{n}$ is also regarded as the zero vector. It is not difficult to establish the weak duality theorem for (P_n) and (D_n). We omit the proof.

Proposition 3.1. *Let* $\mathbf{x}^{(n)} = (\mathbf{x}_1,\cdots,\mathbf{x}_n)$ *and* $(\mathbf{u}^{(n)},\mathbf{w}^{(n)})$ *with* $\mathbf{u}^{(n)} = (\mathbf{u}_1,\cdots,\mathbf{u}_n)$ *and* $\mathbf{w}^{(n)} = (\mathbf{w}_1,\cdots,\mathbf{w}_n)$ *be feasible solutions of* (P_n) *and* (D_n)*, respectively. Then*

$$
\frac{T}{n}\sum_{l=1}^{n}\{1/2\,\mathbf{x}_l^\top D^{(n,l)}\mathbf{x}_l + (\mathbf{c}_l^{(n)})^\top \mathbf{x}_l\} \le \frac{T}{n}\sum_{l=1}^{n}\{-1/2\,\mathbf{u}^\top D^{(n,l)}\mathbf{u} + (\mathbf{b}_l)^\top \mathbf{w}_l\}
$$

and $V(P_n) \le V(D_n)$.

According to the forthcoming Lemma 3.1, the feasible domain of primal problem (P_n) is compact, and hence the existence of its optimal solution can be ensured. Therefore, according to [7], the strong duality property holds true as shown below.

Proposition 3.2. *There exist feasible solution* $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_1,\cdots,\bar{\mathbf{x}}_n)$ *of primal problem* (P_n) *and feasible solution* $(\bar{\mathbf{u}},\bar{\mathbf{w}})$ *of dual problem* (D_n) *with* $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_1,\cdots,\bar{\mathbf{u}}_n)$ *and* $\bar{\mathbf{w}} = (\bar{\mathbf{w}}_1,\cdots,\bar{\mathbf{w}}_n)$ *such that* $\bar{\mathbf{x}} = \bar{\mathbf{u}}$ *and*

$$
\frac{T}{n}\sum_{l=1}^{n}\{1/2\,\bar{\mathbf{x}}_l^\top D^{(n,l)}\bar{\mathbf{x}}_l + (\mathbf{c}_l^{(n)})^\top \bar{\mathbf{x}}_l\} = \frac{T}{n}\sum_{l=1}^{n}\{-1/2\,\bar{\mathbf{u}}^\top D^{(n,l)}\bar{\mathbf{u}} + (\mathbf{b}_l)^\top \bar{\mathbf{w}}_l\}.
$$

Moreover, $\bar{\mathbf{x}}$ *and* $(\bar{\mathbf{u}}, \bar{\mathbf{w}})$ *are optimal solutions of problems* (P_n) *and* (D_n)*, respectively.*

Under the assumption (A2), we can define

$$\sigma = \min\{B_{ij} : B_{ij} > 0\} > 0. \tag{11}$$

We also write

$$\kappa = \max_{j=1,\cdots,q} \sum_{i=1}^{p} K_{ij} \tag{12}$$

and

$$M = \max\{g_i(t) : i = 1, \cdots, p \text{ for } t \in [0, T]\}.$$

[49, Lemma 3.2] has shown that the feasible set of problem (P_n) is uniformly bounded described as follows.

Lemma 3.1. *If* $\mathbf{x}^{(n)} = (\mathbf{x}_1^{(n)}, \mathbf{x}_2^{(n)}, \cdots, \mathbf{x}_n^{(n)})$ *with* $\mathbf{x}_l^{(n)} = (x_{1l}^{(n)}, x_{2l}^{(n)}, \cdots, x_{ql}^{(n)})^\top$ *is a feasible solution of* (P_n)*, then*

$$0 \le x_{jl}^{(n)} \le \frac{M}{\sigma} \cdot e^{\frac{q\kappa T}{\sigma}}, \tag{13}$$

for all $j = 1, \cdots, q$*,* $l = 1, \cdots, n$ *and* $n \in \mathbb{N}$*.*

In general, the sequence of feasible sets $\{F(\mathrm{D}_n)\}_{n=1}^{\infty}$ needs not to be uniformly bounded. It can be shown that there exist uniformly bounded optimal solutions to dual problems (D_n). To see this, let

$$\hat{c} := \frac{M}{\sigma} \cdot e^{\frac{q\kappa T}{\sigma}} \max_{i=1,\cdots,p} \max_{t\in[0,T]} \sum_{j=1}^{q} |d_{ij}(t)| + \max_{j=1,\cdots,q} \max_{t\in[0.T]} |f_j(t)|. \tag{14}$$

Lemma 3.2. *The dual problem* (D_n) *has an optimal solution* $(\widetilde{\mathbf{u}}^{(n)}, \hat{\mathbf{w}}^{(n)})$ *with* $\hat{\mathbf{w}}^{(n)} = (\hat{\mathbf{w}}_1^{(n)}, \cdots, \hat{\mathbf{w}}_n^{(n)})$ *satisfying*

$$0 \le \hat{w}_{il}^{(n)} \le \frac{\hat{c}}{\sigma} \cdot e^{\frac{\kappa}{\sigma}T} \tag{15}$$

for all $i = 1, \cdots, p$ *and* $l = 1, \cdots, n$*.*

Proof. We define $\breve{\mathbf{w}}^{(n)} = (\breve{\mathbf{w}}_1^{(n)}, \breve{\mathbf{w}}_2^{(n)}, \cdots, \breve{\mathbf{w}}_n^{(n)})$ with $\breve{\mathbf{w}}_l^{(n)} = (\breve{w}_{1l}^{(n)}, \breve{w}_{2l}^{(n)}, \cdots, \breve{w}_{pl}^{(n)})^\top$ and

$$\breve{w}_{il}^{(n)} = \frac{\hat{c}}{\sigma} \cdot \left(1 + \frac{T\kappa}{n\sigma}\right)^{n-l} \ge 0 \tag{16}$$

for all $i = 1, \cdots, p$ and $l = 1, \cdots, n$. Let $\widetilde{\mathbf{x}}^{(n)} = (\widetilde{\mathbf{x}}_1^{(n)}, \cdots, \widetilde{\mathbf{x}}_n^{(n)})$ be an optimal solution of (P_n), where the existence of optimal solution can be guaranteed by Lemma 3.1. We shall show that $(\widetilde{\mathbf{x}}^{(n)}, \breve{\mathbf{w}}^{(n)})$ is a feasible solution of dual problem (D_n); that is, if we denote $D_{j\cdot}^{(n,l)}$ by the jth row of $D^{(n,l)}$, then we need to claim

$$\sum_{i=1}^{p}\left[B_{ij}\breve{w}_{il}^{(n)} - \frac{T}{n}K_{ij}\sum_{r=l+1}^{n}\breve{w}_{ir}^{(n)}\right] \geq D_{j\cdot}^{(n,l)}\widetilde{\mathbf{x}}_l^{(n)} + c_{jl}^{(n)},$$

for all $j = 1, \cdots, q$ and $l = 1, \cdots, n$. Since $\breve{w}_{il}^{(n)} \geq 0$, given any j, the assumption (A2) says that there exists $i_j \in \{1, 2, \cdots, p\}$ such that $B_{i_j j} > 0$. Therefore, we have

$$\begin{aligned}\sum_{i=1}^{p}B_{ij}\breve{w}_{il}^{(n)} &\geq B_{i_j j}\cdot\breve{w}_{i_j l}^{(n)} = B_{i_j j}\cdot\frac{\hat{c}}{\sigma}\cdot\left(1+\frac{T\kappa}{n\sigma}\right)^{n-l} \\ &\geq \hat{c}\cdot\left(1+\frac{T\kappa}{n\sigma}\right)^{n-l} \quad \text{(since } 0 < \sigma \leq B_{i_j j}\text{).,}\end{aligned} \tag{17}$$

Since we have

$$\sum_{i=1}^{p}\frac{T}{n}K_{ij}\breve{w}_{ir}^{(n)} = \frac{T}{n}\sum_{i=1}^{p}K_{ij}\frac{\hat{c}}{\sigma}\left(1+\frac{T\kappa}{n\sigma}\right)^{n-r} \leq \frac{T\kappa\hat{c}}{n\sigma}\left(1+\frac{T\kappa}{n\sigma}\right)^{n-r}, \tag{18}$$

it follows that

$$\begin{aligned}&D_{j\cdot}^{(n,l)}\widetilde{\mathbf{x}}_l^{(n)} + c_{jl}^{(n)} + \sum_{i=1}^{p}\frac{T}{n}K_{ij}\sum_{r=l+1}^{n}\breve{w}_{ir}^{(n)} \\ &\leq \hat{c} + \sum_{r=l+1}^{n}\sum_{i=1}^{p}\frac{T}{n}K_{ij}\breve{w}_{ir}^{(n)} \qquad \text{(by (13) and (14))} \\ &\leq \hat{c} + \sum_{r=l+1}^{n}\frac{T\kappa\hat{c}}{n\sigma}\left(1+\frac{T\kappa}{n\sigma}\right)^{n-r} \qquad \text{(by (18))} \\ &= \hat{c}\left[1 + \sum_{r=l+1}^{n}\frac{T\kappa}{n\sigma}\left(1+\frac{T\kappa}{n\sigma}\right)^{n-r}\right] \\ &= \hat{c}\cdot\left(1+\frac{T\kappa}{n\sigma}\right)^{n-l} \leq \sum_{i=1}^{p}B_{ij}\breve{w}_{il}^{(n)} \qquad \text{(by (17)),}\end{aligned}$$

which implies

$$\sum_{i=1}^{p}\left[B_{ij}\breve{w}_{il}^{(n)} - \frac{TK_{ij}}{n}\sum_{r=l+1}^{n}\breve{w}_{ir}^{(n)}\right] \geq D_{j\cdot}^{(n,l)}\widetilde{\mathbf{x}}_l^{(n)} + c_{jl}^{(n)}. \tag{19}$$

for all j and l. This shows that $(\widetilde{\mathbf{x}}^{(n)}, \breve{\mathbf{w}}^{(n)})$ is indeed a feasible solution of dual problem (D_n). Since, for $1 \leq l \leq n$,

$$\left(1 + \frac{T\kappa}{n\sigma}\right)^{n-l} \leq \left(1 + \frac{T\kappa}{n\sigma}\right)^{n} \uparrow e^{\frac{\kappa}{\sigma}T} \text{ as } n \to \infty.$$

from (16), we have

$$\breve{w}_{il}^{(n)} \leq \frac{\hat{c}}{\sigma} \cdot e^{\frac{\kappa}{\sigma}T} \tag{20}$$

for all $n \in \mathbb{N}$, $i = 1, \cdots, p$ and $l = 1, \cdots, n$.

Let $(\widetilde{\mathbf{x}}^{(n)}, \widetilde{\mathbf{w}}^{(n)})$ be an optimal solution of dual problem (D_n), where the existence of optimal solution is guaranteed by Proposition 3.2. Let $\hat{\mathbf{w}}^{(n)} = (\hat{\mathbf{w}}_1^{(n)}, \hat{\mathbf{w}}_2^{(n)}, \cdots, \hat{\mathbf{w}}_n^{(n)})$ with $\hat{\mathbf{w}}_l^{(n)} = (\hat{w}_{1l}^{(n)}, \hat{w}_{2l}^{(n)}, \cdots, \hat{w}_{pl}^{(n)})^\top$ defined by

$$\hat{w}_{il}^{(n)} = \min\left\{\widetilde{w}_{il}^{(n)}, \breve{w}_{il}^{(n)}\right\}$$

for all $i = 1, \cdots, p$ and $l = 1, \cdots, n$. We want to claim that $(\widetilde{\mathbf{x}}^{(n)}, \hat{\mathbf{w}}^{(n)})$ is also a feasible solution of dual problem (D_n). Given any fixed j_0 and l_0, we consider only $B_{ij_0} > 0$ for $i = 1, \cdots, p$. Therefore, we separately discuss the following two cases.

- Suppose that there exists i_0 such that $\hat{w}_{i_0 l_0}^{(n)} = \breve{w}_{i_0 l_0}^{(n)}$ and $B_{i_0 j} > 0$. According to the arguments for deriving (19), we can obtain

$$\sum_{i=1}^{p}\left[B_{ij_0}\hat{w}_{il_0}^{(n)} - \frac{TK_{ij_0}}{n}\sum_{r=l+1}^{n}\hat{w}_{ir}^{(n)}\right] \geq D_{j\cdot}^{(n,l)}\widetilde{\mathbf{x}}_{l_0}^{(n)} + c_{jl_0}^{(n)}.$$

- Suppose that $\hat{w}_{il_0}^{(n)} = \widetilde{w}_{il_0}^{(n)}$ for all i with $B_{ij_0} > 0$. Then, we have

$$\begin{aligned}
&\sum_{i=1}^{p}\left[B_{ij_0}\hat{w}_{il_0}^{(n)} - \frac{TK_{ij_0}}{n}\sum_{r=l+1}^{n}\hat{w}_{ir}^{(n)}\right] \\
&= \sum_{i=1}^{p}\left[B_{ij_0}\widetilde{w}_{il_0}^{(n)} - \frac{TK_{ij_0}}{n}\sum_{r=l+1}^{n}\hat{w}_{ir}^{(n)}\right] \\
&\geq \sum_{i=1}^{p}\left[B_{ij_0}\widetilde{w}_{il_0}^{(n)} - \frac{TK_{ij_0}}{n}\sum_{r=l+1}^{n}\widetilde{w}_{ir}^{(n)}\right] \text{(since } \widetilde{w}_{ir}^{(n)} \geq \hat{w}_{ir}^{(n)} \text{ and } K_{ij_0} \geq 0) \\
&\geq D_{j\cdot}^{(n,l)}\widetilde{\mathbf{x}}_{l_0}^{(n)} + c_{jl_0}^{(n)} \text{ (since } (\widetilde{\mathbf{x}}^{(n)}, \widetilde{\mathbf{w}}^{(n)}) \text{ is a feasible solution of } (\mathrm{D}_n)).
\end{aligned}$$

Therefore, we conclude that $(\widetilde{\mathbf{x}}, \hat{\mathbf{w}}^{(n)})$ is a feasible solution of dual problem (D_n). We also see that $(\widetilde{\mathbf{x}}, \hat{\mathbf{w}}^{(n)})$ is an optimal solution of dual problem (D_n), since

$$\begin{aligned}&\frac{T}{n}\sum_{l=1}^{n}\{-1/2\,(\widetilde{\mathbf{x}}_l^{(n)})^\top D^{(n,l)}\widetilde{\mathbf{x}}_l^{(n)} + (\mathbf{b}_l)^\top \hat{\mathbf{w}}_l^{(n)}\}\\ \le\ &\frac{T}{n}\sum_{l=1}^{n}\{-1/2\,(\widetilde{\mathbf{x}}_l^{(n)})^\top D^{(n,l)}\widetilde{\mathbf{x}}_l^{(n)} + (\mathbf{b}_l)^\top \widetilde{\mathbf{w}}_l^{(n)}\}.\end{aligned}$$

Finally, since $\hat{\mathbf{w}}_l^{(n)} \le \breve{\mathbf{w}}_l^{(n)}$ for $l = 1, \cdots, n$, the bound given in (15) can be realized by (20). This completes the proof. ∎

Remark 3.1. From Lemma 3.2, without loss of generality, if $(\bar{\mathbf{u}}^{(n)}, \bar{\mathbf{w}}^{(n)})$ is an optimal solution of dual problem (D_n), where $\bar{\mathbf{w}}^{(n)} = (\bar{\mathbf{w}}_1^{(n)}, \cdots, \bar{\mathbf{w}}_n^{(n)})$ and $\bar{\mathbf{w}}_l^{(n)} = (\bar{w}_{1l}^{(n)}, \cdots, \bar{w}_{pl}^{(n)})^\top$, we may assume that $\bar{w}_{il}^{(n)} \le \frac{\hat{c}}{\sigma}e^{\frac{\kappa}{\sigma}T}$ for all i and l, since if there are some $\bar{w}_{il}^{(n)}$ with $\bar{w}_{il}^{(n)} > \frac{\hat{c}}{\sigma}e^{\frac{\kappa}{\sigma}T}$, then $\bar{w}_{il}^{(n)}$ can be replaced by $\frac{\hat{c}}{\sigma}(1 + \frac{T\kappa}{n\sigma})^{n-l}$ according to the proof of Lemma 3.2.

4. Constructing Approximate Solutions

Next, we are going to construct the feasible solutions of problems (CQP) and (DCQP) by virtue of the optimal solutions of problems (P_n) and (D_n), respectively. Let $\bar{\mathbf{x}}^{(n)} = (\bar{\mathbf{x}}_1^{(n)}, \cdots, \bar{\mathbf{x}}_n^{(n)})$ be an optimal solution of primal problem (P_n), where $\bar{\mathbf{x}}_l^{(n)} = (\bar{x}_{1l}^{(n)}, \cdots, \bar{x}_{ql}^{(n)})^\top$. We can construct a vector-valued step function $\hat{\mathbf{x}}^{(n)} : [0, T] \mapsto \mathbb{R}^q$ as follows:

$$\hat{\mathbf{x}}^{(n)}(t) = \left(\hat{x}_1^{(n)}(t), \cdots, \hat{x}_q^{(n)}(t)\right)^\top \text{ for } j = 1, \cdots, q$$

with component functions defined by

$$\hat{x}_j^{(n)}(t) = \begin{cases} \bar{x}_{jl}^{(n)}, & \text{if } \dfrac{(l-1)T}{n} \le t < \dfrac{lT}{n} \text{ for } l = 1, \cdots, n \\ \bar{x}_{jn}^{(n)}, & \text{if } t = T. \end{cases} \tag{21}$$

In this case, we say that $\hat{\mathbf{x}}^{(n)}(t)$ is the *natural solution* of problem (CQP) constructed by the optimal solution $\bar{\mathbf{x}}^{(n)}$ of problem (P_n). The feasibility of $\hat{\mathbf{x}}^{(n)}(t)$ will be shown below.

Lemma 4.1. *The function $\hat{\mathbf{x}}^{(n)}(t)$ defined in (21) is a feasible solution of problem (CQP).*

Proof. By the same arguments as [49, Lemma 3.3] we obtain the result. ∎

On the other hand, it is obvious that

$$\begin{aligned}(22)\qquad &\int_0^T \left\{\frac{1}{2}\hat{\mathbf{x}}^{(n)}(t)^\top D(t)\hat{\mathbf{x}}^{(n)}(t) + \mathbf{f}(t)^\top \hat{\mathbf{x}}^{(n)}(t)\right\} dt \\ &\geq \frac{T}{n}\sum_{l=1}^{n}\left\{1/2\,(\bar{\mathbf{x}}_l^{(n)})^\top D^{(n,l)}\bar{\mathbf{x}}_l^{(n)} + (\mathbf{c}_l^{(n)})^\top \bar{\mathbf{x}}_l^{(n)}\right\} = V(\mathrm{P}_n).\end{aligned}$$

Therefore, Lemma 4.1 says that $V(\text{CQP}) \geq V(\mathrm{P}_n)$. By Theorem 2.1, we have

$$(23)\qquad V(\text{DCQP}) \geq V(\text{CQP}) \geq V(\mathrm{P}_n) = V(\mathrm{D}_n)$$

for all $n \in \mathbb{N}$. In the sequel, we are going to prove

$$(24)\qquad \lim_{n\to\infty} V(\mathrm{D}_n) = V(\text{DCQP}).$$

We first adopt some notations. Let

$$\begin{aligned}(25)\qquad \epsilon_n :=& \max_{1\leq j\leq p}\sup_{t\in[0,T]}\left\{f_j(t) - f_j^{(n)}(t)\right\} \\ &+\frac{M}{\sigma}\cdot e^{\frac{q\kappa T}{\sigma}}\max_{i=1,\cdots,q}\sup_{t\in[0,T]}\sum_{j=1}^{q}\left\{d_{ij}(t) - d_{ij}^{(n)}(t)\right\},\end{aligned}$$

$$(26)\qquad \bar{\epsilon}_n = \max_{i=1,\cdots,p}\sup_{t\in[0,T]}\left\{g_i(t) - g_i^{(n)}(t)\right\}$$

and

$$(27)\qquad \rho = \max_{j=1,\cdots,q}\left\{\frac{\sum_{i=1}^{p}K_{ij}}{\sum_{i=1}^{p}B_{ij}}, \frac{1}{\sum_{i=1}^{p}B_{ij}}\right\}.$$

Let $\bar{\mathbf{x}}^{(n)}$ be an optimal solution of (P_n) and let $\hat{\mathbf{x}}^{(n)}(t)$ be the *natural solution* of (CQP) constructed from $\bar{\mathbf{x}}^{(n)}$. Let $(\bar{\mathbf{x}}^{(n)}, \bar{\mathbf{w}}^{(n)})$ be an optimal solution of dual problem (D_n), where $\bar{\mathbf{w}}^{(n)} = (\bar{\mathbf{w}}_1^{(n)}, \cdots, \bar{\mathbf{w}}_n^{(n)})$ and $\bar{\mathbf{w}}_l^{(n)} = (\bar{w}_{1l}^{(n)}, \cdots, \bar{w}_{pl}^{(n)})^\top$. We define

$$(28)\qquad \delta_n = \max_{i=1,\cdots,p}\max_{l=1,\cdots,n}\left\{\frac{T}{n}\bar{w}_{il}^{(n)}\right\}.$$

Let $\mathbf{1} = (1, 1, \cdots, 1)^\top \in \mathbb{R}^p$. We define a function $\hat{\mathbf{w}}^{(n)}(t) : [0, T] \mapsto \mathbb{R}^p$ as follows:

(29) $$\hat{\mathbf{w}}^{(n)}(t) = \begin{cases} \bar{\mathbf{w}}_l^{(n)} + \delta_n \rho e^{\rho(T-t)}\mathbf{1}, & \text{if } t \in \left[\dfrac{l-1}{n}T, \dfrac{l}{n}T\right) \text{ for some } l = 1, \cdots, n \\ \bar{\mathbf{w}}_n^{(n)} + \delta_n \rho \mathbf{1}, & \text{if } t = T. \end{cases}$$

Moreover, define

(30) $$\widetilde{\mathbf{w}}^{(n)}(t) = \hat{\mathbf{w}}^{(n)}(t) + \epsilon_n \rho e^{\rho(T-t)}\mathbf{1}$$

for all $t \in [0, T]$, where ϵ_n is defined as in (25). In this case, we also say that $(\hat{\mathbf{x}}^{(n)}(t), \widetilde{\mathbf{w}}^{(n)}(t))$ is a *natural solution* of problem (DCQP) constructed from the optimal solution $(\bar{\mathbf{x}}^{(n)}, \bar{\mathbf{w}}^{(n)})$ of problem (D_n).

Lemma 4.2. *Let $\bar{\mathbf{x}}^{(n)}$ and $(\bar{\mathbf{x}}^{(n)}, \bar{\mathbf{w}}^{(n)})$ be optimal solutions of (P_n) and (D_n), respectively. Let $\hat{\mathbf{w}}^{(n)}(t)$ and $\widetilde{\mathbf{w}}^{(n)}(t)$ be defined as in (29) and (30), respectively. Then the following statements hold true.*

- (i) *The natural solution $(\hat{\mathbf{x}}^{(n)}(t), \widetilde{\mathbf{w}}^{(n)}(t))$ is a feasible solution of dual problem $(DCQP)$.*
- (ii) *We have*

(31) $$0 \le \widehat{Obj}\left(\hat{\mathbf{x}}^{(n)}(t), \hat{\mathbf{w}}^{(n)}(t)\right) - V(D_n) \le \delta_n \int_0^T \rho e^{\rho(T-t)}\mathbf{g}(t)^\top \mathbf{1} dt,$$

where

(32) $$\begin{aligned} &\widehat{Obj}\left(\hat{\mathbf{x}}^{(n)}(t), \hat{\mathbf{w}}^{(n)}(t)\right) \\ &= \int_0^T \left\{-1/2\, \hat{\mathbf{x}}^{(n)}(t)^\top D^{(n)}(t)\hat{\mathbf{x}}^{(n)}(t) + \mathbf{g}^{(n)}(t)^\top \hat{\mathbf{w}}^{(n)}(t)\right\} dt. \end{aligned}$$

Proof. (i) We first show that

$$B^\top \hat{\mathbf{w}}^{(n)}(t) - \int_t^T K^\top \hat{\mathbf{w}}^{(n)}(s)ds \ge D^{(n)}(t)\hat{\mathbf{x}}^{(n)}(t) + \mathbf{f}^{(n)}(t) \text{ for } t \in [0, T].$$

We consider the following two cases.

- Given any $t \in [\frac{l-1}{n}T, \frac{l}{n}T)$ for some $l = 1, \cdots, n$, we have

$$\begin{aligned} &B^\top \hat{\mathbf{w}}^{(n)}(t) - \int_t^T K^\top \hat{\mathbf{w}}^{(n)}(s)ds \\ &= B^\top \hat{\mathbf{w}}^{(n)}(t) - \int_t^{\frac{l}{n}T} K^\top \hat{\mathbf{w}}^{(n)}(s)ds - \sum_{r=l+1}^{n} \int_{\frac{r-1}{n}T}^{\frac{r}{n}T} K^\top \hat{\mathbf{w}}^{(n)}(s)ds \\ &= B^\top(\bar{\mathbf{w}}_l^{(n)} + \delta_n \rho e^{\rho(T-t)}\mathbf{1}) - \left(\frac{lT}{n} - t\right) K^\top \bar{\mathbf{w}}_l^{(n)} - \delta_n K^\top \mathbf{1} \cdot \int_t^{\frac{l}{n}T} \rho e^{\rho(T-s)}ds \\ &\quad - \frac{T}{n} K^\top \sum_{r=l+1}^{n} \bar{\mathbf{w}}_r^{(n)} - \delta_n K^\top \mathbf{1} \cdot \sum_{r=l+1}^{n} \int_{\frac{r-1}{n}T}^{\frac{r}{n}T} \rho e^{\rho(T-s)}ds \end{aligned}$$

$$
\begin{aligned}
&= B^\top \bar{\mathbf{w}}_l^{(n)} + \delta_n \rho e^{\rho(T-t)} B^\top \mathbf{1} - \left(\frac{lT}{n} - t\right) K^\top \bar{\mathbf{w}}_l^{(n)} - \frac{T}{n} K^\top \sum_{r=l+1}^{n} \bar{\mathbf{w}}_r^{(n)} \\
&\quad -\delta_n K^\top \mathbf{1} \cdot \int_t^T \rho e^{\rho(T-s)} ds \\
&= \left[B^\top \bar{\mathbf{w}}_l^{(n)} - \frac{T}{n} K^\top \sum_{r=l+1}^{n} \bar{\mathbf{w}}_r^{(n)} \right] + \delta_n \rho e^{\rho(T-t)} B^\top \mathbf{1} - \left(\frac{lT}{n} - t\right) K^\top \bar{\mathbf{w}}_l^{(n)} \\
&\quad -\delta_n \left(e^{\rho(T-t)} - 1\right) K^\top \mathbf{1} \\
&\geq D^{(n,l)} \bar{\mathbf{x}}_l^{(n)} + \mathbf{c}_l^{(n)} + \delta_n e^{\rho(T-t)} \left(\rho B - K\right)^\top \mathbf{1} \\
&\quad - \left(\frac{lT}{n} - t\right) K^\top \bar{\mathbf{w}}_l^{(n)} + \delta_n K^\top \mathbf{1} \text{(by the feasibility of } (\bar{\mathbf{x}}^{(n)}, \bar{\mathbf{w}}^{(n)})) \\
&\geq D^{(n,l)} \bar{\mathbf{x}}_l^{(n)} + \mathbf{c}_l^{(n)} + \delta_n \left(\rho B - K\right)^\top \mathbf{1} - \frac{T}{n} K^\top \bar{\mathbf{w}}_l^{(n)} + \delta_n K^\top \mathbf{1} \\
&\geq D^{(n,l)} \bar{\mathbf{x}}_l^{(n)} + \mathbf{c}_l^{(n)} + \delta_n \left(\rho B - K\right)^\top \mathbf{1} \text{ (since } \delta_n \mathbf{1} \geq \tfrac{T}{n} \bar{\mathbf{w}}_l^{(n)} \text{ and } K \geq 0) \\
&\geq D^{(n,l)} \bar{\mathbf{x}}_l^{(n)} + \mathbf{c}_l^{(n)} \quad \text{(by the definition of } \rho) \\
&= D^{(n)}(t) \hat{\mathbf{x}}^{(n)}(t) + \mathbf{f}^{(n)}(t).
\end{aligned}
$$

- For $t = T$, we have

$$
\begin{aligned}
&B^\top \hat{\mathbf{w}}^{(n)}(T) - \int_T^T K^\top \hat{\mathbf{w}}^{(n)}(s) ds \\
&= B^\top \hat{\mathbf{w}}^{(n)}(T) = B^\top (\bar{\mathbf{w}}_n^{(n)} + \delta_n \rho \mathbf{1}) \\
&\geq B^\top \bar{\mathbf{w}}_n^{(n)} \geq D^{(n,n)} \bar{\mathbf{x}}_n^{(n)} + \mathbf{c}_n^{(n)} = D^{(n)}(T) \hat{\mathbf{x}}^{(n)}(T) + \mathbf{f}^{(n)}(T).
\end{aligned}
$$

Furthermore, since

$$
\begin{aligned}
&B^\top \widetilde{\mathbf{w}}^{(n)}(t) - \int_t^T K^\top \widetilde{\mathbf{w}}^{(n)}(s) ds \\
&= B^\top \hat{\mathbf{w}}^{(n)}(t) - \int_t^T K^\top \hat{\mathbf{w}}^{(n)}(s) ds + \epsilon_n \rho e^{\rho(T-t)} B^\top \mathbf{1} - \epsilon_n \int_t^T \rho e^{\rho(T-s)} K^\top \mathbf{1} ds \\
&\geq D^{(n)}(t) \hat{\mathbf{x}}^{(n)}(t) + \mathbf{f}^{(n)}(t) + \epsilon_n \left[\rho e^{\rho(T-t)} B + K - e^{\rho(T-t)} K\right]^\top \mathbf{1} \\
&= D^{(n)}(t) \hat{\mathbf{x}}^{(n)}(t) + \mathbf{f}^{(n)}(t) + \epsilon_n e^{\rho(T-t)} (\rho B - K)^\top \mathbf{1} + \epsilon_n K^\top \mathbf{1} \\
&\geq D^{(n)}(t) \hat{\mathbf{x}}^{(n)}(t) + \mathbf{f}^{(n)}(t) + \epsilon_n \left[\rho B - K + K\right]^\top \mathbf{1} \text{ (since } (\rho B - K)^\top \mathbf{1} \geq 0) \\
&= D^{(n)}(t) \hat{\mathbf{x}}^{(n)}(t) + \mathbf{f}^{(n)}(t) + \epsilon_n \rho B^\top \mathbf{1} \\
&\geq D^{(n)}(t) \hat{\mathbf{x}}^{(n)}(t) + \mathbf{f}^{(n)}(t) + \epsilon_n \mathbf{1} \quad \text{(since } \rho B^\top \mathbf{1} \geq \mathbf{1})
\end{aligned}
$$

$$\begin{aligned}
&\geq D^{(n)}(t)\hat{\mathbf{x}}^{(n)}(t)+\mathbf{f}^{(n)}(t)+\Big\{\mathbf{f}(t)-\mathbf{f}^{(n)}(t)\Big\}+\frac{M}{\sigma}\cdot e^{\frac{q\kappa T}{\sigma}}\sum_{j=1}^{q}\Big\{d_{ij}(t)-d_{ij}^{(n)}(t)\Big\}\,\mathbf{1}\\
&\geq D^{(n)}(t)\hat{\mathbf{x}}^{(n)}(t)+\mathbf{f}(t)+\Big\{D(t)-D^{(n)}(t)\Big\}\,\bar{\mathbf{x}}^{(n)} \text{ (by (13))}\\
&= D(t)\hat{\mathbf{x}}^{(n)}(t)+\mathbf{f}(t) \text{ for all } t\in[0,T],
\end{aligned}$$

we obtain

$$B^\top\widetilde{\mathbf{w}}^{(n)}(t)-\int_t^T K^\top\widetilde{\mathbf{w}}^{(n)}(s)ds\geq D(t)\hat{\mathbf{x}}^{(n)}(t)+\mathbf{f}(t) \text{ for } t\in[0,T].$$

This shows that $(\hat{\mathbf{x}}^{(n)}(t),\widetilde{\mathbf{w}}^{(n)}(t))$ is indeed a feasible solution of dual problem (DCQP).

(ii) We observe that

$$\begin{aligned}
&\widehat{\text{Obj}}(\hat{\mathbf{x}}^{(n)}(t),\hat{\mathbf{w}}^{(n)}(t))\\
&=\int_0^T\Big[-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top D^{(n)}(t)\hat{\mathbf{x}}^{(n)}(t)+\mathbf{g}^{(n)}(t)^\top\hat{\mathbf{w}}^{(n)}(t)\Big]\,dt\\
&=\frac{T}{n}\sum_{l=1}^{n}\Big[-1/2\,(\bar{\mathbf{x}}^{(n)})^\top D^{(n,l)}\bar{\mathbf{x}}^{(n)}\Big]\\
&\quad+(\mathbf{b}_l^{(n)})^\top\int_{\frac{l-1}{n}T}^{\frac{l}{n}T}\bar{\mathbf{w}}_l^{(n)}dt+\delta_n\int_0^T\rho e^{\rho(T-t)}\mathbf{g}^{(n)}(t)^\top\mathbf{1}dt\\
&=V(\mathrm{D}_n)+\delta_n\int_0^T\rho e^{\rho(T-t)}\mathbf{g}^{(n)}(t)^\top\mathbf{1}dt,
\end{aligned}\tag{33}$$

which implies

$$0\leq\widehat{\text{Obj}}(\hat{\mathbf{x}}^{(n)}(t),\hat{\mathbf{w}}^{(n)}(t))-V(\mathrm{D}_n)\leq\delta_n\int_0^T\rho e^{\rho(T-t)}\mathbf{g}(t)^\top\mathbf{1}dt. \text{ (since } \mathbf{g}^{(n)}(t)\leq\mathbf{g}(t)).$$

This completes the proof. ∎

Now, we also have

$$\begin{aligned}
0&\leq\int_0^T\mathbf{g}(t)^\top\widetilde{\mathbf{w}}^{(n)}(t)dt-\int_0^T\mathbf{g}^{(n)}(t)^\top\hat{\mathbf{w}}^{(n)}(t)dt\\
&=\int_0^T\Big[\mathbf{g}(t)-\mathbf{g}^{(n)}(t)\Big]^\top\hat{\mathbf{w}}^{(n)}(t)dt+\epsilon_n\int_0^T\rho e^{\rho(T-t)}\mathbf{g}(t)^\top\mathbf{1}dt\\
&\leq\bar{\epsilon}_n\int_0^T\hat{\mathbf{w}}^{(n)}(t)^\top\mathbf{1}dt+\epsilon_n\int_0^T\rho e^{\rho(T-t)}\mathbf{g}(t)^\top\mathbf{1}dt\\
&=\bar{\epsilon}_n\left[\frac{T}{n}\sum_{r=1}^{n}\mathbf{1}^\top\bar{\mathbf{w}}_r^{(n)}+p\delta_n(e^{\rho T}-1)\right]+\epsilon_n\int_0^T\rho e^{\rho(T-t)}\mathbf{g}(t)^\top\mathbf{1}dt\\
&\leq\bar{\epsilon}_n\,p\,\delta_n\left(n+e^{\rho T}-1\right)+\epsilon_n\int_0^T\rho e^{\rho(T-t)}\mathbf{g}(t)^\top\mathbf{1}dt \text{ (since } \tfrac{T}{n}\bar{\mathbf{w}}_r^{(n)}\leq\delta_n\mathbf{1}).
\end{aligned}\tag{34}$$

Since $(\hat{\mathbf{x}}^{(n)}(t), \widetilde{\mathbf{w}}^{(n)}(t))$ is a feasible solution of dual problem (DCQP), we have

$$V(\text{DCQP}) \leq \int_0^T \left[-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top D\,\hat{\mathbf{x}}^{(n)}(t) + \mathbf{g}(t)^\top \hat{\mathbf{w}}^{(n)}(t)\right] dt,$$

which implies

$$\begin{aligned}
& V(\text{DCQP}) - \widehat{\text{Obj}}(\hat{\mathbf{x}}^{(n)}(t), \hat{\mathbf{w}}^{(n)}(t)) \\
& \leq \int_0^T \left[-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top D(t)\,\hat{\mathbf{x}}^{(n)}(t) + \mathbf{g}(t)^\top \widetilde{\mathbf{w}}^{(n)}(t)\right] dt \\
& \quad - \int_0^T \left[-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top D^{(n)}(t)\,\hat{\mathbf{x}}^{(n)}(t) + {\mathbf{g}^{(n)}(t)}^\top \hat{\mathbf{w}}^{(n)}(t)\right] dt \text{ (by (32))} \\
(35) \quad & = \int_0^T \left\{-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top \left[D(t) - D^{(n)}(t)\right]\,\hat{\mathbf{x}}^{(n)}(t)\right\} dt \\
& \quad + \int_0^T \mathbf{g}(t)^\top \widetilde{\mathbf{w}}^{(n)}(t) dt - \int_0^T {\mathbf{g}^{(n)}(t)}^\top \hat{\mathbf{w}}^{(n)}(t) dt \\
& \leq \bar{\epsilon}_n\, p\, \delta_n \left(n + e^{\rho T} - 1\right) + \epsilon_n \int_0^T \rho e^{\rho(T-t)} \mathbf{g}(t)^\top \mathbf{1} dt \\
& \quad \left(\text{since} \int_0^T \left\{-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top \left[D(t) - D^{(n)}(t)\right]\,\hat{\mathbf{x}}^{(n)}(t)\right\} dt \leq 0 \text{ and by (34)}\right)
\end{aligned}$$

From (23), (31) and (35), we obtain

$$\begin{aligned}
& 0 \leq V(\text{DCQP}) - V(\text{D}_n) \\
(36) \quad & = \left[V(\text{DCQP}) - \widehat{\text{Obj}}(\hat{\mathbf{x}}^{(n)}(t), \hat{\mathbf{w}}^{(n)}(t))\right] + \left[\widehat{\text{Obj}}(\hat{\mathbf{x}}^{(n)}(t), \hat{\mathbf{w}}^{(n)}(t)) - V(\text{D}_n)\right] \\
& \leq \bar{\epsilon}_n p \delta_n (n + e^{\rho T} - 1) + (\delta_n + \epsilon_n) \int_0^T \rho e^{\rho(T-t)} \mathbf{g}(t)^\top \mathbf{1} dt.
\end{aligned}$$

Since $f_i(t)$, $g_i(t)$ and $d_{ij}(t)$ are uniformly continuous on $[0, T]$, we have $\epsilon_n \to 0$ and $\bar{\epsilon}_n \to 0$ as $n \to \infty$. By Lemma 3.2 and (28), we also have $\delta_n \to 0$ and

$$\bar{\epsilon}_n p \delta_n n \leq \bar{\epsilon}_n p T \cdot \frac{\hat{c}}{\sigma} \cdot e^{\frac{T\kappa}{\sigma}} \to 0 \text{ as } n \to \infty.$$

Therefore, using (), we finally obtain

$$\lim_{n \to \infty} V(\text{D}_n) = V(\text{DCQP}).$$

We summarize the above results below.

Theorem 4.1. *The following statements hold true.*

(i) *We have*

$$\lim_{n\to\infty} V(D_n) = V(DCQP).$$

(ii) *We have*

$$0 \le V(DCQP) - V(D_n) \le \varepsilon_n,$$

where

$$\varepsilon_n = \bar{\epsilon}_n p \delta_n (n + e^{\rho T} - 1) + (\delta_n + \epsilon_n) \int_0^T \rho e^{\rho(T-t)} \mathbf{g}(t)^\top \mathbf{1} dt, \tag{37}$$

and ϵ_n, $\bar{\epsilon}_n$, ρ *and* δ_n *are defined in* (25), (26), (27) *and* (28), *respectively.*

By inequality (23) and Theorem 4.1, we have

$$V(\text{DCQP}) \ge V(\text{CQP}) \ge \lim_{n\to\infty} V(\text{D}_n) = V(\text{DCQP}).$$

Therefore, we have

$$V(\text{DCQP}) = V(\text{CQP}) = \lim_{n\to\infty} V(\text{D}_n) = \lim_{n\to\infty} V(\text{P}_n)$$

and

$$0 \le V(\text{CQP}) - V(\text{P}_n) \le \bar{\epsilon}_n p \delta_n (n + e^{\rho T} - 1) + (\delta_n + \epsilon_n) \int_0^T \rho e^{\rho(T-t)} \mathbf{g}(t)^\top \mathbf{1} dt.$$

Moreover, we can establish the estimation for the error bounds given below.

Theorem 4.2. *Let* $\hat{\mathbf{x}}^{(n)}(t)$ *and* $\widetilde{\mathbf{w}}^{(n)}(t)$ *be defined in* (21) *and* (30), *respectively. The error between* $V(CQP)$ *and the objective value of* $\hat{\mathbf{x}}^{(n)}(t)$, *and the error between* $V(DCQP)$ *and the objective value of* $(\hat{\mathbf{x}}^{(n)}(t), \widetilde{\mathbf{w}}^{(n)}(t))$ *are both less than or equal to* ε_n *that is defined in* (37).

Proof. From Lemma 4.1, we see that $\hat{\mathbf{x}}^{(n)}(t)$ is a feasible solution of problem (CQP). By (22) and part (ii) of Theorem 4.1, we have

$$\begin{aligned} 0 &\le V(\text{CQP}) - \int_0^T \left[1/2\, \hat{\mathbf{x}}^{(n)}(t)^\top D(t)\, \hat{\mathbf{x}}^{(n)}(t) + \mathbf{f}(t)^\top \hat{\mathbf{x}}^{(n)}(t) \right] dt \\ &\le V(\text{DCQP}) - V(\text{D}_n) \le \varepsilon_n. \end{aligned}$$

On the other hand, we also have

$$
\begin{aligned}
&\int_0^T \left[-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top D(t)\,\hat{\mathbf{x}}^{(n)}(t) + \mathbf{g}(t)^\top \widetilde{\mathbf{w}}^{(n)}(t)\right] dt \\
=\;& \int_0^T \left[-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top D^{(n)}(t)\,\hat{\mathbf{x}}^{(n)}(t) + \mathbf{g}(t)^\top \hat{\mathbf{w}}^{(n)}(t)\right] dt \\
&+\epsilon_n \int_0^T \rho e^{\rho(T-t)}\mathbf{g}(t)^\top \mathbf{1} dt \\
&+ \int_0^T \left\{-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top \left[D(t) - D^{(n)}(t)\right]\,\hat{\mathbf{x}}^{(n)}(t)\right\} dt \\
\le\;& \int_0^T \left[-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top D^{(n)}(t)\,\hat{\mathbf{x}}^{(n)}(t) + \mathbf{g}^{(n)}(t)^\top \hat{w}^{(n)}(t)dt\right] dt \\
&+ \int_0^T \left[\mathbf{g}(t) - \mathbf{g}^{(n)}(t)\right]^\top \hat{\mathbf{w}}^{(n)}(t)dt + \epsilon_n \int_0^T \rho e^{\rho(T-t)}\mathbf{g}(t)^\top \mathbf{1} dt \\
&(\text{ since } \int_0^T \left\{-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top \left[D(t) - D^{(n)}(t)\right]\,\hat{\mathbf{x}}^{(n)}(t)\right\} dt \le 0) \\
=\;& V(\mathrm{D}_n) + \delta_n \int_0^T \rho e^{\rho(T-t)}\mathbf{g}^{(n)}(t)^\top \mathbf{1} dt \\
&+ \int_0^T \left[\mathbf{g}(t) - \mathbf{g}^{(n)}(t)\right]^\top \hat{\mathbf{w}}^{(n)}(t)dt \\
&+\epsilon_n \int_0^T \rho e^{\rho(T-t)}\mathbf{g}(t)^\top \mathbf{1} dt \text{ (by (33)).}
\end{aligned}
\tag{38}
$$

Since $V(\mathrm{D}_n) \le V(\text{DCQP})$, we obtain

$$
\begin{aligned}
0 \le\;& \int_0^T \left[-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top D(t)\,\hat{\mathbf{x}}^{(n)}(t) + \mathbf{g}(t)^\top \widetilde{\mathbf{w}}^{(n)}(t)\right] dt - V(\text{DCQP}) \\
\le\;& \int_0^T \left[-1/2\,\hat{\mathbf{x}}^{(n)}(t)^\top D(t)\,\hat{\mathbf{x}}^{(n)}(t) + \mathbf{g}(t)^\top \widetilde{\mathbf{w}}^{(n)}(t)\right] dt - V(\mathrm{D}_n) \\
\le\;& \int_0^T [\mathbf{g}(t) - \mathbf{g}^{(n)}(t)]^\top \hat{\mathbf{w}}^{(n)}(t)dt + \delta_n \int_0^T \rho e^{\rho(T-t)}\mathbf{g}^{(n)}(t)^\top \mathbf{1} dt \\
&+\epsilon_n \int_0^T \rho e^{\rho(T-t)}\mathbf{g}(t)^\top \mathbf{1} dt \text{ (by (38))} \\
\le\;& \bar{\epsilon}_n \int_0^T \hat{\mathbf{w}}^{(n)}(t)^\top \mathbf{1} dt + (\delta_n + \epsilon_n) \int_0^T \rho e^{\rho(T-t)}\mathbf{g}(t)^\top \mathbf{1} dt \text{ (by (26))} \\
\le\;& \bar{\epsilon}_n p \delta_n (n + e^{\rho T} - 1) + (\delta_n + \epsilon_n) \int_0^T \rho e^{\rho(T-t)}\mathbf{g}(t)^\top \mathbf{1} dt \text{ (by (29))} \\
=\;& \varepsilon_n.
\end{aligned}
$$

This completes this proof. ∎

5. CONVERGENCE OF APPROXIMATE SOLUTIONS

Now, we shall demonstrate the convergent properties of the sequence $\{\hat{\mathbf{x}}^{(n)}(t)\}$ which is constructed from the optimal solutions $\bar{\mathbf{x}}^{(n)}$ of (P_n).

Let $L^1([0,T],\mathbb{R})$ be the space of real-valued Lebesgue measurable functions on $[0,T]$ with finite L^1 norm. The dual space of the separable Banach space $L^1([0,T],\mathbb{R})$ can be identified with $L^\infty([0,T],\mathbb{R})$. An important property enjoyed by the dual of a separable Banach space is weak-star sequential compactness for sets bounded in the strong topology. According to [12, Theorem 4.12.3] and [20, Lemma 2.1], we have the following useful lemma.

Lemma 5.1. *Let $\{\lambda_n\}$ be a sequence in $L^\infty([0,T],\mathbb{R})$. Suppose that there exists a constant $\hat{\kappa} > 0$ such that $\| \lambda_n \|_\infty \leq \hat{\kappa}$ for $n = 1, 2, \cdots$. Then, the following statements hold true.*

(i) *There exist $\lambda \in L^\infty([0,T],\mathbb{R})$ and a subsequence $\{\lambda_{n_k}\}$ such that λ_{n_k} weakly-star converges to λ, i.e.,*

$$\int_0^T \lambda_{n_k}(t)h(t)dt \to \int_0^T \lambda(t)h(t)dt \text{ as } n_k \to \infty$$

for all $h(t) \in L^1([0,T],\mathbb{R})$.

(ii) *We have*

$$\lambda(t) \leq \limsup_{n_k\to\infty} \lambda_{n_k}(t) \text{ a.e. in } [0,T]$$

and

$$\lambda(t) \geq \liminf_{n_k\to\infty} \lambda_{n_k}(t) \text{ a.e. im } [0,T].$$

Remark 5.2. If $\lambda_n(t) \geq 0$ for all $t \in [0,T]$ and $\lambda_n(t)$ weakly-star converges to $\lambda(t)$, then $\lambda(t) \geq 0$ a.e. in $[0,T]$ by part (ii) of Lemma 5.1.

For the sequence $\{\mathbf{f}^{(n)}\}_{n=1}^\infty$ of vector-valued functions in $L^\infty([0,T],\mathbb{R}^q)$, we say that $\mathbf{f}^{(n)}$ weakly-star converges to $\mathbf{f}^\star \in L^\infty([0,T],\mathbb{R}^q)$ if and only if the sequences of components $\{f_j^{(n)}\}$ weakly-star converge to f_j^* for $j = 1, \cdots, q$. We also need the following result.

Lemma 5.2. *For a symmetric and negative semi-definite matrix $D(t)$ and for any two vector functions $\mathbf{u}(t)$ and $\mathbf{x}(t)$ the following relation holds*

$$\begin{aligned} &\int_0^T \left\{ 1/2\, \mathbf{u}(t)^\top D(t)\mathbf{u}(t) - 1/2\, \mathbf{x}(t)^\top D(t)\mathbf{x}(t) \right\} dt \\ \leq &\int_0^T \left\{ \mathbf{x}(t)^\top D(t)(\mathbf{u}(t) - \mathbf{x}(t)) \right\} dt. \end{aligned} \tag{39}$$

Proof. See [13, Lemma 1]. ∎

The following result demonstrates the convergent properties of the sequence $\{\hat{\mathbf{x}}^{(n)}(t)\}$ defined in (21). It also shows the solvability of problem (CQP).

Theorem 5.1. *The sequence* $\{\hat{\mathbf{x}}^{(n)}(t)\}$ *defined in* (21) *has a convergent subsequence* $\{\hat{\mathbf{x}}^{(n_k)}(t)\}$ *which weakly-star converges to* $\hat{\mathbf{x}}^{\star}(t)$ *such that the limit* $\hat{\mathbf{x}}^{\star}(t)$ *is an optimal solution of* (CQP).

Proof. From (21) and Lemma 3.1, there exists a constant $\hat{\kappa} > 0$ such that $\| \hat{x}_j^{(n)} \|_\infty < \hat{\kappa}$ for all n and $j = 1, \cdots, q$. By Lemma 5.1, there exist $\hat{\mathbf{x}}(t) = (\hat{x}_1(t), \cdots, \hat{x}_q(t))^\top \in L^\infty([0,T], \mathbb{R}^q)$ and a subsequence $\{\hat{\mathbf{x}}^{(n_k)}(t)\}$, where $\hat{\mathbf{x}}^{(n_k)}(t) = (\hat{x}_1^{(n_k)}(t), \cdots, \hat{x}_q^{(n_k)}(t))^\top$, such that $\hat{\mathbf{x}}^{(n_k)}(t)$ weakly-star converges to $\hat{\mathbf{x}}(t)$ in the sense of

$$\int_0^T \hat{x}_j^{(n_k)}(t)h(t)dt \to \int_0^T \hat{x}_j(t)h(t)dt \text{ as } n_k \to \infty \tag{40}$$

for all $h(t) \in L^1([0,T], \mathbb{R})$, and

$$\hat{x}_j(t) \leq \limsup_{n_k \to \infty} \hat{x}_j^{(n_k)}(t) \text{ a.e. in } t \in [0,T] \tag{41}$$

for $j = 1, \cdots, q$. Since $\hat{\mathbf{x}}^{(n_k)}(t)$ is a feasible solution of problem (CQP), for all $t \in [0,T]$, we have

$$B\hat{\mathbf{x}}^{(n_k)}(t) - \int_0^t K\hat{\mathbf{x}}^{(n_k)}(s)ds \leq \mathbf{g}(t) \text{ and } \hat{\mathbf{x}}^{(n_k)}(t) \geq 0. \tag{42}$$

Since $\hat{\mathbf{x}}^{(n_k)}(t) \geq 0$, it follows, by (40) and Remark 5.2, that $\hat{\mathbf{x}}(t) \geq 0$ a.e. in $[0,T]$. For almost all $t \in [0,T]$, we have

$$\begin{aligned} B\hat{\mathbf{x}}(t) &\leq \limsup_{n_k \to \infty} B\hat{\mathbf{x}}^{(n_k)}(t) \text{ (by (41), since } B \geq 0) \\ &\leq \limsup_{n_k \to \infty} \int_0^t K\hat{\mathbf{x}}^{(n_k)}(s)ds \\ &\quad +\mathbf{g}(t) \text{ (by taking limit superior on both sides of (42))} \\ &= \int_0^t K\hat{\mathbf{x}}(s)ds + \mathbf{g}(t) \text{ (by (40))}. \end{aligned} \tag{43}$$

Let $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$, where

$$\mathcal{N}_0 = \{t \in [0,T] : \hat{\mathbf{x}}(t) \not\leq 0\}$$

and

$$\mathcal{N}_1 = \left\{ t \in [0,T] : B\hat{\mathbf{x}}(t) - \int_0^t K\hat{\mathbf{x}}(s)ds \nleq \mathbf{g}(t) \right\}.$$

We see that the set $\mathcal{N}$ has measure zero. Therefore, we define

$$\hat{\mathbf{x}}^\star(t) = \begin{cases} \hat{\mathbf{x}}(t), & \text{if } t \notin \mathcal{N} \\ 0, & \text{if } t \in \mathcal{N}. \end{cases}$$

From (43), it is not hard to verify that $\hat{\mathbf{x}}^\star(t)$ is also a feasible solution of problem (CQP). Since $\mathcal{N}$ has measure zero, by (40), we also see that $\hat{\mathbf{x}}^{(n_k)}(t)$ weakly-star converges to $\hat{\mathbf{x}}^\star(t)$. Finally, we remain to show that $\hat{\mathbf{x}}^\star(t)$ is an optimal solution of problem (CQP). By Lemma 5.2 we have

$$\begin{aligned} &\int_0^T \left\{ 1/2\, \hat{\mathbf{x}}^{(n_k)}(t)^\top D(t)\, \hat{\mathbf{x}}^{(n_k)}(t) + \mathbf{f}(t)^\top \mathbf{x}^{(n_k)}(t) \right\} dt \\ \le &\int_0^T \left\{ 1/2\, \hat{\mathbf{x}}^\star(t)^\top D(t)\, \hat{\mathbf{x}}^\star(t) + \mathbf{f}(t)^\top \mathbf{x}^{(n_k)}(t) \right\} dt \\ &+ \int_0^T \left\{ \mathbf{x}^\star(t)^\top D(t)(\mathbf{x}^{(n_k)}(t) - \mathbf{x}^\star(t)) \right\} dt. \end{aligned} \tag{44}$$

Since $\hat{\mathbf{x}}^{(n_k)}(t)$ weakly-star converges to $\hat{\mathbf{x}}^\star(t)$, it follows that

$$\lim_{n_k \to \infty} \int_0^T \mathbf{f}(t)^\top \mathbf{x}^{(n_k)}(t)dt = \int_0^T \mathbf{f}(t)^\top \mathbf{x}^\star(t)dt \tag{45}$$

and

$$\lim_{n_k \to \infty} \int_0^T \left\{ \mathbf{x}^\star(t)^\top D(t)(\mathbf{x}^{(n_k)}(t) - \mathbf{x}^\star(t)) \right\} dt = 0. \tag{46}$$

Therefore, we obtain

$$\begin{aligned} V(\text{CQP}) = &\lim_{n_k \to \infty} \int_0^T \left\{ 1/2\, \hat{\mathbf{x}}^{(n_k)}(t)^\top D(t)\, \hat{\mathbf{x}}^{(n_k)}(t) \right. \\ &\left. + \mathbf{f}(t)^\top \mathbf{x}^{(n_k)}(t) \right\} dt \text{ (by Theorem 4.2)} \\ \le &\int_0^T \left\{ 1/2\, \hat{\mathbf{x}}^\star(t)^\top D(t)\, \hat{\mathbf{x}}^\star(t) + \mathbf{f}(t)^\top \mathbf{x}^\star(t) \right\} dt \text{ (by (44), (45) and (46))} \\ \le &\, V(\text{CQP}) \text{ (since } \hat{\mathbf{x}}^\star(t) \text{ is a feasible solution of (CQP))}. \end{aligned}$$

Therefore, we conclude that $\hat{\mathbf{x}}^\star$ is indeed an optimal solution of problem (CQP). This completes the proof. ∎

6. Computational Procedure and Numerical Examples

In the sequel, we are going to provide the computational procedure to obtain the approximate solutions of the continuous-time quadratic programming problem (CQP). Of course, the approximate solutions will be the step functions. According to Theorem 4.2, it is possible to obtain the appropriate step functions so that the corresponding objective function value is close enough to the optimal objective function value when n is taken to be sufficiently large.

Remark 6.1. *We define*

$$\theta_n = \bar{\epsilon}_n p \cdot \frac{T\hat{c}}{n\sigma} \cdot e^{\frac{\kappa}{\sigma}T} \cdot \left(n + e^{\rho T} - 1\right) + (\epsilon_n + \frac{T\hat{c}}{n\sigma} \cdot e^{\frac{\kappa}{\sigma}T}) \int_0^T \rho e^{\rho(T-t)} \mathbf{g}(t)^\top \mathbf{1} dt. \tag{47}$$

Since $\delta_n \leq \frac{T\hat{c}}{n\sigma} \cdot e^{\frac{\kappa}{\sigma}T}$ by Remark 3.1, we see that $\varepsilon_n \leq \theta_n$.

Suppose that the error tolerance ϵ is pre-determined by the decision-makers. Given any natural number n, we can calculate θ_n according to (47). If $\theta_n < \epsilon$, then Remark 6.1 says that $\varepsilon_n < \epsilon$. This means that the approximate solution is acceptable, since the error tolerance ϵ is reached. Now, the computational procedure is given below.

- Step 1. Set the error tolerance ϵ and the initial value of natural number $n \in \mathbb{N}$.
- Step 2. Evaluate the values of ϵ_n, $\bar{\epsilon}_n$, $\hat{c}$, σ, ρ and κ as shown in Eqs. (25), (26), (14), (11), (27) and (12), respectively.
- Step 3. Evaluate θ_n according to (47).
- Step 4. If $\theta_n > \epsilon$ then set $n \leftarrow n + 1$ and go to Step 2; otherwise go to Step 5.
- Step 5. Evaluate the values as shown in Eq. (3).
- Step 6. Formulate the finite-dimensional primal-dual pair quadratic programming problems (P_n) and (D_n) using the values obtained in Step 5.
- Step 7. Use the efficient algorithms to obtain the optimal solution $\bar{\mathbf{x}}^{(n)}$ of problem (P_n) and the optimal solution $(\bar{\mathbf{x}}^{(n)}, \bar{\mathbf{w}}^{(n)})$ of problem (D_n).
- Step 8. Set the step function $\hat{\mathbf{x}}^{(n)}(t)$ according to (21), which will be the approximate solution of problem (CQP). The error between $V(\mathrm{CQP})$ and the objective value of approximate solution $\hat{\mathbf{x}}^{(n)}(t)$ is less than the error tolerance ϵ according to Theorem 4.2 and Remark 6.1. Moreover, we can also calculate the error bound ε_n according to (37).

Some numerical examples are provided below by using MATLAB Version 7.0.1 on a PC. Under the given error tolerance ϵ, we first determine $n \in \mathbb{N}$ such that $\theta_n \leq \epsilon$ by using Steps 1-4. From Steps 5-8, we can obtain the corresponding approximate solution $\hat{\mathbf{x}}^{(n)}(t)$ of (CQP) with error bound satisfying $\varepsilon_n \leq \theta_n \leq \epsilon$. We also write Obj_n to denote the objective value of the approximate solution $\hat{\mathbf{x}}^{(n)}(t)$.

Example 6.1.

$$\begin{aligned} &\textit{maximize} && \int_0^1 \left[-x(t)^2 + x(t)\right] dt \\ &\textit{subject to} && 4x(t) - \int_0^t 3x(s)ds \leq e^t - 1 \textit{ for all } t \in [0,1] \\ &&& x(t) \in L^\infty([0,1], \mathbb{R}_+). \end{aligned}$$

The results are summarized in the following table.

ϵ	*Obj*$_n$	ε_n
0.05	*0.1468307*	*0.0091860*
0.01	*0.1472893*	*0.0022986*
0.005	*0.1473657*	*0.0011495*
0.001	*0.1474326*	*0.0001437*
0.0005	*0.1474373*	*0.0000719*

Example 6.2.

$$\begin{aligned} &\textit{maximize} && \int_0^1 \left[(1/2\ t - 3/4) \cdot x(t)^2 + 5t^2 \cdot x(t)\right] dt \\ &\textit{subject to} && 4x(t) - \int_0^t 3x(s)ds \leq e^t - 1 \textit{ for all } t \in [0,1] \\ &&& x(t) \in L^\infty([0,1], \mathbb{R}_+). \end{aligned}$$

The results are summarized in the following table.

ϵ	*Obj*$_n$	ε_n
0.05	*0.6065280*	*0.0216335*
0.01	*0.6078011*	*0.0054134*
0.005	*0.6080135*	*0.0027071*
0.001	*0.6081993*	*0.0003384*
0.0005	*0.6082126*	*0.0001692*

Example 6.3.

$$\textit{maximize} \int_0^1 \left[-2x_1(t)^2 - x_2(t)^2 + x_1(t) + 3x_2(t)\right] dt$$

$$\textit{subject to } 8x_1(t) + 4x_2(t) - \int_0^t [3x_1(s) + 2x_2(s)]ds \leq e^t - 1 \textit{ for all } t \in [0,1]$$

$$x_1(t),\ x_2(t) \in L^\infty([0,1], \mathbb{R}_+).$$

The results are summarized in the following table.

ϵ	*Obj*$_n$	ε_n
0.05	*0.4131803*	*0.0106254*
0.01	*0.4147805*	*0.0026563*
0.005	*0.4150475*	*0.0013281*
0.001	*0.4152478*	*0.0003320*
0.0005	*0.4152812*	*0.0001660*

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