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RANK PRESERVING IN INTEGRAL EXTENSIONS OF COMMUTATIVE C^*-ALGEBRAS

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Abstract. Let A, B be two regular commutative unital Banach algebras such that B is integral over A. In 2003, Dawson and Feinstein showed that the topological stable rank $\mathrm{tsr}(B) = 1$ whenever $\mathrm{tsr}(A) = 1$. In this note, we investigate whether we will have $\mathrm{tsr}(A) = \mathrm{tsr}(B)$ in general. For instance, when A is a commutative unital C^*-algebra, we show that $\mathrm{tsr}(A) \leq \mathrm{tsr}(B)$, and the equality holds at least when the integral extension is separable. In general, A and B have the same Bass stable ranks $\mathrm{Bsr}(A) = \mathrm{Bsr}(B)$.

1. Introduction

Generalizing the concept of the covering dimension (see Section 2 for definitions), Rieffel [10] introduced the concept of the topological stable rank of a C^*-algebra in 1983. Motivated by the fact that closed subspaces have smaller covering dimensions, we may ask whether Banach subalgebras have smaller topological stable ranks. As an evidence, Dawson and Feinstein [5] showed in 2003 that the property having topological stable rank one is inherited by integral extensions of commutative Banach algebras. They first proved this for Arens-Hoffman extensions, and then extended the result to general integral extensions.

Let A be a commutative unital normed algebra, and $\alpha(z)$ be a monic polynomial over A, that is $\alpha(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_{n-1}, \ldots, a_0 \in A$. Here, we do not assume the irreducibility of $\alpha(z)$. No matter $\alpha(z)$ has a root in A or not, we can enlarge A to a commutative normed algebra A_α, the so called *Arens-Hoffman extension* [1] of A with respect to $\alpha(z)$, such that $\alpha(z)$ always has a new root in A_α. As a simple ring extension, A_α is defined to be $A[z]/(\alpha(z))$,

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where $(\alpha(z))$ is the principal ideal in the polynomial ring $A[z]$ generated by $\alpha(z)$. The norm of A_α is defined by

$$\|\sum_{i=0}^{n-1} b_i z^i + (\alpha(z))\| = \sum_{i=0}^{n-1} \|b_i\| t^i,$$

where $\sum_{i=0}^{n-1} b_i z^i \in A[z]$, and t is any but fixed positive number such that $t^n \geq \sum_{i=0}^{n-1} \|a_i\| t^i$. Note that A_α is complete whenever A is. Arens and Hoffman show that this extension is an isometry. Thus the algebraic and topological properties of A_α should be closely tied with those of A.

A natural question arises: Which topological and/or algebraic properties of A will be preserved to A_α under the Arens-Hoffman construction. The readers can refer to Dawson's paper [6, Table 2.2] for a list of partial answers to this question. Note that, in general, Arens-Hoffman extensions of a commutative unital C^*-algebra may not be a C^*-algebra. But from the table [6, Table 2.2], we know if the monic polynomial $\alpha(z)$ is separable, that is, the discriminant (cf. [3]) of $\alpha(z)$ is invertible in A, then the Arens-Hoffman extension preserves these properties: semisimplicity, self-adjointness and supnorm closedness. Therefore, equipped with the supnorm (the Gelfand norm), A_α becomes a C^*-algebra if α is separable. It follows from the Stone-Weierstrass Theorem that $A_\alpha \cong C(\Sigma_{A_\alpha})$, where Σ_{A_α} is the maximal ideal space of A_α. It follows from the open mapping theorem that the Arens-Hoffman norm and the supnorm are equivalent.

Dawson and Feinstein shows

Theorem 1. [5, Theorem 2.1]. *Let A be a commutative unital Banach algebra and $\alpha(z)$ a monic polynomial over A. If $\operatorname{tsr}(A) = 1$, then $\operatorname{tsr}(A_\alpha) = 1$.*

Using Theorem 1, they further show in [5] that if B is a commutative unital Banach algebra, which is an integral extension of A (i.e., every element in B is a root of a monic polynomial in $A[z]$), then $\operatorname{tsr}(A) = 1$ implies $\operatorname{tsr}(B) = 1$. It is interesting to ask the question

> *"whether the topological stable rank is inherited for Arens-Hoffman extensions, or generally, integral extensions of regular commutative unital Banach algebras, that is, whether we will always have $\operatorname{tsr}(A) = \operatorname{tsr}(A_\alpha)$ and $\operatorname{tsr}(A) = \operatorname{tsr}(B)$."*

We shall show in Section 3 that they have equal Bass stable ranks $\operatorname{Bsr}(A) = \operatorname{Bsr}(A_\alpha) = \operatorname{Bsr}(B)$. Assuming A is a commutative unital C^*-algebra, we shall have $\operatorname{tsr}(A) \leq \operatorname{tsr}(A_\alpha)$, and $\operatorname{tsr}(A) = \operatorname{tsr}(A_\alpha)$ if α is separable. Moreover, $\operatorname{tsr}(A) \leq \operatorname{tsr}(B)$, and $\operatorname{tsr}(A) = \operatorname{tsr}(B)$ whenever the integral extension of A to B is separable.

More precisely, we will show that $\dim \Sigma_A = \dim \Sigma_{A_\alpha} = \dim \Sigma_B$ and the assertions follow.

Finally, we would like to point out that although it might be possible to derive our results, Theorems 11 and 12, by using arguments in [9, Chapter 9, Proposition 2.16], here we give a seemingly more direct and more elementary proof, modelling on the one in a paper of Deckard and Pearcy [7, Theorem 1]. We hope this approach would provide more insights into the problem.

2. Preliminaries

Definition 2. ([9, Chapter 3]). Let X be a topological space.

(1) The *order* of a covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of X is the least integer n such that every point of X is contained in at most $n+1$ elements of $\{U_\lambda\}_{\lambda\in\Lambda}$. If no such integer exists, we say that the order is ∞.

(2) The *covering dimension* $\dim X$ of X is the least integer n such that every finite open covering of X has an open refinement of order not exceeding n. If no such integer exists, we say that the covering dimension $\dim X$ is ∞.

Theorem 3. ([9, Chapter 3]). *Let X be a normal topological space.*

(1) *If Y is a closed subspace of X, then $\dim Y \leq \dim X$.*

(2) *If $X = \bigcup_{i\in\mathbb{N}} A_i$ is a countable F_σ-set covering of X, then $\dim X = \sup_{i\in\mathbb{N}} \dim A_i$.*

Theorem 4. ([9, Theorem 3.5.6]). *Let A be a closed subset of a normal space X. If $\dim A \leq n$ and if $\dim F \leq n$ for each closed subset F of X which does not meet A, then $\dim X \leq n$.*

Theorem 5. [9, Corollary 8.1.7]). *If a space X is the inverse limit (i.e. the projective limit) of an inverse system $\{X_\alpha, \pi_{\alpha\beta}\}_{\alpha,\beta\in\Omega}$ of nonempty compact Hausdorff spaces with $\dim X_\alpha \leq n$ for each α in Ω, then $\dim X \leq n$.*

Let A be a unital Banach algebra. We use the notation $Lg_n(A)$ (resp. $Rg_n(A)$) to denote the set of n-tuples of elements of A which generates A as a left (resp. right) ideal, that is, $A = \{\Sigma_{i=1}^n a_i x_i \colon a_i \in A\}$ (resp. $A = \{\Sigma_{i=1}^n x_i a_i \colon a_i \in A\}$) whenever $(x_i) \in Lg_n(A)$ (resp. $Rg_n(A)$). In algebraic K-theory, the elements of $Lg_n(A)$ (resp. $Rg_n(A)$) are traditionally called *left* (resp. *right*) *unimodular rows*.

In this note, we investigate two concepts of dimensions of Banach algebras. The first one is the Bass stable rank introduced by Bass [2].

Definition 6. ([2, Section 1.4]). The *Bass stable rank* $\mathrm{Bsr}(A)$ of a unital Banach algebra A is defined to be the least integer n such that for any $(a_1, a_2, \ldots, a_n, a_{n+1})$ in $Lg_{n+1}(A)$ there is a $(b_1, b_2, \ldots, b_n)$ in A^n with

$$(a_1 + b_1 a_{n+1}, a_2 + b_2 a_{n+1}, \ldots, a_n + b_n a_{n+1}) \in Lg_n(A).$$

If there is no such n then set $\mathrm{Bsr}(A) = \infty$.

The second one is the topological stable rank introduced by Rieffel [10].

Definition 7. ([10, Definition 1.4]). By the *left* (resp. *right*) *topological stable rank* of a unital Banach algebra A, denoted by $\mathrm{ltsr}(A)$ (resp. $\mathrm{rtsr}(A)$), we mean the least integer n such that $Lg_n(A)$ (resp. $Rg_n(A)$) is dense in A^n (in the product topology). If no such integer exists we set $\mathrm{ltsr}(A) = \infty$ (resp. $\mathrm{rtsr}(A) = \infty$). If A does not have an identity, then its topological stable rank is defined to be that of the Banach algebra $\tilde{A}$ obtained from A by adjoining an identity.

For a commutative Banach algebra, it is clear that the left and right topological stable ranks are equal, and so is the case of a Banach algebra with a continuous involution. On the other hand, the left topological stable rank of a unital Banach algebra is 1 if and only if its right topological stable rank is 1. By definition, a unital Banach algebra has topological stable rank one if and only if it has a dense invertible group. We write $\mathrm{tsr}(A)$ for $\mathrm{ltsr}(A)$ in case $\mathrm{ltsr}(A) = \mathrm{rtsr}(A)$.

Theorem 8. ([4], [10]). *If X is a compact Hausdorff space, then*

$$\mathrm{tsr}(C(X)) = \mathrm{Bsr}(C(X)) = \left[\frac{\dim X}{2}\right] + 1.$$

In general, for a regular commutative unital Banach algebra A, we have

$$\mathrm{tsr}(A) \geq \mathrm{Bsr}(A) = \left[\frac{\dim \Sigma_A}{2}\right] + 1,$$

where Σ_A is the maximal ideal space of A, and $[r]$ is the greatest integral part of a real number r.

We also need the following two results to prove our theorems.

Lemma 9. ([7, Lemma 2.2]). *Suppose X is any topological space. Let*

$$P(x, z) = z^n + a_{n-1}(x)z^{n-1} + \cdots + a_1(x)z + a_0(x),$$

where all $a_i \in C(X)$. Fix an x_0 in X and let z_0 be a root of multiplicity μ of the polynomial $P(x_0, z)$ in $z \in \mathbb{C}$. If $\delta > 0$ so that $P(x_0, z)$ has no root other than z_0 satisfying $|z - z_0| < \delta$, then there is a neighborhood U of x_0 such that for all x in U, the polynomial $P(x, z)$ has exactly μ roots (counting multiplicities) satisfying $|z - z_0| < \delta$.

Theorem 10. ([8, Theorem 3.3]). *Let A be a commutative Banach algebra and B an integral extension of A. If $\pi^{-1}(h)$ is an infinite subset of Σ_B for some $h \in \Sigma_A$, then B is incomplete under any norm.*

3. The Results

Theorem 11. *Let A be a commutative unital normed algebra with maximal ideal space Σ_A, and let*

$$\alpha(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be a monic polynomial over A with $n > 1$. Let A_α be the Arens-Hoffman extension of A with respect to $\alpha(z)$. Then

$$\dim \Sigma_A = \dim \Sigma_{A_\alpha}.$$

Proof. As mentioned in the paper of Arens and Hoffman [1], the maximal ideal space Σ_{A_α} of A_α is contained in $\Sigma_A \times \{\lambda \in \mathbb{C}\colon |\lambda| \leq 1\}$. More precisely, Σ_{A_α} consists exactly of all (h, z) such that

$$(h, z)(\alpha) = z^n + a_{n-1}(h)z^{n-1} + \cdots + a_1(h)z + a_0(h) = 0.$$

We define $P\colon \Sigma_A \times \mathbb{C} \to \mathbb{C}$ by

$$P(h, z) = z^n + a_{n-1}(h)z^{n-1} + \cdots + a_1(h)z + a_0(h)$$

and $\pi\colon \Sigma_{A_\alpha} \to \Sigma_A$ by

$$\pi(h, z) = h \quad \text{or} \quad \pi(\tilde{h}) = \tilde{h}|_A, \quad \text{where } \tilde{h} = (h, z) \in \Sigma_{A_\alpha}.$$

Note that for every h_0 in Σ_A, there is a $\delta > 0$ such that the closed balls $\overline{B}(z_i, \delta)$ in $\mathbb{C}$ are pairwise disjoint, where z_i are distinct roots of the complex polynomial $P(h_0, z)$. From Lemma 9, there is a compact neighborhood V_{h_0} of h_0 such that $V_{h_0} = \pi(V_i)$ where $V_i = (V_{h_0} \times \overline{B}(z_i, \delta)) \cap \Sigma_{A_\alpha}$ are pairwise disjoint compact neighborhoods of (h_0, z_i) and $\pi^{-1}(V_{h_0}) = \cup V_i$. Note that for every h' in V_{h_0}, the complex polynomial $P(h', z)$ has exactly k roots (counting multiplicities) in the open balls $B(z_i, \delta)$ as z_i is a root of $P(h_0, z)$ of multiplicity k.

Claim. For every point (h_0, z_0) in Σ_{A_α}, there are σ-compact neighborhoods U_{h_0} of h_0 in Σ_A and $U_{(h_0,z_0)}$ of (h_0, z_0) in Σ_{A_α} such that

$$\dim U_{h_0} = \dim U_{(h_0,z_0)}.$$

We prove the claim by induction on the multiplicity of the root z_0 of the complex polynomial $P(h_0, z)$.

Let (h_0, z_0) be in Σ_{A_α} with z_0 being a root of $P(h_0, z)$ of multiplicity 1. By Lemma 9 with $\delta > 0$ mentioned above, $\pi|_{V_0}\colon V_0 \to V_{h_0}$ is a homeomorphism from the compact neighborhood V_0 of (h_0, z_0) onto the compact neighborhood V_{h_0} of h_0. Set $U_{(h_0,z_0)} = V_0$ and $U_{h_0} = V_{h_0}$. Then $\dim U_{h_0} = \dim U_{(h_0,z_0)}$.

Now suppose that for each (h_0, z_0) in Σ_{A_α} with z_0 being a root of $P(h_0, z)$ of multiplicity less than k, there are σ-compact neighborhoods U_{h_0} of h_0 in Σ_A and $U_{(h_0,z_0)}$ of (h_0, z_0) in Σ_{A_α} such that $\dim U_{h_0} = \dim U_{(h_0,z_0)}$. Let $(h_0, z_0) \in \Sigma_{A_\alpha}$ with z_0 being a root of $P(h_0, z)$ of multiplicity k. By Lemma 9 with $\delta > 0$ mentioned above, there is a compact neighborhood V_{h_0} of h_0 such that for h' in V_{h_0}, the complex polynomial $P(h', z)$ has exactly k roots (counting multiplicities) in the open balls $B(z_0, \delta)$. Furthermore, we can assume that for h' in V_{h_0}, the complex polynomial $P(h', z)$ has no root satisfying $|z - z_0| = \delta$. Let $\mathcal{P}_m \subset V_{h_0}$ be the set of points h' for which $P(h', z)$ has two distinct roots with distance not less than $\frac{1}{m}$ and they lie in the open ball $B(z_0, \delta)$. Let $\mathcal{P} = \bigcup_{m=1}^{\infty} \mathcal{P}_m$.

For h' in $\mathcal{P}$, the complex polynomial $P(h', z)$ has two distinct roots in the open ball $B(z_0, \delta)$. By Lemma 9, h' has a neighborhood such that for every point h in this neighborhood, $P(h, z)$ has two distinct roots in the open ball $B(z_0, \delta)$. Hence $\mathcal{P}$ is relatively open in V_{h_0} and thus $V_{h_0} \setminus \mathcal{P}$ is relatively closed in V_{h_0}. For every h' in $V_{h_0} \setminus \mathcal{P}$, the polynomial $P(h', z)$ has exactly one root of multiplicity k in the open ball $B(z_0, \delta)$. As argued for the case of multiplicity one above, we can prove that $\dim(V_{h_0} \setminus \mathcal{P}) = \dim \pi^{-1}(V_{h_0} \setminus \mathcal{P}) \cap V_0$.

We next show that $\mathcal{P}_m$ is relatively closed in V_{h_0} for $m = 1, 2, \ldots$. If $\mathcal{P}_m$ is nonempty, let $\{h_\lambda\}$ be a net in $\mathcal{P}_m$ such that $h_\lambda \to h_1$ for some h_1 in V_{h_0}, and for each λ, let $z_{1\lambda}$ and $z_{2\lambda}$ be roots of $P(h_\lambda, z)$ satisfying $|z_{1\lambda} - z_{2\lambda}| \geq \frac{1}{m}$ and $|z_{i\lambda} - z_0| < \delta$. Choose a subnet $\{h_\gamma\}$ of $\{h_\lambda\}$ with the property that the nets $\{z_{1\gamma}\}$ and $\{z_{2\gamma}\}$ converge to some z_1 and z_2, respectively. Then, since $P(h, z)$ is jointly continuous in h and z, $P(h_1, z_1) = P(h_1, z_2) = 0$, and $|z_1 - z_2| \geq \frac{1}{m}$. By our previous arrangements we can not have $|z_i - z_0| = \delta$. So $|z_i - z_0| < \delta$ for $i = 1, 2$. Thus $h_1 \in \mathcal{P}_m$, and consequently, $\mathcal{P}_m$ is relatively closed and hence compact.

On the other hand,

$$\mathcal{P}'_m := \pi^{-1}(\mathcal{P}_m) \cap V_0$$

is also compact for each m. For each point (h', z') in $\mathcal{P}'_m$, we have z' is a root of $P(h', z)$ of multiplicity less than k. By the induction hypothesis, there are σ-compact neighborhoods $U_{h'}$ of h' in Σ_A and $U_{(h',z')}$ of (h', z') in Σ_{A_α} such that $\dim U_{h'} = \dim U_{(h',z')}$. Consider the covering $\{U_{(h,z)} \colon (h, z) \in \mathcal{P}'_m\}$ of $\mathcal{P}'_m$. By the compactness of $\mathcal{P}'_m$, it has a finite subcovering $\{U_{(h_i,z_i)} \colon (h_i, z_i) \in \mathcal{P}'_m\}_{i=1}^n$. Let $\tilde{\mathcal{P}}'_m = \bigcup_{i=1}^n U_{(h_i,z_i)}$ and $\tilde{\mathcal{P}}_m = \bigcup_{i=1}^n U_{h_i}$. Note that $\tilde{\mathcal{P}}'_m$ and $\tilde{\mathcal{P}}_m$ are σ-compact and $\dim U_{h_i} = \dim U_{(h_i,z_i)}$. Then the Sum Theorem (Theorem 3) implies that $\dim \tilde{\mathcal{P}}_m = \dim \tilde{\mathcal{P}}'_m$.

Let

$$U_{h_0} = (V_{h_0} \setminus \mathcal{P}) \cup \bigcup_{m=1}^{\infty} \tilde{\mathcal{P}}_m = V_{h_0} \cup \bigcup_{m=1}^{\infty} \tilde{\mathcal{P}}_m$$

and

$$U_{(h_0,z_0)} = (\pi^{-1}(V_{h_0} \setminus \mathcal{P}) \cap V_0) \cup \bigcup_{m=1}^{\infty} \tilde{\mathcal{P}}'_m = V_0 \cup \bigcup_{m=1}^{\infty} \tilde{\mathcal{P}}'_m.$$

Note that U_{h_0} and $U_{(h_0,z_0)}$ are σ-compact neighborhoods of h_0 and (h_0, z_0), respectively. By the Sum Theorem again, we have $\dim U_{h_0} = \dim U_{(h_0,z_0)}$, and we have verified the claim.

Finally, using the compactness of Σ_A and Σ_{A_α}, we see that Σ_{A_α} and Σ_A have finite σ-compact neighborhood covering $\{U_{(h_i,z_i)}\}_{i=1}^n$ and $\{U_{h_i}\}_{i=1}^n$, respectively. Since $\dim U_{h_i} = \dim U_{(h_i,z_i)}$ for $i = 1, \ldots, n$, the Sum Theorem furnishes that $\dim \Sigma_A = \dim \Sigma_{A_\alpha}$. ∎

Theorem 12. *Let A and B be commutative unital Banach algebras such that B is an integral extension of A. Then*

$$\dim \Sigma_A = \dim \Sigma_B.$$

Proof. Similar to the case of the Arens-Hoffman extension, we can embed Σ_B into the product space $\Sigma_A \times \mathbb{C}^{B\setminus A}$ as the subset consisting of points $(h, \{h_b\}_{b\in B\setminus A})$ such that $h(\alpha)(h_b) = 0$ whenever $\alpha(b) = 0$, for all b in $B\setminus A$ and for all α in $A[z]$. In addition, for each $b \in B\setminus A$, fix a minimal monic polynomial α_b in $A[z]$ such that $\alpha_b(b) = 0$. Let $\Lambda = \{\alpha_b\colon b \in B\setminus A\}$ and $\mathfrak{F}$ be the collection of all finite subsets of Λ. Then Σ_B is the limit of the inverse system $\{\Sigma_{A_\alpha}, \pi_{\alpha\beta}\}_{\alpha,\beta\in\mathfrak{F}}$. By Theorems 5 and 11, we have $\dim \Sigma_B \le \sup \dim \Sigma_{A_\alpha} = \dim \Sigma_A$.

Now we prove the reverse inequality. By Theorem 10, we have $\Sigma_A = \bigcup_{i=1}^{\infty} U_i$ where U_i is the collection of h in Σ_A which has at most i extensions in Σ_B. Every U_i is closed in Σ_A and thus compact. Note that

$$U_1 \cong \pi^{-1}(U_1),$$

where $\pi : \Sigma_B \to \Sigma_A$ is defined by $\pi(h, z) = h$. Thus, $\dim U_1 = \dim \pi^{-1}(U_1) \le \dim \Sigma_B$ by Theorem 3(1). Suppose that $\dim U_{i-1} \le \dim \Sigma_B$. For $h \in U_i\setminus U_{i-1}$, by definition h has exactly i extensions $h_1, \ldots, h_i$ to B. Choose a point $b \in B\setminus A$ such that $h_j(b) \ne h_k(b)$, $\forall 1 \le j < k \le i$. By Lemma 9, h has a compact neighborhood U_h such that $U_h \cap U_{i-1} = \emptyset$ and h has an extension belonging to $U_h \times \overline{B}(h_j(b), \delta) \times \mathbb{C}^{B\setminus\{A\cup\{b\}\}}$ for $1 \le j \le i$. Then

$$U_h \cap U_i \cong \pi^{-1}(U_h \cap U_i) \cap (U_h \times \overline{B}(h_1(b), \delta) \times \mathbb{C}^{B\setminus\{A\cup\{b\}\}}).$$

It follows from Theorem 3(1) that $\dim U_h \cap U_i \le \dim \Sigma_B$.

Let V be a closed subset of U_i such that $V \cap U_{i-1} = \emptyset$. By the above arguments, every point h in V has a compact neighborhood U_h such that $\dim U_h \cap V \le$

$\dim U_h \cap U_i \leq \dim \Sigma_B$. By the compactness of V, $V = \bigcup_{m=1}^{n} U_{h_m} \cap V$. Thus $\dim V = \max \dim U_{h_m} \cap V \leq \dim \Sigma_B$. By Theorem 4, $\dim U_i \leq \dim \Sigma_B$. By the induction hypothesis and Theorem 3(2), we have $\dim \Sigma_A = \sup_{i\in\mathbb{N}} \dim U_i \leq \dim \Sigma_B$. ■

Theorem 13. *Let A be a regular commutative unital Banach algebra.*

(1) *Let $\alpha(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial over A with $n > 1$. Then the Bass stable ranks satisfy*

$$\mathrm{Bsr}(A) = \mathrm{Bsr}(A_\alpha).$$

(2) *Let B be a regular integral extension of A. Then the Bass stable ranks satisfy*

$$\mathrm{Bsr}(A) = \mathrm{Bsr}(B).$$

Proof.

(1) Since A is a regular commutative Banach algebra, A_α is also regular (see, e.g. [6]). So case (1) reduces to case (2).

(2) By Theorem 8 and Theorem 12, we have

$$\mathrm{Bsr}(A) = \left[\frac{\dim \Sigma_A}{2}\right] + 1 = \left[\frac{\dim \Sigma_B}{2}\right] + 1 = \mathrm{Bsr}(B). \qquad ■$$

Theorem 14. *Let A be a commutative unital C^*-algebra.*

(1) *Let $\alpha(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial over A with $n > 1$. Then the topological stable ranks satisfy*

$$\mathrm{tsr}(A) \leq \mathrm{tsr}(A_\alpha).$$

They are equal if $\alpha(z)$ is separable.

(2) *Let B be a regular integral extension of A. Then*

$$\mathrm{tsr}(A) \leq \mathrm{tsr}(B).$$

If, in addition, every element in B is a root of a separable monic polynomial over A, then

$$\mathrm{tsr}(A) = \mathrm{tsr}(B).$$

Proof.

(1) By Theorems 8 and 11, we have $\mathrm{tsr}(A) \leq \mathrm{tsr}(A_\alpha)$. If $\alpha(z)$ is separable, A_α is a C^*-algebra in the supnorm. Thus, $\mathrm{tsr}(A) = \mathrm{tsr}(A_\alpha)$.

(2) By Theorem 8 and Theorem 12, we have

$$\operatorname{tsr}(A) = \left[\frac{\dim \Sigma_A}{2}\right] + 1 = \left[\frac{\dim \Sigma_B}{2}\right] + 1 \leq \operatorname{tsr}(B).$$

Suppose that $\operatorname{tsr}(A) = n$. Let $(a_1, a_2, \ldots, a_n) \in B^n$, and a_i be a root of a separable monic polynomial $\alpha_i(z)$ in $A[z]$. Consider the Arens-Hoffman extension $A_{\alpha_1 \ldots \alpha_n}$ of A and let x_i be the root of $\alpha_i(z)$ in the Arens-Hoffman construction $A_{\alpha_1 \ldots \alpha_n}$. Then there is a continuous homomorphism ϕ from $A_{\alpha_1 \ldots \alpha_n}$ into B with respect to the Arens-Hoffman norm, which fixes A and maps x_i to a_i. By (1), we have $\operatorname{tsr}(A_{\alpha_1 \ldots \alpha_n}) = n$. Thus $(x_1, \ldots, x_n)$ can be approximated by elements in $Lg_n(A_{\alpha_1 \ldots \alpha_n})$. By the continuity of ϕ, $(a_1, \ldots, a_n)$ can be approximated by elements in $Lg_n(B)$. This shows that $\operatorname{tsr}(B) = n$. ■

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