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FROM STEINER TRIPLE SYSTEMS TO 3-SUN SYSTEMS

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Abstract. An n-sun is the graph with $2n$ vertices consisting of an n-cycle with n pendent edges which form a 1-factor. In this paper we show that the necessary and sufficient conditions for the decomposition of complete tripartite graphs with at least two partite sets having the same size into 3-suns and give another construction to get a 3-sun system of order n, for $n \equiv 0, 1, 4, 9$ (mod 12). In the construction we metamorphose a Steiner triple system into a 3-sun system. We then embed a cyclic Steiner triple system of order n into a 3-sun system of order $2n - 1$, for $n \equiv 1$ (mod 6).

1. INTRODUCTION

A decomposition of a graph G is a set $\mathbb{H} = \{H_1, H_2, \ldots, H_t\}$ of subgraphs of G such that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_t) = E(G)$ and $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$. For convenience, we say that G can be decomposed into $H_1, H_2, \cdots, H_t$. If H_i is isomorphic to a graph H for each $i = 1, 2, \ldots, t$, then we say that G has an H decomposition. A Steiner triple system of order n (more simply triple system) is a pair (X, T) where X is an n-set and T is a collection of edge disjoint triangles (or triples) which partition the edge set of K_n with the vertex set X. It is well known [3] that the spectrum for Steiner triple systems(STS) is precisely the set of all $n \equiv 1$ or 3 (mod 6). A 3-sun is a graph with six vertices a, b, c, d, e, f consisting of a triangle (a, b, c) and a 1-factor $\{\{a, d\}, \{b, e\}, \{c, f\}\}$. We will denote this 3-sun by $(a, b, c; d, e, f)$. A 3-$sun\ system$ of order n,($3SS(n)$), is a pair (Y, S) where Y is an n-set and S is a collection of edge disjoint 3-suns which partition the edge set of K_n with the vertex set Y. In [6], Yin had shown that the spectrum for 3-sun system is precisely $n \equiv 0, 1, 4, 9$ (mod 12) and if (Y, S) is a 3-sun system of order n then $|S| = n(n-1)/12$.

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In this paper we give the different constructions to get 3-sun systems of order n. Since a 3-sun is a tripartite graph, in Section 2, we consider the decomposition of a complete tripartite graph $K_{p,p,r}$ into 3-suns. We obtain that the necessary and sufficient conditions for the existence of a decomposition of $K_{p,p,r}$ into 3-suns. In Section 3, we use recursive construction to construct 3-sun systems of order n when $n \equiv 0, 4$ (mod 12). For $n \equiv 1$ (mod 12), we construct a cyclic 3-sun system of order n. For $n \equiv 9$ (mod 12), we metamorphose a Kirkman triple system(KTS) of order n into a 3-sun system of order n. Clearly the triangles of a 3-sun system cannot form a triple system. So the following problem is immediate. What is the largest cyclic Steiner triple system can be embedded in the partial triple system consisting of the triangles of a 3-sun system? In Section 4, we embed a cyclic Steiner triple system of order $6m + 1$ into a 3-sun system of order $12m + 1$.

2. Decompose $K_{p,q,r}$ into 3-Suns

Let p, q, r be positive integers. For convenience, we will let $A = \{a_1, a_2, \cdots, a_p\}$, $B = \{b_1, b_2, \cdots, b_q\}, C = \{c_1, c_2, \cdots, c_r\}$ be three partite sets of $K_{p,q,r}$.

Lemma 2.1. *Let p, q, r be positive integers and $p \geq q \geq r$. If $K_{p,q,r}$ has a 3-sun decomposition, then $6 \mid (pq + qr + pr)$ and $r \geq max\{\frac{p}{3}, \frac{pq}{p+q}\}$.*

Proof. If $K_{p,q,r}$ has a 3-sun decomposition, then $6 \mid (pq + qr + pr)$ and there are $\frac{pq+qr+pr}{6}$ 3-suns. Since a 3-sun has three vertices of degree 3 and each belongs to different partite sets, we have $(p + q)r \geq 3 \cdot \frac{pq+qr+pr}{6}$. It implies that $(p + q)r \geq pq$, thus $r \geq \frac{pq}{p+q}$. Since $K_{p,q,r}$ can be decomposed into at most qr 3-suns, we have $\frac{pq+qr+pr}{6} \leq qr$. Combining the inequality $(p+q)r \geq pq$, we obtain that $r \geq \frac{p}{3}$. Therefore, $r \geq max\{\frac{p}{3}, \frac{pq}{p+q}\}$. ∎

If $p = q = r$, then $K_{p,q,r}$ has a 3-sun decomposition provided that p is even. For example, $K_{2,2,2}$ can be decomposed into two 3-suns: $(a_1, b_1, c_1; b_2, c_2, a_2)$ and $(a_2, b_2, c_2;\ b_1, c_1, a_1)$. Since $K_{n,n,n}$ can be decomposed into n^2 triangles from a Latin square of order n, we obtain that $K_{p,p,p}$ has a 3-sun decomposition if and only if p is even.

Next we will consider the decomposition of $K_{p,p,r}$.

Lemma 2.2. *Let $p \geq 2$ and $r \geq 2$ be integers. If $K_{p,p,r}$ has a 3-sun decomposition, then $\frac{p}{2} \leq r \leq \frac{5p}{2}$ and $(1)\ p \equiv 0 \pmod 6$, $(2)\ p \equiv 2 \pmod 6$, $r \equiv 2 \pmod 3$, or $(3)\ p \equiv 4 \pmod 6$, $r \equiv 1 \pmod 3$.*

Proof. By counting the number of edges of $K_{p,p,r}$, if $K_{p,p,r}$ has a 3-sun decomposition, then $6 \mid p(p + 2r)$. It follows that p should be even and $3|p$ or $3|p + 2r$. This implies either $p \equiv 0$ (mod 6), $p \equiv 2$ (mod 6) and $r \equiv 2$ (mod 3) or

$p \equiv 4 \pmod 6$ and $r \equiv 1 \pmod 3$. If $p \geq r$, by Lemma 2.1, $r \geq max\{\frac{p}{3}, \frac{p^2}{p+p}\}$, we have $r \geq \frac{p}{2}$. If $r \geq p$, then $K_{r,p,p}$ can be decomposed into at most p^2 3-suns. We have $p^2 \geq \frac{p^2+2pr}{6}$, thus $r \leq \frac{5p}{2}$. Combining above two results, we obtain $\frac{p}{2} \leq r \leq \frac{5p}{2}$. ∎

Lemma 2.3. *Let p be even. If $K_{p,p,s}$ and $K_{p,p,t}$ have 3-sun decompositions, then $K_{np,np,ms+(n-m)t}$ has a 3-sun decomposition for $0 \leq m \leq n$.*

Proof. The first two partite sets of $K_{np,np,ms+(n-m)t}$ can be partitioned into n groups each group containing p elements and the third partite set can be partitioned into n groups, m of them containing s elements, the others containing t elements. Since $K_{n,n,n}$ can be decomposed into n^2 triangles from a Latin square of order n, $K_{np,np,ms+(n-m)t}$ can be decomposed into n^2 triangles and each triangle corresponds to a $K_{p,p,t}$ or a $K_{p,p,s}$. Thus $K_{np,np,ms+(n-m)t}$ can be decomposed into nm copies of $K_{p,p,s}$ and $n(n-m)$ copies of $K_{p,p,t}$. Since $K_{p,p,s}$ and $K_{p,p,t}$ have a 3-sun decomposition respectively, $K_{np,np,ms+(n-m)t}$ has a 3-sun decomposition. ∎

Lemma 2.4. *Let p and r be positive integers. If $p \equiv 2 \pmod 6, r \equiv 2 \pmod 3$, or $p \equiv 4 \pmod 6, r \equiv 1 \pmod 3$, and $\frac{p}{2} \leq r \leq \frac{5p}{2}$, then $K_{p,p,r}$ has a 3-sun decomposition.*

Proof.

(1) $p \leq r \leq \frac{5p}{2}$. By Lemma 2.2, if $K_{2,2,r}$ has a 3-sun decomposition, then $r = 2$ or 5. $K_{2,2,2}$ has already been done. We can decompose $K_{2,2,5}$ into four 3-suns: $(a_1, b_1, c_1; c_4, c_5, b_2)$, $(a_1, b_2, c_2; c_5, c_3, a_2)$, $(a_2, b_1, c_3; c_5, c_2, a_1)$, $(a_2, b_2, c_4; c_1, c_5, b_1)$. By using the decomposition of $K_{2,2,2}$ and $K_{2,2,5}$, we can get the following construction. Let k be a positive integer. If $p = 6k+2 = (3k+1)\cdot 2$ and $r = 3t+2 \leq 15k+5$, let $m = 5k-t+1$, then $r = m\cdot 2+(3k+1-m)\cdot 5$. By Lemma 2.3, $K_{6k+2,6k+2,r}$ has a 3-sun decomposition. If $p = 6k+4 = (3k+2)\cdot 2$ and $r = 3t+1 \leq 15k+10$, let $m = 5k-t+3$, then $r = m\cdot 2+(3k+2-m)\cdot 5$. By Lemma 2.3, $K_{6k+4,6k+4,r}$ has a 3-sun decomposition.

(2) $\frac{p}{2} \leq r < p$. We can decompose $K_{p,p,r}$ into 3-suns as follows. Let $s = \lfloor \frac{r}{3} \rfloor$. Then $\lfloor \frac{p}{6} \rfloor \leq s \leq \lfloor \frac{p}{3} \rfloor$. Let $q = \lfloor \frac{p}{3} \rfloor - s$.

   (i) For $m = 1, 2, \ldots, q$, $i = 0, 1, 2, \ldots, p-1$,
   $(a_{2m-1+i}, b_{1+i}, c_{m+i}; b_{3m-1+i}, a_{2m+i}, a_{\frac{p}{2}+i})$ and
   $(a_{2q+2m-1+i}, b_{1+i}, c_{q+m+i}; b_{3q+3m-1+i}, a_{2q+2m+i}, b_{q+2m+i})$.
   Notice that the indices of a and b are restricted to $Z_p = \{1, 2, \ldots, p\}$ and the indices of c are restricted to $Z_{\frac{p}{2}} = \{1, 2, \ldots, \frac{p}{2}\}$

   (ii) For $m = 1, 2, \ldots, r - \frac{p}{2}$, $j = 0, 1, 2, \ldots, \frac{p}{2} - 1$,
   $(a_{4q+2m-1+j}, b_{1+j}, c_{\frac{p}{2}+m}; c_{2q+m+j}, a_{4q+2m+j}, b_{\frac{p}{2}+1+j})$ and

$(a_{4q+2m-1+j'}, b_{1+j'}, c_{2q+m+j'}; c_{\frac{p}{2}+m}, a_{4q+2m+j'}, b_{\frac{p}{2}+1+j'})$, $j' = \frac{p}{2}, \frac{p}{2}+1, \ldots, p-1$.

Notice that the indices of a and b are restricted to Z_p and the values of $2q+m+j$ and $2q+m+j'$ are restricted to $Z_{\frac{p}{2}}$. ∎

Lemma 2.5. *Let p and r be positive integers. If $p \equiv 0 \pmod 6$ and $p/2 \le r \le 5p/2$, then $K_{p,p,r}$ has a 3-sun decomposition.*

Proof.

(1) If $p = 6$, then $3 \le r \le 15$. Combining $K_{2,2,2}$ and $K_{2,2,5}$, by Lemma 2.3, we can get that $K_{6,6,r}$ has a 3-sun decomposition for $r = 6, 9, 12, 15$, For the rest of r, the decomposition of $K_{6,6,r}$ can be found in Appendix.

(2) Let $k \ge 2$ be a positive integer. If $p = 6k$, then $3k \le r \le 15k$. Let $i = \lfloor \frac{r}{k} \rfloor$ and $m = r - ik \ge 0$, then r can be written as $m(i+1) + (k-m)i$. By (1) and Lemma 2.3, we obtain that $K_{p,p,r}$ has a 3-sun decomposition. ∎

By Lemma 2.2, 2.4 and 2.5, we obtain

Theorem 2.6. *Let p and r be positive integers. $K_{p,p,r}$ has a 3-sun decomposition if and only if $\frac{p}{2} \le r \le \frac{5p}{2}$ and (1) $p \equiv 0 \pmod 6$, (2) $p \equiv 2 \pmod 6$, $r \equiv 2 \pmod 3$, or (3) $p \equiv 4 \pmod 6$, $r \equiv 1 \pmod 3$.*

We close this section by decomposing $K_{2n,2n,2n}$ into cyclic 3-suns. Let A, B, and C be three partite sets of $K_{2n,2n,2n}$. $K_{2n,2n,2n}$ can be decomposed into *cyclic* 3-suns if there is an automorphism which is a permutation with three orbits and each orbits has length $2n$, see [4]. Let $t = (a_i, b_j, c_k; b_u, c_v, a_w)$ be a 3-sun in $K_{2n,2n,2n}$, where $a_i, a_w \in A$, $b_j, b_u \in B$, and $c_k, c_v \in C$. We define $d_{AB}(t) = \{0_{j-i}, 0_{u-i}\}$, $d_{BC}(t) = \{1_{k-j}, 1_{v-j}\}$, $d_{CA}(t) = \{2_{i-k}, 2_{w-k}\}$, the indices are taken modulo $2n$. Let $d(t) = d_{AB}(t) \cup d_{BC}(t) \cup d_{CA}(t)$. We call $d(t)$ is the difference set of $t = (a_i, b_j, c_k; b_u, c_v, a_w)$. Let $D(H) = \{d(t) | t \in H\}$, where H is a collection of 3-suns in $K_{2n,2n,2n}$. If T contains n 3-suns in $K_{2n,2n,2n}$ and $D(T) = \{0_i, 1_i, 2_i | i = 0, 1, \cdots, 2n-1\}$, then we call that T is a set of *base* 3-suns in $K_{2n,2n,2n}$. That is, $K_{2n,2n,2n}$ can be decomposed into cyclic 3-suns $\bigcup_{x=0}^{2n-1}(T+x) = \{(a_{i+x}, b_{j+x}, c_{k+x}; b_{u+x}, c_{v+x}, a_{w+x}) | \ (a_i, b_j, c_k; b_u, c_v, a_w) \in T, x = 0, 1, \cdots, 2n-1\}$. The indices of a, b, and c are taken modulo $2n$. For example, if $T = \{(a_1, b_1, c_1; b_2, c_2, a_2)\}$ and $D(T) = \{0_0, 0_1, 1_0, 1_1, 2_0, 2_1\}$, then $K_{2,2,2}$ can be decomposed into cyclic 3-suns. Therefore, if we can find the base 3-suns in $K_{2n,2n,2n}$, then $K_{2n,2n,2n}$ can be decomposed into cyclic 3-suns.

Theorem 2.7. *Let n be a positive integer. $K_{2n,2n,2n}$ can be decomposed into cyclic 3-suns.*

Proof. Construct the base 3-suns in $K_{2n,2n,2n}$ as follows:
The following indices of a, b, and c are restricted to the set $\{1, 2, \ldots, 2n\}$.

(1) n is odd. Let $m = \frac{n-1}{2}$,

(i) $k = 0, 1, \ldots, m$, $(a_1, b_{4k+1}, c_{n+2k+1}; b_{4k+2}, c_1, a_{n-2k+1}) \in T$.

(ii) $k = 0, 1, \ldots, m-1$, $(a_1, b_{4k+3}, c_{2k+2}; b_{4k+4}, c_1, a_{2n-2k}) \in T$.

We have $D(T) = \{0_{4k}, 0_{4k+1}, 1_{n-2k}, 1_{2n-4k}, 2_{n-2k}, 2_{2n-4k} | k = 0, 1, \ldots, m\} \cup \{0_{4k+2}, 0_{4k+3}, 1_{2n-2k-1}, 1_{2n-4k-2}, 2_{2n-2k-1}, 2_{2n-4k-2} | k = 0, 1, \ldots, m-1\} = \{0_i, 1_i, 2_i | i = 0, 1, \ldots, 2n-1\}$.

(2) n is even. Let $m = \frac{n}{2}$,

(i) $(a_1, b_1, c_{n+1}; b_2, c_1, a_{n+1}) \in T$.

(ii) $k = 1, 2, \ldots, m-1$, $(a_1, b_{4k+1}, c_{n+2k+1}; b_{4k+2}, c_{n+6k+1}, a_{4k+1}) \in T$.

(iii) $k = 0, 1, \ldots, m-1$, $(a_1, b_{4k+3}, c_{2k+2}; b_{4k+4}, c_{6k+4}, a_{4k+3}) \in T$.

We have $D(T) = \{0_0, 0_1, 1_n, 1_0, 2_n, 2_0\} \cup \{0_{4k}, 0_{4k+1}, 1_{n-2k}, 1_{n+2k}, 2_{n-2k}, 2_{n+2k} | k = 1, 2, \ldots, m-1\} \cup \{0_{4k+2}, 0_{4k+3}, 1_{2n-2k-1}, 1_{2k+1}, 2_{2n-2k-1}, 2_{2k+1} | k = 0, 1, 2, \ldots, m-1\} = \{0_i, 1_i, 2_i | i = 0, 1, \ldots, 2n-1\}$.

Therefore, $K_{2n,2n,2n}$ can be decomposed into cyclic 3-suns. ■

3. 3-Sun System of Order n

In this section, we will construct the 3-sun system of order n, $3SS(n)$, i.e., decomposing K_n into 3-suns. The spectrum of $3SS(n)$ is precisely $n \equiv 0, 1, 4, 9$ (mod 12). First we will see the construction of $3SS(n)$ for $n \equiv 0, 4$ (mod 12), by using the decomposition of complete tripartite graphs into 3-suns. Let the vertex set of K_n be $\{1, 2, \ldots, n\}$.

Example 3.1.

(a) $3SS(12) = \{(1, 3, 4; 9, 11, 12), (1, 5, 11; 2, 8, 12), (1, 7, 10; 8, 12, 3),$
$(2, 6, 12; 4, 5, 10), (2, 8, 11; 5, 9, 4), (3, 5, 12; 6, 10, 1), (3, 7, 9; 2, 5, 12),$
$(4, 6, 9; 7, 11, 2), (4, 8, 10; 5, 12, 2), (6, 7, 8; 1, 2, 3), (9, 10, 11; 5, 6, 7)\}$.

(b) $3SS(24) = \{(1, 2, 4; 8, 9, 23), (1, 3, 7; 16, 24, 8), (1, 5, 6; 22, 8, 13),$
$(1, 9, 21; 10, 17, 5), (1, 11, 18; 12, 20, 3), (1, 13, 23; 14, 19, 5), (2, 3, 5; 8, 10, 18),$
$(2, 6, 7; 21, 8, 14), (2, 10, 22; 11, 18, 6), (2, 12, 19; 13, 21, 4), (2, 14, 24; 15, 20, 6),$
$(3, 4, 6; 8, 11, 17), (3, 11, 23; 12, 19, 7), (3, 15, 17; 16, 21, 7), (3, 13, 20; 14, 22, 5),$
$(4, 5, 7; 8, 12, 20), (4, 12, 24; 13, 20, 8), (4, 14, 21; 15, 23, 6), (4, 16, 18; 9, 22, 8),$
$(5, 9, 19; 10, 23, 1), (5, 13, 17; 14, 21, 1), (5, 15, 22; 16, 24, 7), (6, 10, 20; 11, 24, 2),$
$(6, 14, 18; 15, 22, 2), (6, 16, 23; 9, 17, 8), (7, 11, 21; 12, 17, 3), (7, 15, 19; 16, 23, 3),$
$(7, 9, 24; 10, 18, 1), (8, 12, 22; 13, 18, 4), (8, 16, 20; 9, 24, 4), (8, 14, 19; 15, 17, 6),$
$(8, 10, 17; 11, 19, 2), (9, 10, 12; 14, 21, 6), (9, 11, 16; 20, 5, 14), (9, 13, 15; 3, 14, 18),$
$(10, 11, 13; 14, 22, 7), (10, 15, 16; 4, 14, 19), (11, 12, 15; 14, 23, 1),$
$(12, 13, 16; 14, 24, 2), (17, 18, 21; 19, 13, 8), (17, 22, 23; 4, 19, 10),$
$(17, 20, 24; 12, 1, 19), (18, 20, 22; 19, 15, 3), (18, 23, 24; 7, 19, 11),$
$(20, 21, 23; 19, 16, 2), (21, 22, 24; 19, 9, 5)\}$.

Lemma 3.2. *If* $n \equiv 0 \pmod{12}$, *then there exists a* 3*-sun system of order* n.

Proof. By Example 3.1, there are 3-sun systems of order 12 and 24 respectively. Let $n = 12m$ where $m \geq 3$. Let $m = 3s + p$ where $s \geq 1$ and $0 \leq p \leq 2$. K_n can be viewed as the union of two K_{12s}'s, one $K_{12s+12p}$ and one $K_{12s,12s,12s+12p}$. By Lemma 2.5, $K_{12s,12s,12t}$ can be decomposed into 3-suns if $\frac{s}{2} \leq t \leq \frac{5s}{2}$. By Example 3.1 and the above construction, K_n can be recursively decomposed into 3-suns as $n > 24$, except $n = 60$. Since K_{60} can be viewed as the union of one K_{12}, two K_{24}'s and one $K_{24,24,12}$, K_{60} has a 3-sun system. The proof is completed. ■

Example 3.3.

(a) $3SS(16)$ = $\{(1,2,4;13,8,11)$, $(1,5,9;6,12,13)$, $(1,14,15;8,3,5)$, $(1,3,16;7,10,5)$, $(2,3,5;14,9,13)$, $(2,6,10;7,13,14)$, $(2,15,16;9,4,6)$, $(3,4,6;15,10,14)$, $(3,7,11;8,14,15)$, $(4,5,7;16,11,15)$, $(4,8,12;9,15,16)$, $(5,6,8;10,12,16)$, $(6,7,9;11,13,16)$, $(7,8,10;12,14,1)$, $(8,9,11;13,15,2)$, $(9,10,12;14,16,3)$, $(10,11,13;15,1,4)$, $(11,12,14;16,2,5)$, $(12,13,15;1,3,6)$, $(13,14,16;2,4,7)\}$.

(b) $3SS(28)$ = $\{(1,2,4;22,10,14)$, $(1,3,28;9,13,6)$, $(1,6,25;13,19,8)$, $(1,5,10;12,17,23)$, $(1,8,15;7,19,22)$, $(1,20,24;14,3,8)$, $(1,26,27;11,4,7)$, $(2,3,5;23,11,15)$, $(2,6,11;13,18,24)$, $(2,7,26;14,20,9)$, $(2,9,16;8,20,23)$, $(2,21,25;15,4,9)$, $(2,27,28;12,5,8)$, $(3,4,6;24,12,16)$, $(3,7,12;14,19,25)$, $(3,8,27;15,21,10)$, $(3,10,17;9,21,24)$, $(3,22,26;16,5,10)$, $(4,5,7;25,13,17)$, $(4,8,13;15,20,26)$, $(4,11,18;10,22,25)$, $(4,23,27;17,6,11)$, $(4,9,28;16,22,11)$, $(5,6,8;26,14,18)$, $(5,9,14;16,21,27)$, $(5,12,19;11,23,26)$, $(5,24,28;18,7,12)$, $(6,7,9;27,15,19)$, $(6,10,15;17,22,28)$, $(6,13,20;12,24,27)$, $(7,8,10;28,16,20)$, $(7,11,16;18,23,1)$, $(7,14,21;13,25,28)$, $(8,9,11;14,17,21)$, $(8,12,17;22,24,2)$, $(9,10,12;15,18,22)$, $(9,13,18;23,25,3)$, $(10,11,13;16,19,23)$, $(10,14,19;24,26,4)$, $(11,12,14;17,20,24)$, $(11,15,20;25,27,5)$, $(12,13,15;18,21,25)$, $(12,16,21;26,28,6)$, $(13,14,16;19,22,26)$, $(13,17,22;27,1,7)$, $(14,15,17;20,23,27)$, $(14,18,23;28,2,8)$, $(15,16,18;21,24,28)$, $(15,19,24;26,3,9)$, $(16,17,19;22,25,1)$, $(16,20,25;27,4,10)$, $(17,18,20;23,26,2)$, $(17,21,26;28,5,11)$, $(18,19,21;24,27,3)$, $(18,22,27;1,6,12)$, $(19,20,22;25,28,4)$, $(19,23,28;2,7,13)$, $(20,21,23;26,1,5)$, $(21,22,24;27,2,6)$, $(22,23,25;28,3,7)$, $(23,24,26;1,4,8)$, $(24,25,27;2,5,9)$, $(25,26,28;3,6,10)\}$.

Lemma 3.4. $3SS(n)$ *exists if* $n = 40, 52, 64$.

Proof. K_{40} can be viewed as the union of two K_{12}'s, one K_{16} and one $K_{12,12,16}$. K_{52} can be viewed as the union of two K_{12}'s, one K_{28} and one $K_{12,12,28}$. K_{64} can be viewed as the union of one K_{16}, two K_{24}'s and one $K_{24,24,16}$. According to Example 3.1 and 3.3, K_{12}, K_{16}, K_{24}, and K_{28} can be decomposed into 3-suns. By Lemma 2.5, $K_{12,12,16}$, $K_{12,12,28}$, and $K_{24,24,16}$ can be decomposed into 3-suns. Hence, K_n can be decomposed into 3-suns for $n = 40, 52, 64$. ■

Lemma 3.5. *If* $n \equiv 4 \pmod{12}$, *then there exists a* 3*-sun system of order* n.

Proof. Let $n = 12m+4$. By Example 3.3 and Lemma 3.4, we have $3SS(16)$, $3SS(28)$, $3SS(40)$, $3SS(52)$, and $3SS(64)$. Let $m = 3s + p$ where $s \geq 2$ and $0 \leq p \leq 2$. $K_{36s+12p+4}$ can be viewed as the union of two K_{12s}'s, one $K_{12(s+p)+4}$ and one $K_{12s,12s,12(s+p)+4}$. By Lemma 3.2 and recursive construction, K_{12s} and $K_{12(s+p)+4}$ can be decomposed into 3-suns. By Lemma 2.5, $K_{12s,12s,12(s+p)+4}$ can be decomposed into 3-suns. Hence, the proof is completed. ■

Next, we will construct cyclic 3-sun systems of order n for $n \equiv 1$ (mod 12). A 3-sun system $3SS(n)$ on the elements of $Z_n = \{1, 2, \dots, n\}$ is said to be *cyclic* if whenever $(a, b, c; x, y, z)$ is a 3-sun, so also is $(a+1, b+1, c+1; x+1, y+1, z+1)$.

Example 3.6.

(a) The 3-suns $(i, i+1, i+3; i+4, i+6, i+9)$, $1 \leq i \leq 13$, form a cyclic $3SS(13)$.

(b) The 3-suns $(i, i+1, i+5; i+9, i+12, i+17)$, $(i, i+2, i+8; i+13, i+16, i+23)$, $(i, i+3, i+10; i+16, i+20, i+28)$, $1 \leq i \leq 37$, form a cyclic $3SS(37)$.

By [1,5], suppose that $\{1, 2, \dots, 3m\}$ can be partitioned into m triples $\{a, b, c\}$ such that $a + b = c$ or $a + b + c \equiv 0$ (mod $6m + 1$). Then the triples $\{0, a, a + b\}$ form a $(6m + 1, 3, 1)$ difference system and so lead to the construction of cyclic $STS(6m+1)$. A *Skolem triple system* of order m is a partition of $\{1, 2, \dots, 3m\}$ into m triples $\{i, a_i, i + a_i\}$, $1 \leq i \leq m$. An *O'Keefe triple system* of order m is a partition of $\{1, 2, \dots, 3m-1, 3m+1\}$ into m triples $\{i, a_i, i+a_i\}$, $1 \leq i \leq m$. It is well-known that if $m \equiv 0, 1$ (mod 4) then a Skolem triple system of order m exists and if $m \equiv 2, 3$ (mod 4) then an O'Keefe triple system of order m exists. Let $t = (a, b, c; d, e, f)$ be a 3-sun in K_n. Since t contains a triangle (a, b, c) and one 1-factor $\{\{a, d\}, \{b, e\}, \{c, f\}\}$, we obtain two difference triples $\{b - a, c - b, a - c\}$ and $\{d - a, e - b, f - c\}$ where the values are taken modulo n. Next, we will metamorphose a cyclic $STS(12m + 1)$ into a cyclic $3SS(12m + 1)$.

Lemma 3.7. *If* $n \equiv 1 \pmod{12}$*, then there exists a cyclic* $3SS(n)$.

Proof. Let $n = 12k + 1$.

(1) k is even.
If $k = 2$, then the 3-suns in a cyclic $3SS(25)$ are constructed as follows.
For $i = 1, 2, \dots, 25$,
$(i, i+1, i+12; i+2, i+8, i+21)$ and $(i, i+3, i+8; i+4, i+9, i+18)$.
It is easy to check that the union of four difference triples is the set $\{1, 2, \dots, 12\}$.
If $k \geq 4$, then the 3-suns are constructed as follows.
For $i = 1, 2, \dots, n$,
$(i, i+1, i+6k; i+k, i+4k, i+11k-1)$,

$(i, i+2k-1, i-1+\frac{9k}{2}; i+2k, i+5k-1, i-1+\frac{19k}{2})$,
and $j = 1, 2, \ldots, \frac{k}{2}-1$,
$(i, i+2j, i+3k+j; i+2j+1, i+5k+j-1, i+8k+2j)$,
$(i, i+k+2j-1, i+\frac{7k}{2}+j-1; i+k+2j, i+\frac{11k}{2}+j-2, i+9k+2j-2)$.
Since from each 3-sun we can get two difference triples, these difference triples form a $Skolem\ triple\ system$ of order $2k$ when k is even and $k \geq 2$, see[1]. Therefore, we have a cyclic $3SS(12k+1)$, k is even.

(2) k is odd.
In Example 3.6, we have a cyclic $3SS(13)$ and a cyclic $3SS(37)$. We consider when $k \geq 5$, the 3-suns are constructed as follows.
For $i = 1, 2, \ldots, n$,
$(i, i+2j, i+3k+j+1; i+2j+1, i+5k+j-1, i+8k+2j+1)$, where $j = 1, 2, \ldots, \frac{k-1}{2}$.
$(i, i+k+2j-1, i+\frac{7k+1}{2}+j; i+k+2j+2, i+\frac{11k-3}{2}+j, i+9k+2j+2)$, where $j = 1, 2, \ldots, \frac{k-5}{2}$.
$(i, i+2k-1, i+5k; i+2k-4, i+4k+2, i+9k-1)$,
$(i, i+k+2, i+6k+1; i+2k-2, i+3k+4, i+10k+1)$, and
$(i, i+1, i+\frac{11k+3}{2}; i+2k, i+2k+2, i+\frac{19k+5}{2})$
Since from each 3-sun we can get two difference triples, these difference triples form an $O'Keefe\ triple\ system$ of order $2k$ when k is odd and $k \geq 5$, see[1]. Therefore, we have a cyclic $3SS(12k+1)$, k is odd. ■

Next we will metamorphose a $KTS(12k+9)$ into a $3SS(12k+9)$. A *parallel class* in a Steiner triple system (S,T) is a set of triples in T that partitions S. If the triples in T can be partitioned into parallel classes, then we say $STS(v)$ is a *Kirkman triple system of order* v, denoted by $KTS(v)$. It is well-known [2] that there exists a $KTS(v)$ if and only if $v \equiv 3$ (mod 6).

Lemma 3.8. *If* $n \equiv 9 \pmod{12}$, *then there is a* 3*-sun system of order* n.

Proof. Let $n = 12k+9$ where $k \geq 0$. Then there exists a $KTS(12k+9)$ with $6k+4$ parallel classes. Let (S,T) be a $KTS(12k+9)$. π and π' are any two distinct parallel classes in T. Consider $\pi \cup \pi'$. If $(x,y,z) \in \pi$ and $(x,a,b) \in \pi'$, then the edges $\{x,y\}$, $\{y,z\}$, and $\{x,z\}$ can not be contained in any triple in π'. That means y, z, a, and b are distinct. Using this property, we give a direction to each edge, such that each triple (x,a,b) in π' forms a directed cycle $\langle x,a,b\rangle$ with the edge set $\{(x,a),(a,b),(b,x)\}$. Similarly, we have $\langle y,c,d\rangle$ and $\langle z,e,f\rangle$ in π'. Any triangle in π with its out-edge from π' forms a 3-sun. Thus we can get a 3-sun $(x,y,z;a,c,e)$ in $\pi \cup \pi'$. Therefore, the edge-set of the union of any two distinct parallel classes of $KTS(12k+9)$ can be decomposed into $4k+3$ 3-suns. Hence, we obtain a 3-sun system of order n. ■

Example 3.9. There is a $3SS(9)$ constructed from a $KTS(9)$.

Proof. Let (Z_9, T) be a $KTS(9)$ with 4 parallel classes π_1, π_2, π_3, and π_4, where $\pi_1 = \{(1,2,3),\ (4,5,6), (7,8,9)\}$, $\pi_2 = \{(1,4,7), (2,5,8), (3,6,9)\}$, $\pi_3 = \{(1,5,9), (2,6,7), (3,4,8)\}$, and $\pi_4 = \{(1,6,8), (2,4,9), (3,5,7)\}$. We give a direction to each edge in π_2 and π_4 as follows: $\pi_2' = \{\langle 1,4,7\rangle, \langle 2,5,8\rangle, \langle 3,6,9\rangle\}, \pi_4' = \{\langle 1,6,8\rangle, \langle 2,4,9\rangle, \langle 3,5,7\rangle\}$. Then the edge-set of $\pi_1 \cup \pi_2'$ can be decomposed into three 3-suns, $(1,2,3;4,5,6)$, $(4,5,6;7,8,9)$, and $(7,8,9;1,2,3)$. $\pi_3 \cup \pi_4'$ can be decomposed into three 3-suns, $(1,5,9;6,7,2)$, $(2,6,7;4,8,3)$, and $(3,4,8;5,9,1)$. ■

By Lemma 3.2, 3.5, 3.7, and 3.8, we obtain the following theorem.

Theorem 3.10. *There exists a 3-sun system of order n, if and only if $n \equiv 0,1,4,9 \pmod{12}$.*

4. Embedding a Cyclic Steiner Triple System in a 3-Sun System

Let (Y, S) be a 3-sun system of order n and P be the collection of triangles in S. Then (Y, P) is a partial triple system of order n. We say that the Steiner triple system (X, T) is $embedded$ in a 3-sun system (Y, S) provided $X \subseteq Y$ and $T \subseteq P$. Subsequently, we give a construction for a 3-sun system of order $12m + 1$ embedding a cyclic Steiner triple system of order $6m + 1$.

Theorem 4.11. *Let m be a positive integer. Let (X, T) be a cyclic Steiner triple system of order $6m + 1$. Then there is a 3-sun system (Y, S) of order $12m + 1$, such that (X, T) is embedded in (Y, S).*

Proof. Let $X = \{v_1, v_2, \ldots, v_{6m}, v_{6m+1}\}$, $U = \{u_1, u_2, \ldots, u_{6m}\}$ and $X \cap U = \emptyset$. Set $Y = X \cup U$. Let (X, T) be a cyclic $STS(6m+1)$. Suppose $E_1, E_2, \ldots$, and E_m are base triples in T. For convenience, we give an order for the elements in each base triple such that $E_i = \langle v_{a_i^1}, v_{a_i^2}, v_{a_i^3}\rangle$, for all $i = 1, 2, \ldots, m$, and $a_i^1 < a_i^2 < a_i^3$.
Define a collection S of 3-suns over Y as follows:

(1) For $i = 1, 2, \ldots, m$, $j = 0, 1, 2, \ldots, 6m$.
Define $t_{i,j}^k := a_i^k + j \in Z_{6m+1} = \{1, 2, \ldots, 6m+1\}$, for all $k = 1, 2, 3$.
$B_{i,j} := (v_{t_{i,j}^1}, v_{t_{i,j}^2}, v_{t_{i,j}^3}; u_{2m+3i+t_{i,j}^1-3}, u_{2m+3i+t_{i,j}^2-2}, u_{2m+3i+t_{i,j}^3-1})$ where the indices of u are restricted to $Z_{6m} = \{1, 2, \ldots, 6m-1, 6m\}$.
Therefore, there are $m(6m+1)$ 3-suns.

(2) Define $\alpha_k = v_k$, $k = 1, 2, \ldots, 6m$.
For $i = 1, 2, \ldots, m-1$ and $j = 0, 1, 2, \ldots, 6m-1$,
$B'_{i,j} := (u_{1+j}, u_{2m-2(i-1)+j}, \alpha_{2-i+j}; u_{2m+1-2(i-1)+j}, u_{4m+1-(i-1)+j}, u_{5m+1+j})$
And $B'_{m,j} := (u_{1+j}, u_{2+j}, \alpha_{5m+2+j}; u_{3+j}, \beta_j, u_{5m+1+j})$ where

$$\beta_j := \begin{cases} v_{6m+1} & \text{if } j = 0,1,2,\ldots,2m-2, 5m-1, 5m, \ldots, 6m-1. \\ u_{3m+2+j} & \text{if } j = 2m-1, 2m, \ldots, 5m-3, 5m-2. \end{cases}$$

The indices of α and u are restricted to $Z_{6m} = \{1, 2, \ldots, 6m\}$. Hence, there are $6m^2$ 3-suns.

From (1), the base triples in T is the triangles of $B_{i,0}$, for $i = 1, 2, \ldots, m$. Therefore, (X, T) is embedded in (Y, S). ■

Example 4.12. Let $X = \{v_1, v_2, \ldots, v_7\}$, $U = \{u_1, u_2, \ldots, u_6\}$ and $Y = X \cup U$. Let (X, T) be a cyclic $STS(7)$. If $\{v_1, v_2, v_4\}$ is a base triple in T. Let $E_i = \langle v_{a_1^1}, v_{a_1^2}, v_{a_1^3} \rangle = \langle v_1, v_2, v_4 \rangle$, and $S = \{B_{1,j}, B'_{1,j'} | j = 0, 1, \ldots, 6, j' = 0, 1, \ldots, 5\}$. By the construction in Theorem 4.1, we can get:
$B_{1,0} = (v_1, v_2, v_4; u_3, u_5, u_2)$, $B_{1,1} = (v_2, v_3, v_5; u_4, u_6, u_2)$, $B_{1,2} = (v_3, v_4, v_6; u_5, u_1, u_4)$, $B_{1,3} = (v_4, v_5, v_7; u_6, u_2, u_5)$, $B_{1,4} = (v_5, v_6, v_1; u_1, u_3, u_5)$, $B_{1,5} = (v_6, v_7, v_2; u_2, u_4, u_6)$, $B_{1,6} = (v_7, v_1, v_3; u_3, u_4, u_1)$, $B'_{1,0} = (u_1, u_2, v_1; u_3, v_7, u_6)$, $B'_{1,1} = (u_2, u_3, v_2; u_4, u_6, u_1)$, $B'_{1,2} = (u_3, u_4, v_3; u_5, u_1, u_2)$, $B'_{1,3} = (u_4, u_5, v_4; u_6, u_2, u_3)$, $B'_{1,4} = (u_5, u_6, v_5; u_1, v_7, u_4)$, $B'_{1,5} = (u_6, u_1, v_6; u_2, v_7, u_5)$. Then (Y, S) is a $3SS(13)$ and (X, T) is a cyclic $STS(7)$ embedded in a $3SS(13)$.

5. Conclusion and Open Question

There are further questions to be asked.

(1) If $p > q > r \geq 2$, what is the necessary and sufficient condition for the decomposition of $K_{p,q,r}$ into 3-suns ?

(2) Can one embed any Steiner triple system into a 3-sun system?

Appendix

A. $K_{6,6,3}$ can be decomposed into 12 3-suns as follows:
$\{(a_1, b_1, c_1; b_2, a_2, b_3), (a_2, b_2, c_2; b_3, a_3, b_1), (a_3, b_3, c_3; b_1, a_1, b_2),$
$(a_4, b_4, c_1; b_5, a_5, a_2), (a_5, b_5, c_2; b_6, a_6, a_3), (a_6, b_6, c_3; b_4, a_4, a_1),$
$(a_4, b_1, c_3; b_2, a_5, b_4), (a_5, b_2, c_1; b_3, a_6, b_5), (a_6, b_3, c_2; b_1, a_4, b_6),$
$(a_1, b_4, c_2; b_5, a_2, a_4), (a_2, b_5, c_3; b_6, a_3, a_5), (a_3, b_6, c_1; b_4, a_1, a_6)\}.$

B. $K_{6,6,4}$ can be decomposed into 14 3-suns as follows:
$\{(a_1, b_1, c_1; b_2, c_4, a_3), (a_2, b_2, c_2; b_3, c_3, a_3), (a_3, b_3, c_3; b_1, a_4, b_4),$
$(a_4, b_4, c_1; c_4, a_6, b_5), (a_6, b_5, c_2; c_4, a_5, b_6), (a_6, b_6, c_3; b_2, a_5, b_1),$
$(a_4, b_1, c_2; c_3, a_5, b_3), (a_5, b_2, c_4; b_4, a_4, b_3), (a_6, b_3, c_1; b_1, a_5, b_2),$
$(a_1, b_4, c_2; b_3, a_3, a_5), (a_2, b_5, c_3; b_1, a_3, a_5), (a_3, b_6, c_4; b_2, a_1, a_2),$
$(a_1, b_5, c_4; c_3, a_4, b_4), (a_2, b_6, c_1; b_4, a_4, a_5)\}.$

C. $K_{6,6,5}$ can be decomposed into 16 3-suns as follows:
$\{(a_1, b_1, c_1; b_2, a_2, b_5), (a_2, b_2, c_2; b_3, a_3, b_6), (a_3, b_3, c_3; c_2, a_4, a_1),$
$(a_4, b_4, c_4; b_5, a_1, a_5), (a_2, b_5, c_4; b_6, a_6, b_1), (a_3, b_6, c_4; c_1, a_5, b_2),$
$(a_4, b_1, c_2; c_3, a_3, b_5), (a_6, b_1, c_3; c_2, a_5, b_6), (a_4, b_2, c_5; c_1, c_3, a_2),$

$(a_6, b_2, c_1; c_4, a_5, b_4)$, $(a_5, b_3, c_1; c_2, a_6, b_6)$, $(a_5, b_5, c_5; c_3, a_3, b_3)$, $(a_6, b_4, c_5; b_6, a_5, b_1)$, $(a_2, b_4, c_3; c_1, a_3, b_5)$, $(a_1, b_3, c_2; b_5, c_4, b_4)$, $(a_1, b_6, c_5; c_4, a_4, a_3)\}$.

D. $K_{6,6,7}$ can be decomposed into 20 3-suns as follows:
$\{(a_1, b_1, c_1; c_7, a_4, a_5)$, $(a_2, b_2, c_2; c_1, a_1, a_4)$, $(a_3, b_3, c_3; b_4, a_2, a_5)$, $(a_4, b_4, c_4; b_6, a_1, b_5)$, $(a_5, b_5, c_5; b_2, a_2, a_1)$, $(a_6, b_6, c_6; c_3, c_1, a_2)$, $(a_2, b_1, c_3; c_5, a_5, b_6)$, $(a_3, b_2, c_4; b_6, c_7, b_1)$, $(a_4, b_3, c_5; c_3, c_6, b_4)$, $(a_5, b_4, c_6; b_6, a_2, a_1)$, $(a_6, b_5, c_7; c_5, a_4, b_4)$, $(a_3, b_1, c_5; c_2, c_6, b_2)$, $(a_4, b_2, c_6; c_7, c_3, a_3)$, $(a_5, b_3, c_7; c_2, c_1, b_6)$, $(a_6, b_4, c_1; b_2, c_3, a_4)$, $(a_1, b_5, c_2; b_6, c_3, b_4)$, $(a_2, b_6, c_4; c_7, c_5, a_6)$, $(a_6, b_1, c_2; b_3, c_7, b_6)$, $(a_1, b_3, c_4; c_3, c_2, a_5)$, $(a_3, b_5, c_1; c_7, c_6, b_2)\}$.

E. $K_{6,6,8}$ can be decomposed into 22 3-suns as follows:
$\{(a_1, b_1, c_1; b_2, c_8, a_2)$, $(a_2, b_2, c_2; b_4, c_1, a_6)$, $(a_3, b_3, c_3; c_6, a_2, a_6)$, $(a_4, b_4, c_4; b_1, c_1, a_5)$, $(a_5, b_5, c_5; c_8, a_2, b_6)$, $(a_6, b_6, c_6; b_2, c_3, a_2)$, $(a_2, b_1, c_3; c_5, c_7, a_1)$, $(a_3, b_2, c_4; c_7, c_5, b_1)$, $(a_4, b_3, c_5; b_5, c_1, b_4)$, $(a_5, b_4, c_6; c_1, c_2, a_1)$, $(a_6, b_5, c_7; b_3, c_3, b_4)$, $(a_1, b_6, c_8; c_5, a_3, a_2)$, $(a_3, b_1, c_5; c_8, c_2, a_6)$, $(a_4, b_2, c_6; c_1, c_8, b_1)$, $(a_5, b_3, c_7; b_6, c_6, a_4)$, $(a_6, b_4, c_8; b_1, a_3, a_4)$, $(a_1, b_5, c_2; c_7, c_8, a_5)$, $(a_2, b_6, c_4; c_7, c_1, a_6)$, $(a_3, b_5, c_1; c_2, c_6, a_6)$, $(a_4, b_6, c_2; c_3, c_7, b_3)$, $(a_1, b_3, c_4; b_4, c_8, b_5)$, $(a_5, b_2, c_3; b_1, c_7, b_4)\}$.

F. $K_{6,6,10}$ can be decomposed into 26 3-suns as follows:
$\{(a_1, b_1, c_1; b_3, c_8, b_6)$, $(a_2, b_2, c_2; b_4, a_1, a_4)$, $(a_3, b_3, c_3; c_1, c_2, b_4)$, $(a_4, b_4, c_4; c_3, c_1, b_5)$, $(a_5, b_5, c_5; c_2, c_3, b_6)$, $(a_6, b_6, c_6; c_4, c_3, b_5)$, $(a_2, b_1, c_3; b_3, a_6, a_1)$, $(a_3, b_2, c_4; b_5, c_9, a_2)$, $(a_4, b_3, c_5; c_1, c_4, b_4)$, $(a_5, b_4, c_6; c_4, c_2, b_3)$, $(a_6, b_5, c_7; c_3, c_8, a_3)$, $(a_1, b_6, c_8; c_4, c_7, a_4)$, $(a_3, b_1, c_5; b_4, c_2, b_2)$, $(a_4, b_2, c_6; b_6, c_1, b_1)$, $(a_5, b_3, c_7; c_1, c_8, a_2)$, $(a_6, b_4, c_8; c_2, c_7, a_3)$, $(a_1, b_5, c_9; c_5, c_{10}, a_2)$, $(a_2, b_6, c_{10}; c_5, c_9, a_5)$, $(a_4, b_1, c_7; b_5, c_4, a_1)$, $(a_5, b_2, c_8; b_6, c_3, a_2)$, $(a_6, b_3, c_9; c_1, c_{10}, a_4)$, $(a_1, b_4, c_{10}; c_6, c_9, a_4)$, $(a_2, b_5, c_1; c_6, c_2, b_3)$, $(a_3, b_6, c_2; c_6, c_4, a_1)$, $(a_5, b_1, c_9; c_3, c_{10}, a_3)$, $(a_6, b_2, c_{10}; c_5, c_7, a_3)\}$.

G. $K_{6,6,11}$ can be decomposed into 28 3-suns as follows:
$\{(a_1, b_1, c_1; b_2, c_8, b_5)$, $(a_2, b_2, c_2; c_7, c_{11}, a_4)$, $(a_3, b_3, c_3; c_{11}, c_2, b_4)$, $(a_4, b_4, c_4; c_3, c_{11}, b_5)$, $(a_5, b_5, c_5; c_{11}, c_3, b_6)$, $(a_6, b_6, c_6; c_3, c_4, b_5)$, $(a_2, b_1, c_3; b_3, a_6, a_1)$, $(a_3, b_2, c_4; b_5, c_9, a_2)$, $(a_4, b_3, c_5; c_{11}, c_4, b_4)$, $(a_5, b_4, c_6; c_4, c_2, b_3)$, $(a_6, b_5, c_7; c_4, c_8, b_6)$, $(a_1, b_6, c_8; c_4, c_{11}, a_4)$, $(a_3, b_1, c_5; c_1, c_2, b_2)$, $(a_4, b_2, c_6; b_6, c_1, b_1)$, $(a_5, b_3, c_7; c_1, c_8, a_3)$, $(a_6, b_4, c_8; c_2, c_7, a_3)$, $(a_1, b_5, c_9; c_5, c_{10}, b_6)$, $(a_2, b_6, c_{10}; c_5, c_3, a_5)$, $(a_4, b_1, c_7; b_5, c_4, a_1)$, $(a_5, b_2, c_8; b_6, c_3, a_2)$, $(a_6, b_3, c_9; c_1, c_{10}, a_4)$, $(a_1, b_4, c_{10}; c_6, c_9, a_4)$, $(a_2, b_5, c_{11}; c_6, c_2, a_6)$, $(a_3, b_6, c_2; c_6, c_1, a_5)$, $(a_5, b_1, c_9; c_3, c_{10}, a_3)$, $(a_6, b_2, c_{10}; c_5, c_7, a_3)$, $(a_1, b_3, c_{11}; c_2, c_1, b_1)$,

$(a_2, b_4, c_1; c_9, a_3, a_4)\}$.

H. $K_{6,6,13}$ can be decomposed into 32 3-suns as follows:
$\{(a_1, b_1, c_1; c_{13}, c_2, a_4)$, $(a_2, b_2, c_2; c_1, c_3, b_3)$, $(a_3, b_3, c_3; b_4, c_4, a_5)$, $(a_4, b_4, c_4; c_8, c_1, b_1)$, $(a_5, b_5, c_5; c_{10}, c_2, a_6)$, $(a_6, b_6, c_6; c_{12}, a_5, a_3)$, $(a_2, b_1, c_3; b_3, c_{12}, a_4)$, $(a_3, b_2, c_4; c_8, c_1, a_5)$, $(a_4, b_3, c_5; c_9, c_6, b_6)$, $(a_5, b_4, c_6; c_{13}, c_2, a_2)$, $(a_6, b_5, c_7; c_{13}, c_3, a_3)$, $(a_1, b_6, c_8; c_3, c_1, a_2)$, $(a_3, b_1, c_5; c_1, c_6, b_2)$, $(a_4, b_2, c_6; c_{11}, c_7, a_1)$, $(a_5, b_3, c_7; c_{12}, c_1, a_2)$, $(a_6, b_4, c_8; c_1, c_3, b_5)$, $(a_1, b_5, c_9; c_2, c_4, a_2)$, $(a_2, b_6, c_{10}; c_4, c_3, a_4)$, $(a_4, b_1, c_7; c_{13}, c_8, b_6)$, $(a_5, b_2, c_8; c_1, c_9, b_3)$, $(a_6, b_3, c_9; c_2, c_{10}, b_4)$, $(a_1, b_4, c_{10}; c_4, c_5, a_3)$, $(a_2, b_5, c_{11}; c_{13}, c_6, b_4)$, $(a_3, b_6, c_{12}; c_9, c_4, a_4)$, $(a_5, b_1, c_9; c_2, c_{10}, b_6)$, $(a_6, b_2, c_{10}; c_3, c_{11}, b_5)$, $(a_1, b_3, c_{11}; c_5, c_{13}, b_6)$, $(a_2, b_4, c_{12}; c_5, c_7, b_3)$, $(a_3, b_5, c_{13}; c_{11}, c_1, b_4)$, $(a_4, b_6, c_2; b_5, c_{13}, a_3)$, $(a_6, b_1, c_{11}; c_4, c_{13}, a_5)$, $(a_1, b_2, c_{12}; c_7, c_{13}, b_5)\}$.

I. $K_{6,6,14}$ can be decomposed into 34 3-suns as follows:
$\{(a_1, b_1, c_1; c_{13}, c_2, a_4)$, $(a_2, b_2, c_2; c_1, c_3, b_3)$, $(a_3, b_3, c_3; c_{10}, c_4, a_5)$, $(a_4, b_4, c_4; b_5, c_1, b_1)$, $(a_5, b_5, c_5; c_{14}, c_2, a_6)$, $(a_6, b_6, c_6; c_{12}, a_5, a_3)$, $(a_2, b_1, c_3; c_9, c_{12}, a_4)$, $(a_3, b_2, c_4; c_2, c_1, a_5)$, $(a_4, b_3, c_5; c_9, c_6, b_6)$, $(a_5, b_4, c_6; c_{13}, c_2, a_2)$, $(a_6, b_5, c_7; c_{13}, c_3, a_3)$, $(a_1, b_6, c_8; c_3, c_2, a_2)$, $(a_3, b_1, c_5; c_8, c_6, b_2)$, $(a_4, b_2, c_6; c_{11}, c_7, a_1)$, $(a_5, b_3, c_7; c_{12}, c_1, a_1)$, $(a_6, b_4, c_8; c_1, c_3, b_5)$, $(a_1, b_5, c_9; c_2, c_4, b_2)$, $(a_2, b_6, c_{10}; c_4, c_3, a_4)$, $(a_4, b_1, c_7; c_2, c_{14}, b_6)$, $(a_5, b_2, c_8; c_1, c_{14}, b_1)$, $(a_6, b_3, c_9; c_2, c_{10}, b_4)$, $(a_1, b_4, c_{10}; c_4, c_5, a_5)$, $(a_2, b_5, c_{11}; c_{14}, c_6, b_4)$, $(a_3, b_6, c_{12}; c_9, c_4, a_4)$, $(a_5, b_1, c_9; c_2, c_{10}, b_6)$, $(a_6, b_2, c_{10}; c_3, c_{11}, b_5)$, $(a_1, b_3, c_{11}; c_5, c_8, b_6)$, $(a_2, b_4, c_{12}; c_5, c_7, b_3)$, $(a_3, b_5, c_{13}; c_{11}, c_1, b_6)$, $(a_4, b_6, c_{14}; c_8, c_1, b_5)$, $(a_6, b_1, c_{11}; c_4, c_{13}, a_5)$, $(a_1, b_2, c_{12}; c_{14}, c_{13}, b_5)$, $(a_2, b_3, c_{13}; c_7, c_{14}, a_4)$, $(a_3, b_4, c_{14}; c_1, c_{13}, a_6)\}$.

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