TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 2, pp. 511-519, April 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

ON-LINE 3-CHOOSABLE PLANAR GRAPHS

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Abstract. This paper proves that if G is a triangle-free planar graph in which no 4-cycle is adjacent to a 4-cycle or a 5-cycle, then G is on-line 3-choosable.

1. INTRODUCTION

Given a graph G and a function f from $V(G)$ to $\mathbb{N}$. An *f-assignment* of G is a mapping L which assigns each vertex v of G a set $L(v)$ of $f(v)$ integers as permissible colours. Given a list assignment L of G, an *L-colouring* of G is a mapping $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for each vertex v and $c(u) \neq c(v)$ for each edge uv. We say G is *L-colourable* if there exists an L-colouring of G. We say G is *f-choosable* if for every f-assignment L, G is L-colourable. If $f(v) = k$ for all $v \in V(G)$, then f-choosable is called *k-choosable*. The *choice number* $\mathrm{ch}(G)$ of G is the least number k such that G is k-choosable. List colouring of graphs was introduced in the 1970's by Vizing [5] and independently by Erdős, Rubin and Taylor [2], and has been studied extensively in the literature [4]. On-line list colouring of graphs was introduced by Schauz [3].

The on-line list colouring of graphs is defined through a two-person game.

Definition 1. Given a graph G and a mapping $f : V(G) \to \mathbb{N}$, the on-line f-list colouring game on G is a game with two players: Alice and Bob. At the beginning, all vertices of G are uncoloured. In the ith move, Alice chooses a nonempty subset V_i of uncoloured vertices of G and assign colour i as a permissible colour to each vertex of V_i. Bob chooses an independent set X_i contained in V_i and colour vertices of X_i by colour i. If for some integer m, at the end of the mth step, there is a vertex v which has been assigned $f(v)$ permissible colours, i.e., is contained in $f(v)$ of the V_i's, but is not coloured, i.e., not contained in any of the X_i's, then Alice wins the game. Otherwise, in the end, each vertex v is assigned in at most $f(v)$ colours and all vertices are coloured and Bob wins the game.

Received August 16, 2010, accepted December 4, 2010.
Communicated by Hung-Lin Fu.
2010 *Mathematics Subject Classification*: 05C10, 05C15.
Key words and phrases: On-line list colouring, Planar graph, Triangle-free.
[1]The author was supported in part by the National Center of Theoretical Sciences.

Definition 2. Suppose $f : V(G) \to \mathbb{N}$. We say G is *on-line f-choosable* if Bob has a winning strategy in any f-list colouring game on G. We say G is *on-line k-choosable* if G is on-line f-choosable for the constant function $f = k$. The *on-line choice number* $\mathrm{ch}^{\mathrm{OL}}(G)$ of G is the least number k such that G is on-line k-choosable.

It follows from definition that for any graph G, $\mathrm{ch}^{\mathrm{OL}}(G) \geq \mathrm{ch(G)}$. It is recently proved by Dvorák, Lidický and Skrekovski [1] that if G is a triangle-free planar graph without 4-cycles adjacent to a 4- or 5-cycle, then G is 3-choosable. In this paper, we strengthen this result and show that such graphs are on-line 3-choosable.

2. The Proof

For a subset U of $V(G)$, let 1_U be the characteristic function of U and $f|_U$ be the restriction of f to U. Lemmas 3, 4, 5, 6 7 are proved in [3].

Lemma 3. *If G is on-line f-choosable and $g(x) \geq f(x)$ for all $x \in V(G)$, then G is on-line g-choosable.*

Lemma 4. *Let $A = \{x : f(x) > deg_G(x)\}$. If $G - A$ is on-line $f|_{G-A}$-choosable, then G is on-line f-choosable.*

Lemma 5. *Let u and v be two nonadjacent vertices. If G is on-line f-choosable, then $G+uv$ is on-line $(f+f(u)1_{\{v\}})$-choosable. If $f(u) = 1$ and $G+uv$ is on-line $(f+1_{\{v\}})$-choosable, then G is on-line f-choosable.*

Lemma 6. *Suppose $G = (V, E)$ is a graph and A is an independent set such that $f(v) = 1$ for all $v \in A$. Let $g : V(G) - A \to \mathbb{N}$ be defined as $g(x) = f(x) - |A \cap N_G(x)|$. Then G is on-line f-choosable if and only if $G - A$ is on-line g-choosable.*

Lemma 7. *Suppose G_1 and G_2 are on-line f_1 and f_2 choosable where $f_2(x) = 1$ for $\forall x \in V(G_1) \cap V(G_2)$. Let $g : V(G_1) \cup V(G_2) \to \mathbb{N}$ be defined as $g(x) = f_1(x)1_{V(G_1)} + f_2(x)1_{V(G_2)-V(G_1)}$. Then $G_1 \cup G_2$ is on-line g-choosable.*

Now we shall use these lemmas to prove the following result:

Theorem 8. *Assume G is a triangle-free plane graph in which no 4-cycle is adjacent to a 4-cycle or a 5-cycle. Let C be its outer face. Assume P is a path of length at most 3 such that $E(P) \subseteq E(C)$, $V(P) \neq V(C)$ and $f : V(G) \to \mathbb{N}$ satisfies the following conditions:*

1. *$f(v) = 3$ for all $v \in V(G) - V(C)$.*
2. *$2 \leqslant f(v) \leqslant 3$ for all $v \in V(C) - V(P)$.*

3. $f(v) = 1$ *for all* $v \in V(P)$.
4. *The set* $\{v : f(v) = 2\}$ *is an independent set in* G.
5. *Each vertex* v *with* $f(v) = 2$ *has at most one neighbor in* P.

Then the graph $G - E(P)$ *is on-line* f*-choosable.*

Theorem 8 has the following easy consequence.

Corollary 9. *Let* G *be a triangle-free plane graph in which no* 4*-cycle is adjacent to a* 4*-cycle or a* 5*-cycle, and its outer face is bounded by an induced cycle* C *of length at most* 7*. If* $f(x) = 1$ *for* $x \in V(C)$ *and* $f(x) = 3$ *for* $x \in V(G) - V(C)$*, then* $G - E(C)$ *is on-line* f*-choosable.*

Proof. Let u_1vu_2 be an arbitrary subpath of C and P be the subpath of C which contains the other vertices in C. Let g be the function defined as $g(v) = f(v) + 2$, $g(u_i) = f(u_i) + 1$ for $i = 1, 2$, and $g(x) = f(x)$ otherwise. The graph $G - P$ and the function g satisfy the assumptions of Theorem 8. It follows that $G - E(P)$ is on-line g-choosable. By applying Lemma 5, $G - E(C)$ is on-line f-choosable. ■

Proof of Theorem 8. Suppose $G - E(P)$ is a smallest counterexample, i.e., $G - E(P)$ has the minimum number of edges for which there is a mapping f satisfying the condition of Theorem 8 and $G - E(P)$ is not on-line f-choosable. We first derive several properties of this counterexample. By applying Lemma 4, each vertex v of G has degree at least $f(v)$. A cycle K in G is separating if $K \neq C$ and the interior of K contains at least one vertex. A chord of cycle K is an edge in G joining two non-consecutive vertices of K.

Lemma 10. *If* K *is a separating cycle in* G*, then the length of* K *is at least* 8.

Proof. Suppose the length of K at most 7. Since G is triangle-free and no 4-cycle is adjacent to a 5-cycle, K has no chord. Let G_1 be the subgraph of G drawn inside K and G_2 be the subgraph of G drawn outside K. By the minimality of $G - E(P)$, $G_2 - E(P)$ is on-line $f|_{G_2}$-choosable. By the minimality of $G - E(P)$ and Corollary 9, $G_1 - E(K)$ is on-line g-choosable where $g = f|_{G_1}$, except that $g(x) = 1$ for $x \in V(K)$. By applying Lemma 7, we conclude that $G - E(P)$ is on-line f-choosable, a contradiction. ■

Lemma 11. *The graph* G *is* 2*-connected.*

Proof. Suppose v is a cut vertex of G and G_1 and G_2 are nontrivial induced subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$. Assume $P \subseteq G_1$. By the minimality of $G - E(P)$ together with f, $G_1 - E(P)$ is on-line $f|_{G_1}$-choosable and G_2 is on-line g-choosable where $g = f|_{G_2}$ except that $g(v) = 1$. Assume $P \nsubseteq G_1$ and $P \nsubseteq G_2$. Then $v \in V(P)$. Let P_i be the subpath of P in

G_i for $i = 1, 2$. By the minimality of $G - E(P)$ together with f, $G_1 - E(P_1)$ is on-line $f|_{G_1}$-choosable and $G_2 - E(P_2)$ is on-line $f|_{G_2}$-choosable. By applying Lemma 7, $G - E(P)$ is on-line f-choosable. This is a contradiction. ■

By applying Lemma 11, C is a cycle.

Lemma 12. *The cycle C has no chord.*

Proof. Suppose $e = uv$ is a chord of C, separating G into two subgraphs G_1 and G_2 intersecting in e. Assume first that $P \subseteq G_1$. By the minimality of $G - E(P)$, $G_1 - E(P)$ is on-line $f|_{G_1}$-choosable and $G_2 - e$ is on-line g-choosable where $g = f|_{G_2}$ except that $g(u) = g(v) = 1$. By applying Lemma 7, $G - E(P)$ is on-line f-choosable. Assume $P \nsubseteq G_i$ for $i = 1, 2$. Let P_i be the subpath of P in G_i for $i = 1, 2$. Since G is triangle-free and has no 4-cycle adjacent to a 4- or 5-cycles, either G_1 or G_2 has three vertices in $C - P$. Assume G_2 has three vertices in $C - P$. By the minimality of $G - E(P)$ together with f, $G_1 - E(P_1)$ is on-line $f|_{G_1}$-choosable and $G_2 - (E(P_2) \cup \{e\})$ is on-line g-choosable where $g = f|_{G_2}$ except that $g(u) = g(v) = 1$. By applying Lemma 7, $G - E(P)$ is on-line f-choosable. This is a contradiction. ■

By applying the previous lemma, C is an induced cycle.

Lemma 13. $l(C) \geqslant 5$ *where $l(C)$ means the length of C.*

Proof. Assume $V(C) = \{p, q, r, s\}$ and $f(p) = 3$. We consider the on-line g-list colouring game on graph G, where $g(x) = 1$ for $x \in V(C)$ and $g(x) = 3$ for $x \in V(G) - V(C)$. Let $G^{'} = G - C$ and $h = g|_{G-p} - 1_{N_{G'}(p)}$. Then $(G - p) - \{qr, rs\}$ together with h satisfies the assumptions of Theorem 8. By the minimality of $G - E(P)$, $(G - p) - \{qr, rs\}$ is on-line h-choosable. By applying Lemmas 5 and 6, $G - E(P)$ is on-line f-choosable. This is a contradiction. ■

For $k \geq 2$, a k-chord of C is a path $Q = q_0 q_1 ... q_k$ of length k such that $V(C) \cap V(Q) = \{q_0, q_k\}$.

Lemma 14. *The graph G has no 2-chord uvw such that $f(u) = 2$ or $\{u, w\} \subseteq V(P)$.*

Proof. Assume $P' = uvw$ is such a 2-chord of C. Let G_1 and G_2 be the two subgraphs of G separated by P'. Without loss of generality, we may assume that $|V(G_1) \cap V(P)| \geqslant |V(G_2) \cap V(P)|$. Let P_i be the subpath of P in G_i for $i = 1, 2$. By the minimality of $G - E(P)$, $G_1 - E(P_1)$ is on-line $f|_{G_1}$-choosable and $G_2 - (E(P_2) \cup E(P'))$ is on-line g-choosable where $g = f|_{G_2}$ except that $g(u) = g(v) = g(w) = 1$. By applying Lemma 7, $G - E(P)$ is on-line f-choosable. This is a contradiction. ■

Lemma 15. *The graph G has no 3-chord $uvwx$ such that $u \notin V(P)$ and $f(x) = 2$.*

Proof. Assume $P' = uvwx$ is a 2-chord of C. Let G_1 and G_2 be the two subgraphs of G separated by P'. As $u, x \notin V(P)$, we may assume that $P \subseteq G_1$. By the minimality of $G - E(P)$, $G_1 - E(P)$ is on-line $f|_{G_1}$-choosable. Let $g = f|_{G_2}$ except that $g(u) = g(v) = g(w) = g(x) = 1$. Since $f(x) = 2$, all neighbours y of x in G_2 have $f(y) = 3$. As G has no 2-chord, we conclude that each vertex y with $g(y) = 2$ has at most one neighbour in P'. So $G_2 - E(P')$ together with g satisfy the conditions of Theorem 8. By the minimality of $G - E(P)$, $G_2 - E(P')$ is on-line g-choosable. By applying Lemma 7, $G - E(P)$ is on-line f-choosable. This is a contradiction. ■

By applying Lemma 13, $l(C) \geqslant 5$. By applying Lemma 7 if needed, we can assume that $|V(P)| = 4$. Suppose $P = p_1p_2p_3p_4$. Let $x_1x_2x_3x_4x_5$ be the part of the facial walk of C, where x_1 is the neighbour of p_4 on C different from p_3.

Lemma 16. *If $K = x_1x_2x_3v_1$ or $K = x_2x_3x_4v_1$ is a cycle, where $v_1 \notin V(C)$, then there are no edges between $V(C) - V(K)$ and $\{v_1\}$.*

Proof. Assume $K = x_1x_2x_3v_1$ (respectively, $K = x_2x_3x_4v_1$) is a cycle, where $v_1 \notin V(C)$. If v_1 is adjacent to y where $y \in V(C) - V(K)$, then let G_1 and G_2 be the two subgraphs of G separated by the path yv_1x_3 (respectively, by the path yv_1x_4), where G_1 contains x_2. Let $P_1 = P$ and $P_2 = yv_1x_3$ (respectively, $P_2 = yv_1x_4$). Let $g = f|_{G_2}$ except that $g(y) = g(v_1) = g(x_3) = 1$ (respectively, $g(y) = g(v_1) = g(x_4) = 1$). Note that the out-face of G_2 has length at least 5, because G has no 4-cycle adjacent to a 5-cycle. So $G_2 - E(P_2)$ and g satisfy the condition of Theorem 8. By the minimality of $G - E(P_1)$, $G_1 - E(P_1)$ is on-line $f|_{G_1}$-choosable, and $G_2 - E(P_2)$ is on-line g-choosable. By applying Lemma 7, $G - E(P)$ is on-line f-choosable. ■

Lemma 17. *If $K = x_1x_2x_3v_1v_2$ or $K = x_2x_3x_4v_1v_2$ is a cycle, where $v_1, v_2 \notin V(C)$, then there are at most one edge between $V(P)$ and $\{v_1, v_2\}$. In addition, there are no edges between $V(C) - (V(P) \cup V(K))$ and $\{v_1, v_2\}$.*

Proof. Assume $K = x_1x_2x_3v_1v_2$ (respectively, $K = x_2x_3x_4v_1v_2$) is a cycle, where $v_1, v_2 \notin V(C)$. By applying Lemma 14, G has no 2-chord connecting two vertices of P. So each of v_1, v_2 is adjacent to at most one vertex of P. Assume there are more than one edge between P and $\{v_1, v_2\}$. Then each of v_1, v_2 is adjacent to one vertex of P.

Let z be a vertex in $V(C) - (V(P) \cup V(K))$. If v_1 is adjacent to y where y is p_1 or z, then let G_1 and G_2 be the two subgraphs of G separated by the path yv_1x_3 (respectively, by the path yv_1x_4), where G_1 contains x_2. Let $P_1 = P$

and $P_2 = yv_1x_3$ (respectively, $P_2 = yv_1x_4$). Let $g = f|_{G_2}$ except that $g(y) = g(v_1) = g(x_3) = 1$ (respectively, $g(y) = g(v_1) = g(x_4) = 1$). Note that the out-face of G_2 has length at least 5, because G has no 4-cycle adjacent to a 5-cycle. So $G_2 - E(P_2)$ and g satisfy the condition of Theorem 8. By the minimality of $G - E(P_1)$, $G_1 - E(P_1)$ is on-line $f|_{G_1}$-choosable, and $G_2 - E(P_2)$ is on-line g-choosable. By applying Lemma 7, $G - E(P)$ is on-line f-choosable.

We know that v_2 is not adjacent to p_4, for otherwise G has a triangle or a 4-cycle adjacent to a 5-cycle. By the planarity, we must have v_1 adjacent to p_2 and v_2 adjacent to p_3. But then there is a 4-cycle adjacent to a 5-cycle, contrary to our assumption. ■

We choose a set $X_1 \subseteq \{x_1, x_2, x_3, x_4\}$ as follows:

Case 1. If $l(C) \leqslant 7$, then $X_1 = V(C) - \{p_1, p_2, p_3, p_4\}$.

Case 2. If $f(x_2) = f(x_4) = 2$ and $f(x_1) = f(x_3) = 3$, then $X_1 = \{x_1, x_2, x_3\}$.

Case 3. If $f(x_5) \leqslant 2$, $f(x_1) = f(x_3) = 2$ and $f(x_2) = f(x_4) = 3$, then $X_1 = \{x_1, x_2, x_3, x_4\}$.

Case 4. If $f(x_2) = f(x_3) = 3$, then $X_1 = \{x_1\}$.

Case 5. If $f(x_2) = 2$ and $f(x_1) = f(x_3) = f(x_4) = 3$, then $X_1 = \{x_1, x_2\}$.

Case 6. If $f(x_1) = f(x_3) = 2$ and $f(x_2) = f(x_4) = f(x_5) = 3$, then $X_1 = \{x_1, x_2, x_3\}$.

Let $X_2 \subseteq V(G) - V(C)$ be defined as follows: $z \in X_2$ if there exists a 2-chord xzy or 3-chord $xzvy$ such that $x, y \in X_1$ (by Lemma 10, the chord is unique).

First we consider Cases 1, 2, 3.

Let $m = \max\{i : x_i \in X_1\}$. Let $G' = G - E'$, where

$$E' = \{uv : u \in X_1 \cup X_2, v \in X_1 \cup X_2 \cup V(P)\} \cup \{x_m x_{m+1}\}.$$

Let $g = f$ except that $g(x) = 1$ for $x \in X_1 \cup X_2$.

We shall show that if $G' - E(P)$ is on-line g-choosable, then $G - E(P)$ is on-line f-choosable.

In Case 1, if $l(C) = 5$, then by our assumption (5), $f(x_1) = 3$. Apply Lemma 5 to G' and p_1x_1, then to $(G' + p_1x_1)$ and p_4x_1, we conclude that if $G' - E(P)$ is on-line g-choosable, then $G - E(P)$ is on-line f-choosable.

If $l(C) = 6$, then by our assumption (4), for some $i \in \{1, 2\}$, $f(x_i) = 3$. Without loss of generality, assume that $f(x_1) = 3$ and $f(x_2) \geq 2$. By Lemma 3, we may assume that $f(x_2) = 2$. Apply Lemma 5 to G' and p_4x_1, then to $(G' + p_4x_1)$ and x_2x_1, and finally to $(G' + p_4x_1 + x_2x_1)$ and p_1x_2, we conclude that if $G' - E(P)$ is on-line g-choosable, then $G - E(P)$ is on-line f-choosable. In Figure 1, the dotted edges are edges added to G'. If the dotted edge xy has an arrow pointed to y, it means that when this edge is added to G', the function value at vertex y increases according to Lemma 5. To apply Lemma 5, the added edges are added one by one. The order in which the edges are added does make a difference. However, in each of the graphs, it is easy to find an appropriate order to add these edges. We need to choose the order of adding the edges carefully.

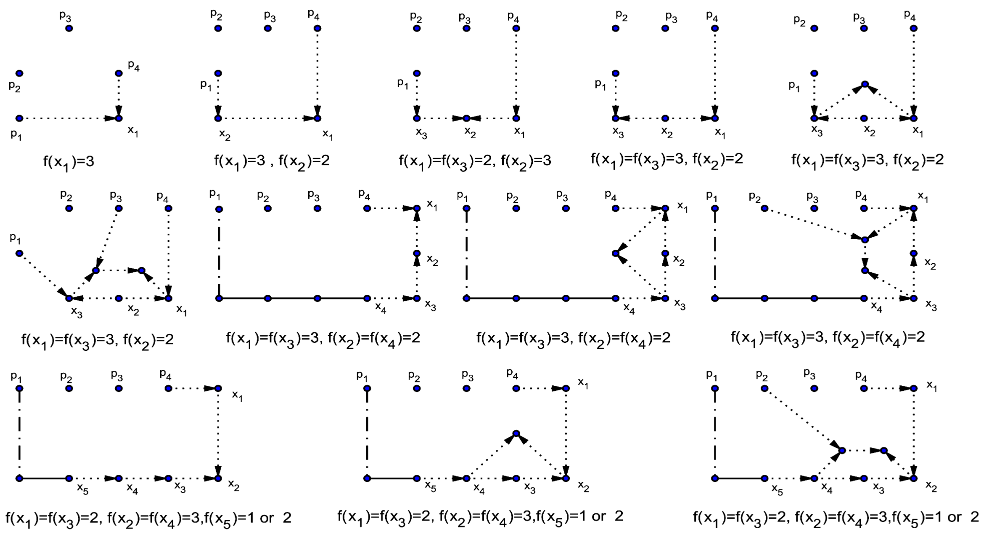

Fig. 1. For the proofs of Cases 1, 2, 3.

The case that $l(C) = 7$ is proved in the same way, see the 3nd, 4rd 5th and 6th graphs of Figure 1. Note that in the 5th graph X_2 contains one vertex, and it follows from Lemma 16 that there is no edge connecting X_2 and P. In the 5th graph, X_2 contains two vertices, and it follows from Lemma 17 that there is at most one edge connecting P and X_2. The 6th graph indicates one of the two possibilities connecting P and X_2 by an edge.

The proofs for Case 2 and 3 are also by applying Lemma 5 and are indicated by the 7th, 8th, 9th graphs, and the 10th, 11th and 12th graphs in Figure 1, respectively.

Let G'' be the graph obtained from G' by deleting vertices in the set $X_1 \cup X_2$. Let $h = f|_{G''} - 1_{N_{G'}(X_1 \cup X_2)}$. By Lemma 6, if $G'' - E(P)$ is on-line h-choosable, then $G' - E(P)$ is on-line g-choosable.

So it remains to prove that $G'' - E(P)$ is on-line h-choosable. By induction hypothesis, it suffices to check that $G'' - E(P)$ and h satisfy the conditions of Theorem 8. Let C' be the outer face of G''. As G has no 4-cycle adjacent to a 4- or 5-cycle, we conclude that no vertex in C' is adjacent two vertices in $X_1 \cup X_2$.

By Lemmas 16 and 17, there are no edges between $X_1 \cup X_2$ and $V(C) - X_1$. So, $h(x) \geq 2$ for all $x \in V(C') - V(P)$. Since there is no edge between $V(P)$ and $X_1 \cup X_2$ in G', $h(x) = 1$ for all $x \in V(P)$. Assume there exist two adjacent vertices u, v in C' with $h(u) = h(v) = 2$. Then $f(u) = 3$ or $f(v) = 3$. If $f(u) = f(v) = 3$, then there is a 4-cycle adjacent to 4- or 5-cycle, contrary to our assumption. Otherwise, there is a 2-chord or 3-chord containing u, v, contrary to Lemma 14 or 15. By applying Lemma 14, each vertex v with $h(v) = 2$ has at most one neighbor in P. So $G'' - E(P)$ and h indeed satisfy the condition of Theorem 8. This completes the proof of Cases 1, 2, 3.

For Cases 4, 5, 6, let G' be the graph obtained from G by deleting edges in the set $\{uv : u \in X_1, v \in X_1 \cup V(P)\}$. Let $g = f$ except that $g(x) = 1$ for $x \in X_1$.

By Lemma 5, if $G' - E(P)$ is on-line g-choosable, then $G - E(P)$ is on-line f-choosable (see Figure 2). Let G'' be the graph obtained from G' by deleting vertices in the set X_1. Let $h = f|_{G''} - 1_{N_{G'}(X_1)}$. By Lemma 6, if $G'' - E(P)$ is on-line h-choosable, then $G' - E(P)$ is on-line g-choosable. Similarly as in Cases 1,2,3, it is easy to verify that $G'' - E(P)$ and h satisfy the conditions of Theorem 8. So by the induction hypothesis, $G'' - E(P)$ is on-line h-choosable. This completes the proof of Theorem 8.

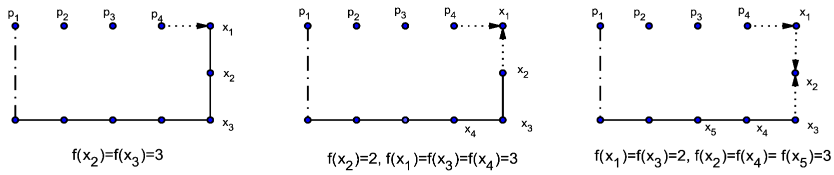

Fig. 2. For the proofs of Cases 4, 5, 6.

By Lemma 3, 4 and Theorem 8, we can get the following theorem.

Theorem 18. *If G is a triangle-free planar graph without 4-cycles adjacent to a 4- or 5-cycle, then G is on-line 3-choosable.*

ACKNOWLEDGMENTS

The author thank the referees for many valuable and constructive suggestions.

REFERENCES

1. Z. Dvořák, B. Lidický and R. Skrekovski, 3-Choosability of triangle-free planar graphs with constraints on 4-cycles, *SIAM J. Discrete Math.*, **24(3)** (2010), 934-945.
2. P. Erdős, A. L. Rubin and H. Taylor, Choosability in graphs, *Congr. Numer.*, **26** (1980), 125-157.

3. U. Schauz, Mr. Paint and Mrs. Correct, *Electron. J. Combin.*, Vol. 16(1), R77, 2009.

4. Z. Tuza, Graph colorings with local constraints - A survey, *Discuss. Math. Graph Theory*, **17** (1997), 161-228.

5. V. G. Vizing, Vertex colorings with given colors, *Metody Diskret. Analiz. Novosibirsk*, **29** (1976), 3-10.

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