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MULTIPLE SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS IN UNBOUNDED CYLINDER DOMAINS

Tsing-San Hsu* and Huei-Li Lin

Abstract. In this paper, we show that if $Q(x)$ satisfies some suitable conditions, then the quasilinear elliptic Dirichlet problem $-\Delta_p u + |u|^{p-2}u = Q(x)|u|^{q-2}u$ in an unbounded cylinder domain Ω has at least two solutions in which one is a positive ground state solution and the other is a nodal solution.

1. Introduction and Main Results

Throughout this article, let $x = (y, z)$ be the generic point of $\mathbb{R}^N$ with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, $N = m + n \geq 3$, $m \geq 0$, $n \geq 1$, $2 \leq p < N$ and $2 \leq p < q < p^* = Np/(N-p)$. In this paper, we concerned with the existence of solutions of the quasilinear elliptic equation:

$$(1.1) \qquad \begin{cases} -\Delta_p u + |u|^{p-2}u = Q(x)\,|u|^{q-2}\,u & \text{in } \Omega, \\ \quad u \in W_0^{1,p}(\Omega), \ u \not\equiv 0, \end{cases}$$

where $\omega \subseteq \mathbb{R}^m$ is a bounded smooth domain, $0 \in \Omega = \omega \times \mathbb{R}^n \subseteq \mathbb{R}^N$ is an unbounded cylinder domain, $\Delta_p u$ is the p-Laplacian operator, that is,

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}),$$

and $Q(x)$ is a positive, bounded and continuous function in $\overline{\Omega}$. Moreover, $Q(x)$ satisfies assumption $(A1)$ below.

$(A1)$ $\quad Q(x) \geq Q_\infty > 0$ in $\overline{\Omega}$, $Q(x) \not\equiv Q_\infty$ and

$$\lim_{|z|\to\infty} Q(x) = Q_\infty \text{ uniformly for } y \in \overline{\omega}.$$

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*Corresponding author.

For $p = 2$, It is well-known that Equation (1.1) has infinitely many solutions if Ω is bounded (see [19], and the references therein). Here, we only interest in unbounded domains. If $\Omega = \mathbb{R}^N$, the existence of solutions of Equation (1.1) has been investigated, among others, in [3, 4, 6, 7, 15-17, 25] (where general nonlinearities are considered). In [25], Zhu proved the multiplicity of the solutions of Equation (1.1) as follows. Assume $N \geq 5$ and $Q(x)$ satisfies the assumption $(A1)$, Equation (1.1) has a positive ground state solution. Moreover, if $Q(x)$ satisfies $Q(x) \geq Q_\infty + C|x|^{-\gamma}$ as $|x| \to \infty$, where $C, \gamma > 0$ are some constants, then Equation (1.1) has a nodal solution. Let us recall that, by a nodal solution we mean the solution of Equation (1.1) with change of sign.

More recently, Hsu [10] extended the results of Zhu [25] with $\Omega = \mathbb{R}^N$ to $\Omega = \omega \times \mathbb{R}^n$. In the present paper, motivated by [10] we extend the results of Hsu [10] with $p = 2$ to $2 \leq p < N$. When $Q(x) \equiv Q_\infty$ for all $x \in \mathbb{R}^N$, Li-Yan [14, Theorem 3.1] and Serrin-Tang [21, p.899] showed the existence of a positive ground state solution $w \in W^{1,p}(\mathbb{R}^N)$. In addition, w has the asymptotic behavior (see Lemma 3.5). In our article, we deal with Equation (1.1) in an unbounded cylinder domain for $2 \leq p < N$. First, we use the Global Compactness Lemma by Benci-Cerami [5] (or Alves-Carrião-Medeiros [1]) to obtain a positive "ground state solution". In order to prove that Equation (1.1) has an another solution which is nodal, we need to estimate the asymptotic behavior of solutions. For $p = 2$, any positive solution of Equation (1.1) has the exponential decay at infinity by the standard elliptic regularity theorem and the maximum principle. For $p > 2$, it is more difficulties to deal with that any positive solution of Equation (1.1) also has the asymptotic behavior in an unbounded cylinder domain (see section 3). We will apply the arguments in [9, 18, 20, 22, 24] to establish the asymptotic behavior of any positive solution of Equation (1.1) (see Lemma 3.3). To the best of our knowledge, the results of this paper are new for the case $2 < p < N$ and $\Omega = \omega \times \mathbb{R}^n$.

We now state the main results of this paper.

Theorem 1.1. *Suppose $N \geq 3$, $2 \leq p < N$ and $Q(x)$ satisfies assumption $(A1)$, then Equation (1.1) possesses a positive ground state solution in unbounded cylinder domains.*

Theorem 1.2. *Suppose $N \geq 3$, $2 \leq p < N$, $Q(x)$ satisfies assumption $(A1)$ and there exist positive constants $\delta < (\frac{1+\lambda_1}{p-1})^{1/p}$, C_0 and R_0 such that*

$$Q(x) \geq Q_\infty + C_0 \exp(-\delta\,|z|) \text{ for } |z| \geq R_0, \text{ uniformly for } y \in \overline{\omega},$$

where λ_1 is the first eigenvalue of the Dirichlet problem $-\Delta_p$ in ω. Then Equation (1.1) possesses a nodal solution in unbounded cylinder domains in addition to a positive solution.

Theorem 1.3. *Suppose* $\Omega = \mathbb{R}^N$, $N \geq 3$, $2 \leq p < N$ *and* $Q(x)$ *satisfies assumption* $(A1)$, *then Equation* (1.1) *possesses a positive ground state solution in* $\mathbb{R}^N$.

Theorem 1.4. *Suppose* $\Omega = \mathbb{R}^N$, $N \geq 3$, $2 \leq p < N$, $Q(x)$ *satisfies assumption* $(A1)$ *and there exist positive constants* $\delta < (\frac{1}{p-1})^{1/p}$, C_0 *and* R_0 *such that*

$$Q(x) \geq Q_\infty + C_0 \exp(-\delta\,|x|) \text{ for } |x| \geq R_0.$$

Then Equation (1.1) *possesses a nodal solution in* $\mathbb{R}^N$ *in addition to a positive solution.*

This paper is organized as follows. In section 2, we give preliminary results and a Global Compactness Lemma. In section 3, we establish some regularity lemmas and asymptotic behavior of the solution of Equation (1.1). In section 4, we prove the existence of a positive ground state solution. In section 5, we show the existence of another solution which is nodal.

2. Preliminaries

In this paper, we always assume that Ω is an unbounded cylinders or $\mathbb{R}^N$ $(N \geq 3)$ and C, C_0, C_1, $C_2, \ldots$ denote (possibly different) positive constants unless otherwise specified. Now we begin our discussion by giving some definitions and some known results. First we recall the definition of $W^{1,p}(\Omega)$,

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) | \partial_i u \in L^p(\Omega), i = 1, 2, \ldots, N\},$$
$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)},$$

where $\|\cdot\|_{L^p(\Omega)}$ denotes the norm in $L^p(\Omega)$. The space $W_0^{1,p}(\Omega)$ is the completion of the space $D(\Omega)$ of C^∞-functions with compact support with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. Associated with Equation (1.1), we consider the energy functionals a, b and J, for $u \in W_0^{1,p}(\Omega)$

$$a(u) = \int_\Omega (|\nabla u|^p + u^p)\, dx,$$
$$b(u) = \int_\Omega Q\,(x)\,|u|^q\, dx,$$
$$J(u) = \frac{1}{p} a(u) - \frac{1}{q} b(u).$$

Define

$$\alpha = \inf_{u \in \mathbf{M}(\Omega)} J(u),$$

where $\mathbf{M}(\Omega) = \left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} \mid a(u) = b(u) \right\}$. By Huang-Li [12, Theorem 2, 4], there is a positive ground state solution w of Equation (2.1)

$$(2.1) \qquad \begin{cases} -\Delta_p u + |u|^{p-2} u = Q_\infty |u|^{q-2} u & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

We also define

$$\begin{aligned} a^\infty(u) &= \int_\Omega (|\nabla u|^p + u^p)\, dx, \\ b^\infty(u) &= \int_\Omega Q_\infty |u|^q \, dx, \\ J^\infty(u) &= \frac{1}{p} a^\infty(u) - \frac{1}{q} b^\infty(u), \\ \alpha^\infty &= \inf_{u \in \mathbf{M}^\infty(\Omega)} J^\infty(u), \end{aligned}$$

where $\mathbf{M}^\infty(\Omega) = \left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} \mid a^\infty(u) = b^\infty(u) \right\}$.

We need the following definition and lemmas to prove the main theorems.

Definition 2.1. For $\beta \in \mathbb{R}$, a sequence $\{u_k\}$ is a (PS)$_\beta$–sequence in $W_0^{1,p}(\Omega)$ for J if $J(u_k) = \beta + o_k(1)$ and $J'(u_k) = o_k(1)$ strongly in $W^{-1,p'}(\Omega)$ as $k \to \infty$ where $W^{-1,p'}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$ and $1/p + 1/p' = 1$.

Lemma 2.2. *Let $\beta \in \mathbb{R}$ and let $\{u_k\}$ be a (PS)$_\beta$–sequence in $W_0^{1,p}(\Omega)$ for J, then $\{u_k\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Moreover,*

$$a(u_k) = b(u_k) + o_k(1) = \frac{qp}{q-p}\beta + o_k(1) \quad \textit{as } k \to \infty$$

and $\beta \geq 0$.

Proof. By $p \geq 2$, we have that

$$\sqrt[p]{a(u_k)} \leq 1 \text{ if } a(u_k) \leq 1 \text{ and } \sqrt[p]{a(u_k)} \leq \sqrt{a(u_k)} \text{ if } a(u_k) \geq 1.$$

For sufficiently large k, we have

$$\begin{aligned} |\beta| + 2 + \sqrt{a(u_k)} &\geq |\beta| + 1 + \sqrt[p]{a(u_k)} \\ &\geq J(u_k) - \frac{1}{q} \langle J'(u_k), u_k \rangle \\ &= \left(\frac{1}{p} - \frac{1}{q} \right) a(u_k). \end{aligned}$$

It follows that $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$. Since $\{u_k\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$, then $\langle J'(u_k), u_k \rangle = o_k(1)$ as $k \to \infty$. Thus,

$$\beta + o_k(1) = J(u_k) = \left(\frac{1}{p} - \frac{1}{q}\right) a(u_k) + o_k(1) = \left(\frac{1}{p} - \frac{1}{q}\right) b(u_k) + o_k(1),$$

that is, $a(u_k) = b(u_k) + o_k(1) = \frac{qp}{q-p}\beta + o_k(1)$ as $k \to \infty$ and $\beta \geq 0$. ■

Lemma 2.3. *(i) For each $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists a number $s_u > 0$ such that $s_u u \in \mathbf{M}(\Omega)$.*
(ii) Let $\{u_k\}$ be a (PS)$_\beta-$sequence in $W_0^{1,p}(\Omega)$ for J with $\beta > 0$. Then there is a sequence $\{s_k\}$ in $\mathbb{R}^+$ such that $\{s_k u_k\} \subset \mathbf{M}(\Omega)$, $s_k = 1 + o_k(1)$ and $J(s_k u_k) = \beta + o_k(1)$ as $k \to \infty$. In particular, the statement holds for J^∞.

Proof. (i) For $s \geq 0$ and $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, let

$$h_u(s) = J(su) = \frac{1}{p} a(u) s^p - \frac{1}{q} b(u) s^q.$$

Then $h'_u(s) = a(u)s^{p-1} - b(u)s^{q-1}$. Let $s_u = (\frac{a(u)}{b(u)})^{1/(q-p)}$, then $s_u > 0$ and $h'_u(s_u) = 0$, that is, $s_u u \in \mathbf{M}(\Omega)$.
(ii) By Lemma 2.2 and $\beta > 0$, we may assume $\{u_k\}$ is in $W_0^{1,p}(\Omega) \setminus \{0\}$ for all k. Thus, by (i) there exists a sequence $\{s_k\}$ in $\mathbb{R}^+$ such that $\{s_k u_k\} \subset \mathbf{M}(\Omega)$, that is, $s_k^p a(u_k) = s_k^q b(u_k)$ for each k. Since $a(u_k) = b(u_k) + o_k(1)$ and $J(u_k) = \beta + o_k(1)$ as $k \to \infty$, we have that $s_k = 1 + o_k(1)$ as $k \to \infty$. Hence, $J(s_k u_k) = \beta + o_k(1)$ as $k \to \infty$. ■

Lemma 2.4. *There exists a constant $c > 0$ such that $\|u\|_{W_0^{1,p}(\Omega)} \geq c > 0$ for each $u \in \mathbf{M}(\Omega)$, where c is independent of u.*

Proof. For each $u \in \mathbf{M}(\Omega)$, by the Sobolev inequality, we get

$$\|u\|_{W_0^{1,p}(\Omega)}^p = \int_\Omega Q(x)|u|^q dx \leq C_1 \|u\|_{W_0^{1,p}(\Omega)}^q.$$

This implies that $\|u\|_{W_0^{1,p}(\Omega)} \geq C_1{}^{-1/(q-p)} = c > 0$ for each $u \in \mathbf{M}(\Omega)$. ■

Remark 2.5. From the above lemma, we can easily deduce that there exists a constant $\mu_1 > 0$, independent of u, such that

$$\int_\Omega |u|^q dx > \mu_1 \text{ for each } u \in \mathbf{M}(\Omega).$$

Lemma 2.6. *Let $u \in \mathbf{M}(\Omega)$ satisfy $J(u) = \min_{v \in \mathbf{M}(\Omega)} J(v) = \alpha$. Then u is a nonzero solution of Equation (1.1) in Ω.*

Proof. We define $g(v) = a(v) - b(v)$ for $v \in W_0^{1,p}(\Omega) \setminus \{0\}$. Note that $\langle g'(u), u\rangle = (p-q)\, a(u) \neq 0$. Since the minimum of J is achieved at u and is constrained on $\mathbf{M}(\Omega)$, by the Lagrange multiplier theorem, there exists a number $\lambda \in \mathbb{R}$ such that $J'(u) = \lambda g'(u)$ in $W^{-1,p'}(\Omega)$. Then we have

$$0 = \left\langle J'(u), u\right\rangle = \lambda \left\langle g'(u), u\right\rangle.$$

Thus, $\lambda = 0$ and $J'(u) = 0$ in $W^{-1,p'}(\Omega)$. Therefore, u is a nonzero solution of Equation (1.1) in Ω such that $J(u) = \alpha$. ■

Lemma 2.7. *Let u be a sign-changing solution of Equation (1.1). Then $J(u) \geq 2\alpha$. In particular, the result holds for J^{∞}.*

Proof. Define $u^{+} = \max\{u, 0\}$ and $u^{-} = \max\{-u, 0\}$. Since u is a sign-changing solution of Equation (1.1), then u^{-} is nonnegative and nonzero. Multiply Equation (1.1) by u^{-} and integrate it to obtain

$$\int_{\Omega} (|\nabla u|^{p-2}\nabla u \nabla u^{-} + |u|^{p-2}uu^{-})dx = \int_{\Omega} Q(x)\,|u|^{q-2}\,uu^{-}dx,$$

that is, $u^{-} \in \mathbf{M}(\Omega)$ and $J(u^{-}) \geq \alpha$. Similarly, $J(u^{+}) \geq \alpha$. Hence,

$$J(u) = J(u^{+}) + J(u^{-}) \geq 2\alpha.$$

■

Lemma 2.8. *(Global Compactness Lemma) Let $\{u_k\}$ be a (PS)$_\beta$−sequence in $W_0^{1,p}(\Omega)$ for J and $u_0 \in W_0^{1,p}(\Omega)$ such that, $u_k \rightharpoonup u_0$ weakly in $W_0^{1,p}(\Omega)$. Then either*

(i) $u_k \to u_0$ strongly in $W_0^{1,p}(\Omega)$ or

(ii) there are a subsequence $\{u_k\}$, an integer $l \geq 0$, sequences $\{x_k^i\}_{k=1}^{\infty} \subseteq \mathbb{R}^N$ of the form $(0, z_k^i) \in \Omega$ with $|x_k^i| \to \infty$ as $k \to \infty$, functions and $w_i \neq 0$ in $W_0^{1,p}(\Omega)$ for $1 \leq i \leq l$ such that

$$\begin{aligned} -\Delta_p u_0 + |u_0|^{p-2}u_0 &= Q(x)\,|u_0|^{q-2}\,u_0 \text{ in } W^{-1,p'}(\Omega), \\ -\Delta_p w_i + |w_i|^{p-2}w_i &= Q_\infty\,|w_i|^{q-2}\,w_i \text{ in } W^{-1,p'}(\Omega), \\ u_k &= u_0 + \sum\nolimits_{i=1}^{l} w_i(\cdot - x_k^i) + o_k(1) \text{ strongly in } W_0^{1,p}(\Omega), \\ J(u_k) &= J(u_0) + \sum_{i=1}^{l} J^{\infty}(w_i) + o_k(1). \end{aligned}$$

Proof. The proof can be obtained by using the arguments in Alves [2], Benci-Cerami [5] or see Alves-Carrião-Medeiros [1, Lemma 3.3]. ■

3. Asymptotic Behavior

In this section, we will prove the $C^{1,\alpha}_{loc}$ regularity as well as the asymptotic behavior of the weak solutions of Equation (1.1).

Lemma 3.1. *Let* $1 < p < N$, $2 \leq q \leq p^*$ *and* $u \in W_0^{1,p}(\Omega)$ *be a weak solution of Equation* (1.1). *Then* $u \in L^s(\Omega)$ *for* $s \in [p, +\infty)$. *Moreover,* $u \in L^\infty(\Omega)$ *and decays uniformly to zero, as* $|x| \to \infty$.

Proof. The proof is based on the classical Moser's iteration scheme as it was adapted by Ôtani for the bounded domain case in [18, Theorem II]. Let S denote the Sobolev embedding constant defined by

$$\|v\|_{L^{p^*}(\Omega)} \leq S\|\nabla v\|_{L^p(\Omega)} \text{ for all } v \in W_0^{1,p}(\Omega). \tag{3.1}$$

Let $k \in \mathbb{N}$ and $L = \|Q\|_{L^\infty(\Omega)} S$. Then we introduce the sequences

$$\begin{aligned} &q_{k+1} = q_k^* p^*/p, \quad q_k^* = q_k - q + p, \quad q_1 = p^*, \\ &L_{k+1} = L^{p/q_k^*}(q_k - q + 1)^{-1/q_k^*}(q_k^*/p)^{p/q_k^*} L_k^{q_k/q_k^*}, \quad L_1 = \|u\|_{L^{p^*}(\Omega)}, \end{aligned} \tag{3.2}$$

We claim that, for every $k \in \mathbb{N}$, the following estimate is true:

$$\|u\|_{L^{q_k}(\Omega)} \leq L_k. \tag{3.3}$$

For $k = 1$, (3.3) is obvious. We suppose that (3.3) holds for some k. We define, for $n \in \mathbb{N}$, the C^1 real function ψ_n, as

$$\psi_n(t) = \begin{cases} t, & |t| \leq n, \\ n+1, & |t| \geq n+2, \end{cases} \qquad 0 \leq \psi_n'(t) \leq 1. \tag{3.4}$$

Setting $u_n = \psi_n(u)$ we obtain that $|u_n|^{l-2}u_n$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, for all $l \in [2, +\infty)$. Multiplying Equation (1.1) by $|u_n|^{q_k - q}u_n$ and integrating over Ω, we derive

$$\begin{aligned} &(q_k - q + 1)\int_\Omega |\nabla u|^p \psi_n'(u)|u_n|^{q_k - q} dx + \int_\Omega |u_n|^{q_k - q + 1}|u|^{p-1} dx \\ = &\int_\Omega Q(x)|u_n|^{q_k - q + 1}|u|^{q-1} dx. \end{aligned} \tag{3.5}$$

The definition of u_n implies that

$$\int_\Omega Q(x)|u_n|^{q_k - q + 1}|u|^{q-1} dx \leq \int_\Omega |Q(x)||u|^{q_k} dx \leq \|Q\|_{L^\infty(\Omega)}\|u\|^{q_k}_{L^{q_k}(\Omega)}. \tag{3.6}$$

On the other hand, from (3.1) and (3.4) it follows that:

$$
\begin{aligned}
&(q_k-q+1)\int_\Omega |\nabla u|^p \psi_n'(u)|u_n|^{q_k-q}dx\\
&\geq (q_k-q+1)\int_\Omega |\nabla u_n|^p|u_n|^{q_k-q}dx\\
&\geq (q_k-q+1)(p/q_k^*)^p\int_\Omega |\nabla(|u_n|^{q_k^*/p})|^p|u_n|^{q_k-q}dx\\
&\geq S^{-p}(q_k-q+1)(p/q_k^*)^p\||u_n|^{q_k^*/p}\|^p_{L^{p^*}(\Omega)}.
\end{aligned}
\tag{3.7}
$$

Hence from $(3.5)-(3.7)$ we deduce

$$
\|u_n\|^{q_k^*}_{L^{q_{k+1}}(\Omega)} = \||u_n|^{q_k^*/p}\|^p_{L^{p^*}(\Omega)} \leq \|Q\|_{L^\infty(\Omega)}S^p(q_k-q+1)^{-1}(q_k^*/p)^pL_k^{q_k},
$$

which implies that

$$
\|u_n\|^{q_k^*}_{L^{q_{k+1}}(\Omega)} \leq L_{k+1}.
$$

Let $n\to+\infty$ and by induction, we prove that (3.3) holds for any $k\in\mathbb{N}$.
Setting

$$
\zeta = p^*\log L(p^*-\min\{p^*(q-p)/(p^*-p),0\}).
$$

We get the following estimate:

$$
\begin{aligned}
L_k \leq\ &(p^*/p)^{k-1}L_1+\{\zeta((p^*/p)-1)\\
&+p^*\log(p^*/p)\}((p^*/p)^{k-1}-1)/((p^*/p)-1)^2.
\end{aligned}
$$

Then the solution of Equation (1.1) satisfies the following L^∞ estimate:

$$
\|u\|_{L^\infty(\Omega)} \leq \lim_{k\to+\infty}\|u\|_{L^{q_k}(\Omega)} \leq e^d,
\tag{3.8}
$$

where $d=[L_1+\{\zeta((p^*/p)-1)+p^*\log(p^*/p)\}/((p^*/p)-1)]/(p^*-(p^*/p))$. Since $u\in L^p(\Omega)\cap L^\infty(\Omega)$, using the interpolation inequality, we prove that $u\in L^s(\Omega)$ for all $s\in[p,\infty]$. By a similar argument used to prove Theorem 1 of Serrin [20] (see also Gilbarg-Trudinger [9, Theorem 8.17]), for any open ball $B_{2r}(x)\subset\Omega$ of radius $2r$ centered at $x\in\Omega$ and some constant $C(N,q_2)$, the function $u\in W_0^{1,p}(\Omega)$ such that

$$
-\Delta_p u \leq h(x)
$$

in the weak sense, satisfies the estimate

$$
\|u\|_{L^\infty(B_r(x))} \leq C\{\|u\|_{L^p(B_{2r}(x))}+\|h\|_{L^{q_2}(B_{2r}(x))}\}.
$$

Thus, for any solution of Equation (1.1) we have

$$\|u\|_{L^\infty(B_r(x))} \leq C\{\|u\|_{L^p(B_{2r}(x))} + \|u^{q-1}\|_{L^{q_2}(B_{2r}(x))}\}.$$

By the preceding results we know that $u^{q-1} \in L^{q_2}(\Omega)$. Hence, the decay of u follows. ■

For $r > 0$, we denote

$$B_r = \{(y, z) \in \Omega | |z| \leq r\}.$$

Then we have the following regularity lemma.

Lemma 3.2. *If $u \in W_0^{1,p}(\Omega)$ be a weak solution of Equation (1.1), then $u \in C_{loc}^{1,\alpha}(\overline{\Omega} \cap B_r)$, where $r > 0$ and $\alpha = \alpha(r) \in (0, 1)$.*

Proof. The proof is a direct consequence of Lemma 3.1 and the results of Tolksdorf [22]. ■

Finally, we are going to prove the exponential decay.

Lemma 3.3. *Let u be a positive solution of Equation (1.1) in an unbounded cylinder $\Omega = \omega \times \mathbb{R}^n \subseteq \mathbb{R}^{m+n}$, $m \geq 2$, $n \geq 1$ and ϕ be the first positive eigenfunction of the Dirichlet problem $-\Delta_p\phi = \lambda_1\phi^{p-1}$ in ω, then for any $\varepsilon > 0$, there exist constants $C_\varepsilon, \widetilde{C_\varepsilon} > 0$ such that for all $(y, z) \in \overline{\Omega}$,*

$$\widetilde{C_\varepsilon}\phi(y)\exp\left(-(\frac{1+\lambda_1+\varepsilon}{p-1})^{1/p}|z|\right) \leq u(x) \leq C_\varepsilon\phi(y)\exp\left(-(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p}|z|\right).$$

Proof. We divide the proof into the following steps:

Step 1. First, we claim that for any $\varepsilon > 0$ with $0 < \varepsilon < 1 + \lambda_1$, there exists a constant $C_\varepsilon > 0$ such that

$$u(x) \leq C_\varepsilon\phi(y)\exp\left(-(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p}|z|\right) \text{ for all } (y, z) \in \overline{\Omega}.$$

Without loss of generality, we may assume $\varepsilon < 1$, and let $\beta_\varepsilon = (\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p}$. Now given $\varepsilon > 0$, by Lemma 3.1, we may choose ρ large enough such that

$$Q(x)u^{q-1}(x) \leq \varepsilon u^{p-1}(x), \text{ for } |z| \geq \rho.$$

Consequently,

$$-\Delta_p u + (1-\varepsilon)u^{p-1} \leq Q(x)u^{q-1} - \varepsilon u^{p-1} \leq 0, \text{ for all } |z| \geq \rho.$$

Let $q = (q_y, q_z) \in \partial\Omega$, and B be a small ball in Ω such that $q \in \partial B$. Since $\phi(y) > 0$ for $x = (y, z) \in B$, $\phi(q_y) = 0$, $u(x) > 0$ for $x \in B$, $u(q) = 0$, by the

Hopf lemma (see [24, Theorem 5]), we have that $\frac{\partial\phi}{\partial y}(q_y) < 0$, $\frac{\partial u}{\partial \nu}(q) < 0$, where ν is the outward unit normal vector at (q_y, q_z). Thus

$$\lim_{(y,z)\to(q_y,q_z)} \frac{u(y,z)}{\phi(y)} = \frac{\frac{\partial u}{\partial\nu}(q_y,q_z)}{\frac{\partial\phi}{\partial y}(q_y)} > 0,$$

where $(y,z)\in\Omega$ and $(y,z)\to(q_y,q_z)$ normally. Note that $u(y,z)\phi^{-1}(y) > 0$ for $(y,z)\in\Omega$, thus

$$u(y,z)\phi^{-1}(y) > 0 \text{ for } (y,z)\in\overline{\Omega}.$$

Since $\phi(y)e^{-\beta_\varepsilon|z|}$ and $u(x)$ are $C^1\left(\overline{\Omega}\right)$, if set

$$C_\varepsilon = \sup_{(y,z)\in\overline{\Omega},|z|\le\rho} \left(u(y,z)\phi^{-1}(y)e^{\beta_\varepsilon|z|}\right),$$

then $C_\varepsilon > 0$ and

$$C_\varepsilon\phi(y)e^{-\beta_\varepsilon|z|} \ge u(y,z) \text{ for } (y,z)\in\overline{\Omega},\ \ |z|\le\rho.$$

Let $\Phi(y,z) = C_\varepsilon\phi(y)e^{-\beta_\varepsilon|z|}$, for $(y,z)\in\overline{\Omega}$. Then, for $(y,z)\in\overline{\Omega}$, $|z|\ge\rho$, we have

$$\begin{aligned} & -\Delta_p\Phi(y,z) + (1-\varepsilon)\Phi^{p-1}(y,z) \\ = \ & (1+\lambda_1-\varepsilon-(p-1)\beta_\varepsilon^p + \tfrac{n-1}{|z|}\beta_\varepsilon^{p-1})\Phi^{p-1}(y,z) \\ \ge \ & 0. \end{aligned}$$

Since $p > 1$, we have that the function $\zeta:\mathbb{R}^N\to\mathbb{R}$, $\zeta(x) = |x|^p$ is convex, thus

$$(|x_1|^{p-2}x_1 - |x_2|^{p-2}x_2)(x_1-x_2) \ge 0 \text{ for all } x_1, x_2\in\mathbb{R}^N.$$

We now take as a test function $\eta = \max\{u-\Phi, 0\}\in W^{1,p}(\Omega_\rho)$, where $\Omega_\rho = \{(y,z)\in\Omega||z|>\rho\}$. Hence, combining these estimates, we get

$$\begin{aligned} 0 \ &\ge \int_\Omega \Big((|\nabla u|^{p-2}\nabla u - |\nabla\Phi|^{p-2}\nabla\Phi)\eta + (1-\varepsilon)(u^{p-1}-\Phi^{p-1})\eta\Big)dx \\ &\ge (1-\varepsilon)\int_{\{x\in\Omega|u\ge\Phi\}} (u^{p-1}-\Phi^{p-1})(u-\Phi)dx \ge 0 \text{ for all } x\in\Omega_\rho. \end{aligned}$$

Therefore, the set $\{x=(y,z)\in\Omega||z|\ge\rho \text{ and } u(x)\ge\Phi(x)\}$ is empty. From this we can easily get this claim.

Step 2. Given $\varepsilon > 0$, let $\gamma_\varepsilon = (\frac{1+\lambda_1+\varepsilon}{p-1})^{1/p}$ and

$$g(z) = (n-1)\gamma_\varepsilon^{p-1}|z|^{-1} - \varepsilon.$$

We can choose $\rho_0 > 0$ such that $g(z)\le 0$ for $|z|\ge\rho_0$. As in step 1, if we set

$$\widetilde{C_\varepsilon} = \inf_{(y,z)\in\overline{\Omega},|z|\le\rho_0} \left(u(y,z)\phi^{-1}(y)e^{\gamma_\varepsilon|z|}\right),$$

then $\widetilde{C_\varepsilon} > 0$ and

$$\widetilde{C_\varepsilon}\phi(y)e^{-\gamma_\varepsilon|z|} \le u(y,z) \text{ for } (y,z) \in \overline{\Omega},\ |z| \le \rho_0.$$

Now, let $\Psi(y,z) = \widetilde{C_\varepsilon}\phi(y)e^{-\gamma_\varepsilon|z|}$, for $(y,z) \in \overline{\Omega}$. If $x = (y,z) \in \Omega$, $|z| \ge \rho_0$, we have

$$-\Delta_p u(x) + u^{p-1}(x) = Q(x)u^{q-1}(x) \ge 0, \text{ and}$$
$$-\Delta_p \Psi(x) + \Psi^{p-1}(x) = g(z)\Psi^{p-1}(x) \le 0.$$

Repeating the same arguments as in setp 1, we also obtain that

$$u(x) \ge \widetilde{C_\varepsilon}\phi(y)\exp\left(-(\frac{1+\lambda_1+\varepsilon}{p-1})^{1/p}|z|\right) \text{ for } (y,z) \in \overline{\Omega}.$$

This completes the proof. ■

Remark 3.4. In the case $Q(x) \equiv Q_\infty > 0$, we have that every positive solution of Equation (2.1) has the same asymptotic behavior as in Lemma 3.3.

By adopting the similar argument as in the above lemmas and Li-Yan [14, Theorem 3.1], we obtain the following asymptotic behavior result of the solutions of Equation (1.1) in $\mathbb{R}^N$ at infinity.

Lemma 3.5. *Any positive solution $w \in W^{1,p}(\mathbb{R}^N)$ of Equation (1.1) with $2 \le p < N$ has the following asymptotic behavior.*
(i) *$w \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for some $0 < \alpha < 1$ and $\lim_{|x|\to\infty} w(x) = 0$,*
(ii) *for any $\varepsilon > 0$, there exist constants C_1, $C_2 > 0$ such that*

$$C_1\exp\left(-(\frac{1+\varepsilon}{p-1})^{1/p}|x|\right) \le w(x) \le C_2\exp\left(-(\frac{1-\varepsilon}{p-1})^{1/p}|x|\right) \textit{ for all } x \in \mathbb{R}^N.$$

Remark 3.6. Using the same arguments as in the above lemma, we can get that any positive weak solution of Equation (2.1) in $\mathbb{R}^N$ also has the same asymptotic behavior at infinity in Lemma 3.5.

4. Existence of the Ground State Solution

Lemma 4.1. *If $\alpha < \alpha^\infty$, then α attains a minimizer u_0, that is, there exists a positive ground state solution u_0 of Equation (1.1).*

Proof. By Ekeland's vaiational principle [8] and the definition of α, there exists a minimizing sequence $\{u_k\} \subset \mathbf{M}(\Omega)$ such that

$$J(u_k) \to \alpha \text{ and } J'|_{\mathbf{M}(\Omega)}(u_k) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } k \to \infty$$

where $1/p + 1/p' = 1$.
Let us define $g(u) = a(u) - b(u)$ for all $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. Then we have

$$\mathbf{M}(\Omega) = \{u \in W_0^{1,p}(\Omega)\backslash\{0\} \mid g(u) = 0\}.$$

Thus there exists a sequence $\{\theta_k\} \subset \mathbb{R}$ such that

$$J'(u_k) = \theta_k g'(u_k) + o_k(1) \text{ as } k \to \infty.$$

Since $u_k \in \mathbf{M}(\Omega)$ we have

$$\langle J'(u_k), u_k\rangle = \theta_k \langle g'(u_k), u_k\rangle + \langle o_k(1), u_k\rangle = 0,$$

and

$$\langle g'(u_k), u_k\rangle = (p - q)a(u_k) \neq 0 \text{ for all } k \in \mathbb{N}.$$

Hence, $\theta_k \to 0$ as $k \to \infty$. This implies $J'(u_k) \to 0$ in $W^{-1,p'}(\Omega)$ as $k \to \infty$. Thus $\{u_k\}$ is a $(PS)_\alpha$ sequence for J. By Lemma 2.8 and $\alpha < \alpha^\infty$, we can obtain (by choosing a subsequence if necessary)

$$u_k \to u_0 \text{ strongly in } W_0^{1,p}(\Omega) \text{ as } k \to \infty.$$

Thus $J(u_0) = \alpha$ and by Lemma 2.6 we have that u_0 is a nonzero solution of Equation (1.1). By Lemma 2.7, u_0 has constant sign in Ω. Without loss of generality, we may assume that $u_0^- \equiv 0$. Thus $u_0 \geq 0$ in Ω. By Lemmas 3.1, 3.2 and the Harnack's inequality [23, Theorem 1.1], we can show that $u_0 \in L^\infty(\Omega) \cap C_{loc}^{1,\alpha}(\Omega)$ for some $0 < \alpha < 1$ and $u_0 > 0$ in Ω. This completes the proof. ■

Lemma 4.2. *If w is a positive ground state solution of Equation (2.1) and Q satisfies assumption $(A1)$, then we have*

$$\sup_{s \geq 0} J(sw) < \alpha^\infty.$$

Proof. Let $B_1 = \{(y, z) \in \Omega | |z| \leq 1\}$, then we have

$$J(sw) \leq \frac{s^p}{p} \int_\Omega (|\nabla w|^p + w^p)dx - C\frac{s^q}{q} \int_{B_1} w^q dx.$$

Therefore, there exists a number $s_1 > 0$ such that

$$J(sw) < 0 \text{ for } s \geq s_1.$$

Since J is continuous in $W_0^{1,p}(\Omega)$, then there exists a number $s_0 > 0$ such that

$$J(sw) < \alpha^\infty \text{ for } 0 \leq s < s_0.$$

Then we only need to prove

$$\sup_{s_0 \le s \le s_1} J(sw) < \alpha^\infty.$$

For $s_0 \le s \le s_1$, since Q satisfies assumption $(A1)$ and $\sup_{s\ge 0} J^\infty(sw) = J^\infty(w) = \alpha^\infty$, then

$$\begin{aligned} J(sw) &= \frac{s^p}{p}\int_\Omega (|\nabla w|^p + |w|^p)dx - \frac{s^q}{q}\int_\Omega Q(x)w^q dx \\ &\le J^\infty(sw) - \frac{s_0^q}{q}\int_\Omega (Q(x) - Q_\infty)w^q dx \\ &< \alpha^\infty. \end{aligned}$$

Hence, we have

$$\sup_{s\ge 0} J(sw) < \alpha^\infty. \qquad \blacksquare$$

Theorem 4.3. *Assume that Q satisfies the condition $(A1)$, then Equation (1.1) has a positive ground state solution v_1.*

Proof. By Lemma 2.3 (i), there exists a number $s_w > 0$ such that $s_w w \in \mathbf{M}(\Omega)$. From the definition of α, we get that $\alpha \le J(s_w w)$. Applying Lemma 4.2, we have $\alpha < \alpha^\infty$. Thus, by Lemma 4.1 there exists a positive ground state solution v_1 of Equation (1.1). ■

5. Existence of Nodal Solution

In this section, Q satisfies assumption $(A1)$, and the following assumption $(A2)$ below.

$(A2)$ there exist positive constants $\delta < (\frac{1+\lambda_1}{p-1})^{1/p}$, C_0 and R_0 such that

$$Q(x) \ge Q_\infty + C_0 \exp(-\delta\,|z|) \text{ for } |z| \ge R_0, \text{ uniformly for } y \in \overline{\omega}.$$

Recall that μ_1 is the same positive constant as in Remark 2.5 and

$$\mathbf{M}(\Omega) = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} \mid a(u) = b(u)\}.$$

We define

$$\mathbf{M}_0 = \left\{u \in W_0^{1,p}(\Omega) \mid u^\pm \in \mathbf{M}(\Omega)\right\},$$

$$\chi = \{u \in W_0^{1,p}(\Omega) \mid \int_\Omega |u^\pm|^q dx > \mu\} \text{ and } \mathcal{N} = \mathbf{M}(\Omega) \cap \chi,$$

where $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and $\mu \in (0, \frac{\mu_1}{4})$.

Lemma 5.1. (i) *If* $u \in W_0^{1,p}(\Omega)$ *changes sign, then there are positive numbers* $s^{\pm}(u) = s^{\pm}$ *such that* $s^+u^+ \pm s^-u^- \in \mathbf{M}(\Omega)$,
(ii) *There exists a constant* $c' > 0$ *such that* $\|u^{\pm}\|_{W_0^{1,p}(\Omega)} \geq c' > 0$ *for each* $u \in \mathcal{N}$.

Proof. (i) Since u^+ and u^- are nonzero, by Lemma 2.3 (i), it is easy to obtain the result.
(ii) For each $u \in \mathcal{N} = \mathbf{M}(\Omega) \cap \chi$, by the definition of χ and the Sobolev inequality we have

$$\mu < \int_\Omega |u^{\pm}|^q dx \leq C \left\|u^{\pm}\right\|^q_{W_0^{1,p}(\Omega)}.$$

This implies that $\|u^{\pm}\|_{W_0^{1,p}(\Omega)} \geq (\frac{\mu}{C})^{1/q} = c' > 0$ for each $u \in \mathcal{N}$. ∎

Define

$$\gamma = \inf_{u \in \overline{\mathcal{N}}} J(u).$$

Lemma 5.2. *There exists a sequence* $\{u_k\} \subset \overline{\mathcal{N}}$ *such that* $J(u_k) = \gamma + o_k(1)$ *and* $J'(u_k) = o_k(1)$ *strongly in* $W^{-1,p'}(\Omega)$ *as* $k \to \infty$.

Proof. See Alves-Carrião-Medeiros [1, Lemma 5.1]. ∎

Lemma 5.3. *Let* f *and* g *be real-valued functions in* Ω. *If* $g(x) > 0$ *in* Ω, *then one has the following inequalities.*
(i) $(f+g)^+ \geq f^+$, (ii) $(f+g)^- \leq f^-$, (iii) $(f-g)^+ \leq f^+$, (iv) $(f-g)^- \geq f^-$.

Lemma 5.4. *Let* $\{u_k\} \subset \overline{\mathcal{N}}$ *be a (PS)*$_\gamma-$*sequence in* $W_0^{1,p}(\Omega)$ *for* J *satisfying*

$$\alpha < \gamma < \alpha + \alpha^\infty \,(< 2\alpha^\infty).$$

Then there exists $v_2 \in \mathbf{M}_0$ *such that* u_k *converges to* v_2 *strongly in* $W_0^{1,p}(\Omega)$. *Moreover,* v_2 *is a higher energy solution of Equation* (1.1) *such that* $J(v_2) = \gamma$.

Proof. By the definition of the (PS)$_\gamma-$sequence in $W_0^{1,p}(\Omega)$ for J, it is easy to see that $\{u_k\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$ and satisfies

$$\int_\Omega (\left|\nabla u_k^{\pm}\right|^p + \left|u_k^{\pm}\right|^p) dx = \int_\Omega Q(x) \left|u_k^{\pm}\right|^q dx + o_k(1).$$

By Lemma 5.1 (ii), there exists a $C > 0$ such that

$$c' \leq \int_\Omega (\left|\nabla u_k^{\pm}\right|^p + \left|u_k^{\pm}\right|^p) dx = \int_\Omega Q(x) \left|u_k^{\pm}\right|^q dx + o_k(1).$$

Let v_2 be the weak limit of $\{u_k\}$ in $W_0^{1,p}(\Omega)$. By the Compactness Global Lemma 2.8, we have either $u_k \to v_2$ strongly in $W_0^{1,p}(\Omega)$ or $\gamma = J(v_2) + \sum_{i=1}^{l} J^\infty(w_i)$,

where v_2 is a solution of Equation (1.1) in Ω and w_i is a solution of Equation (2.1) in Ω. Since $J^{\infty}(w_i) \geq \alpha^{\infty}$ for each $i \in \mathbb{N}$ and $\alpha < \alpha^{\infty}$, we have $l \leq 1$. Now we want to show that $l = 0$. On the contrary, suppose that $l = 1$.
(i) w_1 is a sign-changing solution of Equation (2.1): by Lemma 2.7, we have $\gamma \geq 2\alpha^{\infty}$, which is a contradiction.
(ii) w_1 is a constant sign solution of Equation (2.1): we may assume $w_1 > 0$. By the Compactness Global Lemma 2.8, there exists a sequence $\left\{x_k^1\right\}$ in Ω such that $\left|x_k^1\right| \to \infty$, and

$$\left\|u_k - v_2 - w_1\left(\cdot - x_k^1\right)\right\|_{W_0^{1,p}(\Omega)} = o_k(1) \text{ as } k \to \infty.$$

By the Sobolev continuous embedding inequality, we obtain

$$\left\|u_k - v_2 - w_1\left(\cdot - x_k^1\right)\right\|_{L^q(\Omega)} = o_k(1) \text{ as } k \to \infty.$$

Since $w_1 > 0$, by Lemma 5.3, then

$$\left\|(u_k - v_2)^-\right\|_{L^q(\Omega)} = o_k(1) \text{ as } k \to \infty.$$

Suppose $v_2 \equiv 0$, we obtain $\left\|u_k^-\right\|_{L^q(\Omega)} = o_k(1)$ as $k \to \infty$. Then

$$0 < c' \leq \int_{\Omega} Q(x)\left|u_k^-\right|^q dx = o_k(1),$$

which is a contradiction. Hence, $v_2 \not\equiv 0$. We have $\gamma = J(v_2) + J^{\infty}(w_1) \geq \alpha + \alpha^{\infty}$, which is a contradiction.
By (i) and (ii), then $l = 0$. Thus, $\|u_k - v_2\|_{W_0^{1,p}(\Omega)} = o_k(1)$ as $k \to \infty$ and $J(v_2) = \gamma$. Similarly, by Lemma 5.3, we obtain that v_2 is a sign-changing solution of Equation (1.1) in Ω. By Lemma 2.7, $2\alpha < \gamma$. ■

Recall that w is the positive ground state solution of Equation (2.1) in Ω. Let $w_k(x) = w(x+e_k)$, where $e_k = (0, 0, ...0, k) \in \mathbb{R}^N$ and denote $\tilde{e_k} = (0, 0, ...0, k) \in \mathbb{R}^n$ and $B_R = \{(y, z) \in \Omega | |z| \leq R\}$ for $R > 0$. Then we have the following results.

Lemma 5.5. *There are $k_0 \in \mathbb{N}$, real numbers t_1^* and t_2^* such that for $k \geq k_0$*

$$t_1^* v_1 - t_2^* w_k \in \mathbf{M}_0 \textit{ and } \gamma \leq J(t_1^* v_1 - t_2^* w_k),$$

where $\frac{1}{p} \leq t_1^*, t_2^* \leq p$.

Proof. See Alves-Carrião-Medeiros [1, Proposition 5.1]. ■

Lemma 5.6. *For all v, $w \in \mathbb{R}^N$ with $N \geq 1$ and $p \geq 2$, we have*

$$(|v|^{p-2}v - |w|^{p-2}w)(v - w) \geq |v - w|^p.$$

Proof. See Jianfu [13, Lemma 4.2]. ∎

Lemma 5.7. *Let Θ be a domain in $\mathbb{R}^n$. If $f : \Theta \to \mathbb{R}$ satisfies*

$$\int_{\Theta} |f(x) \exp(\sigma |x|)| \, dx < \infty \text{ for some } \sigma > 0,$$

then

$$\left(\int_{\Theta} f(x) \exp(-\sigma |x + \tilde{e_k}|) \, dx \right) \exp(\sigma k)$$
$$= \int_{\Theta} f(x) \exp(-\sigma x_n) \, dx + o_k(1) \text{ as } k \to \infty,$$

or

$$\left(\int_{\Theta} f(x) \exp(-\sigma |x - \tilde{e_k}|) \, dx \right) \exp(\sigma k)$$
$$= \int_{\Theta} f(x) \exp(\sigma x_n) \, dx + o_k(1) \text{ as } k \to \infty.$$

Proof. We know $\sigma |\tilde{e_k}| \leq \sigma |x| + \sigma |x + \tilde{e_k}|$, then

$$|f(x) \exp(-\sigma |x + \tilde{e_k}|) \exp(\sigma |\tilde{e_k}|)| \leq |f(x) \exp(\sigma |x|)| \, .$$

Since $-\sigma |x + \tilde{e_k}| + \sigma |\tilde{e_k}| = -\sigma \frac{\langle x, \tilde{e_k} \rangle}{|\tilde{e_k}|} + o_k(1) = -\sigma x_n + o_k(1)$ as $k \to \infty$, the lemma follows from the Lebesgue dominated convergence theorem. ∎

Lemma 5.8. *There exists a $k_0^* \in \mathbb{N}$ such that for $k \geq k_0^* \geq k_0$*

$$\gamma \leq \sup_{\frac{1}{p} \leq t_1, t_2 \leq p} J(t_1 v_1 - t_2 w_k) < \alpha + \alpha^{\infty},$$

where v_1 is a ground state solution of Equation (1.1) in Ω.

Proof. Since

$$J(t_1 v_1 - t_2 w_k) = \frac{1}{p} a(t_1 v_1 + t_2 w_k) - \frac{1}{q} b(t_1 v_1 - t_2 w_k).$$

holds and by Lemma 5.6 and using the inequality

$$\left(s - t\right)^q \geq s^q + t^q - C_1 (s^{q-1} t + s t^{q-1}\right) ,$$

for any s, $t > 0$ and some positive constant C_1, then we get

$$J(t_1 v_1 - t_2 w_k) \leq I_1 + I_2 - I_3,$$

where

$$I_1 = \frac{1}{p}\int_\Omega (|\nabla(t_1v_1)|^{p-2}\nabla(t_1v_1) - |\nabla(t_2w_k)|^{p-2}\nabla(t_2w_k))(\nabla(t_1v_1) - \nabla(t_2w_k))dx,$$

$$I_2 = \frac{1}{p}\int_\Omega (|t_1v_1|^{p-2}t_1v_1 - (t_2w_k)^{p-2}(t_2w_k))(t_1v_1 - t_2w_k)dx,$$

and

$$I_3 = \frac{1}{q}b(t_1v_1) + \frac{1}{q}b(t_2w_k) - C_1\int_\Omega ((t_1v_1)^{q-1}(t_2w_k) + t_1v_1(t_2w_k)^{q-1})dx.$$

Since v_1 is a positive solution of Equation (1.1) in Ω and w_k is related with a positive ground state of Equation (2.1), we have

$$\begin{aligned}\sup_{\frac{1}{p}\le t_1,t_2\le p} J(t_1v_1 - t_2w_k) \le & \sup_{t_1\ge 0} J(t_1v_1) + \sup_{t_2\ge 0} J^\infty(t_2w) \\ & - \frac{1}{p^q q}\int_\Omega (Q(x) - Q_\infty)\, w_k^q dx \\ & + C_2\left(\int_\Omega v_1^{q-1}w_k + w_k^{q-1}v_1\right)dx.\end{aligned}$$

(i) First, by the Hölder inequality and applying Lemma 3.3,

$$\begin{aligned}\int_{B_{R_0}} v_1^{q-1}w_k dx \le & \left(\int_{B_{R_0}} v_1^q dx\right)^{\frac{q-1}{q}}\left(\int_{B_{R_0}} w_k^q dx\right)^{\frac{1}{q}} \\ \le & C_3\left(\int_\omega\int_{\{z||z|\le R_0\}} \phi^q(y)\exp\left(-q(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p}|z+\tilde{e_k}|\right)dydz\right)^{\frac{1}{q}} \\ \le & C_4\exp\left((\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p}k\right).\end{aligned}$$

Applying Lemma 5.7, there exists a $k_1 \ge k_0$ such that for $k \ge k_1$

$$\begin{aligned}&\int_{\Omega\setminus R_0} v_1^{q-1}w_k dx \\ \le & C_5\int_{\{z||z|\ge R_0\}} \exp\left(-(q-1)\,(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p}|z|\right) \\ & \exp\left(-(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p}|z+\tilde{e_k}|\right)dz \\ \le & C_6\exp(-(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p}k).\end{aligned}$$

Similarly, we also obtain

$$\int_{B_{R_0}} w_k^{q-1} v_1 dx \leq C_7 \exp(-(q-1)(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p} k),$$

$$\int_{B_{R_0}} |Q(x)-Q_\infty| w_k^q dx \leq C_8 \exp(-q(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p} k),$$

and there exists a $k_2 \geq k_1$ such that for $k \geq k_2$

$$\int_{\Omega \setminus R_0} w_k^{q-1} v_1 dx \leq C_9 \exp(-(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p} k).$$

(ii) Since Q satisfies assumption $(A2)$ and $0 < \delta < (\frac{1+\lambda_1}{p-1})^{1/p}$, by Lemma 5.7, there exists a $k_3 \geq k_2$ such that for $k \geq k_3$

$$\int_{\Omega \setminus B_{R_0}} (Q(x)-Q_\infty) w_k^q \geq C_{10} \exp(-\delta k).$$

By (i), (ii) and $2 \leq p < q < p^*$, choosing $\varepsilon > 0$, such that $(\frac{1+\lambda_1-\varepsilon}{p-1})^{1/p} > \delta$, we can find a $k_0^* \geq k_3 \geq k_0$ such that for $k \geq k_0^*$

$$C_2 \int_\Omega \left(v_1^{q-1} w_k + w_k^{q-1} v_1 \right) dx - \frac{1}{p^q q} \int_\Omega (Q(x)-Q_\infty) w_k^q dx < 0.$$

Since $J(v_1) = \sup_{t \geq 0} J(tv_1)$ and $J^\infty(w) = \sup_{t \geq 0} J^\infty(tw)$, we have for $k \geq k_0^*$

$$\sup_{\frac{1}{p} \leq t_1, t_2 \leq p} J(t_1 v_1 - t_2 w_k) < J(v_1) + J^\infty(w) = \alpha + \alpha^\infty. \qquad \blacksquare$$

Now, we begin to show the proof of our main results

First, we consider that Ω is an unbounded cylinder domain. Theorem 1.1 follows from Theorem 4.3. Theorem 1.2 follows immediately from Lemmas 5.2, 5.4, 5.5, 5.8, and Theorem 1.1. With the same argument, we also have that Theorem 1.3 and Theorem 1.4 hold for $\Omega = \mathbb{R}^N$.

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Tsing-San Hsu and Huei-Li Lin
Center for General Education
Chang Gung University
Kwei-San, Tao-Yuan 333, Taiwan
E-mail: tshsu@mail.cgu.edu.tw