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SOME CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH CONIC REGIONS

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Abstract. The purpose of the present paper is to introduce and investigate the function classes k-$\mathcal{SP}(\alpha, \beta)$ and k-$\mathcal{UCV}(\alpha, \beta)$ of analytic functions associated with conic regions in the open unit disk $\mathbb{U}$, which generalize the function classes defined and studied in a series of earlier papers by Kanas *et al.* [11, 12, 13, 14]. In particular, we consider the extremal problems for each of the above-mentioned function classes. The Fekete-Szegö problem is also considered for functions in the class k-$\mathcal{SP}(\alpha, \beta)$. Moreover, we investigate some mapping properties for each of the function classes k-$\mathcal{SP}(\alpha, \beta)$ and k-$\mathcal{UCV}(\alpha, \beta)$.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions of the form:

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

We denote by $\mathcal{S}$ the class of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{U}$. Let us recall the following definitions of the familiar classes of k-uniformly convex functions and k-starlike functions as follows:

$$k\text{-}\mathcal{UCV} := \left\{ f : f \in \mathcal{S} \ \text{ and } \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U};\ k \geqq 0) \right\}$$

and

$$k\text{-}\mathcal{ST} := \left\{ f : f \in \mathcal{S} \ \text{ and } \ \Re\left(\frac{zf'(z)}{f(z)}\right) > k \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathbb{U};\ k \geqq 0) \right\}.$$

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The function classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ were introduced and investigated by Kanas and Wiśniowska in [13] and [14], respectively (see also [11]). For a fixed $k \geqq 0$, the class k-$\mathcal{UCV}$ is defined purely geometrically as a subclass of univalent functions which map the intersection of $\mathbb{U}$ with any disk centered at the point $z = \zeta \;\; (|\zeta| < k)$ onto a convex domain. The notion of k-uniform convexity is a natural extension of that of the classical convexity. We observe that, if $k = 0$, then the center ζ is the origin and the class k-$\mathcal{UCV}$ reduces to the class $\mathcal{CV}$, which is the well-known class of all convex functions in $\mathbb{U}$, which are *normalized* as in (1). Moreover, for $k = 1$, it coincides with the class $\mathcal{UCV}$ of uniformly convex functions introduced by Goodman [4] and studied extensively by Rønning [24] (and *independently* by Ma and Minda [19, 20]). The class k-$\mathcal{ST}$ is related to the class k-$\mathcal{UCV}$ by means of the well-known Alexander transformation between the usual classes of convex and starlike functions (see also [12-14 19, 24]). Some more interesting developments involving the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ were presented by Lecko and Wiśniowska [17], Kanas [6-10] and others [1, 2, 22, 23, 25] (see also [3], [26] and [27]).

We now introduce the subclasses k-$\mathcal{UCV}(\alpha, \beta)$ and k-$\mathcal{ST}(\alpha, \beta)$ of the univalent function class $\mathcal{S}$ as follows.

Definition 1. Let α, β and k be nonnegative real numbers satisfying the following inequalities:

$$0 \leqq \beta < \alpha \leqq 1 \quad \text{and} \quad k(1-\alpha) < 1-\beta.$$

Then a function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{ST}(\alpha, \beta)$ if it satisfies the condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) - \beta > k\left|\frac{zf'(z)}{f(z)} - \alpha\right| \qquad (z \in \mathbb{U}). \tag{2}$$

Definition 2. Let α, β and k be nonnegative real numbers satisfying the following inequalities:

$$0 \leqq \beta < \alpha \leqq 1 \quad \text{and} \quad k(1-\alpha) < 1-\beta.$$

Then a function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{UCV}(\alpha, \beta)$ if it satisfies the condition:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) - \beta > k\left|1 + \frac{zf''(z)}{f'(z)} - \alpha\right| \qquad (z \in \mathbb{U}). \tag{3}$$

In particular, we note that the classes k-$\mathcal{UCV}(1, 0)$ and k-$\mathcal{ST}(1, 0)$ coincide with the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$, respectively. Furthermore, the classes k-$\mathcal{UCV}(1, \beta)$

and k-$\mathcal{ST}(1, \beta)$ are the same as the classes studied by Nishiwaki *et al.* [23] and Shams *et al.* [25].

In the present paper, we find the explicit form of functions in each of the general classes k-$\mathcal{UCV}(\alpha, \beta)$ and k-$\mathcal{ST}(\alpha, \beta)$. We also consider the Fekete-Szegö problems for the class k-$\mathcal{ST}(\alpha, \beta)$. Moreover, we investigate some other interesting properties and characteristics of the classes k-$\mathcal{UCV}(\alpha, \beta)$ and k-$\mathcal{ST}(\alpha, \beta)$.

2. Extremal Functions in the Class $\mathcal{P}(p_{k,\alpha,\beta})$

Let us consider the function $p(z)$ given by

$$p(z) = \frac{zf'(z)}{f(z)} \qquad \text{or} \qquad p(z) = 1 + \frac{zf''(z)}{f'(z)}.$$

We can thus rewrite the conditions (2) or (3) in the form:

$$k|p(z) - \alpha| < \Re\{p(z)\} - \beta \qquad (z \in \mathbb{U}).$$

It follows that the range of the function $p(z)$ is a conic domain given by

$$\Omega_{k,\alpha,\beta} = \{w : w \in \mathbb{C} \quad \text{and} \quad k|w - \alpha| < \Re(w) - \beta\}$$

or

$$\Omega_{k,\alpha,\beta} = \left\{w : w \in \mathbb{C} \quad \text{and} \quad k\sqrt{[\Re(w) - \alpha]^2 + [\Im(w)]^2} < \Re(w) - \beta\right\},$$

where

$$0 \leqq \beta < \alpha \leqq 1 \quad \text{and} \quad k(1 - \alpha) < 1 - \beta.$$

We note that the conic domain $\Omega_{k,\alpha,\beta}$ is such that $1 \in \Omega_{k,\alpha,\beta}$ and that its boundary $\partial\Omega_{k,\alpha,\beta}$ is a curve defined by

$$\partial\Omega_{k,\alpha,\beta} := \left\{w : w = u + iv \quad \text{and} \quad k^2(u - \alpha)^2 + k^2v^2 = (u - \beta)^2\right\}.$$

In what follows, we need the principle of subordination between analytic functions.

Definition 3. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic (Schwarz) function $\mathfrak{w}(z)$ in $\mathbb{U}$, satisfying the following conditions:

$$\mathfrak{w}(0) = 0 \qquad \text{and} \qquad |\mathfrak{w}(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\mathfrak{w}(z)\big).$$

We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We denote by $\mathcal{P}$ the family of *normalized* analytic and Carathéodory functions. Furthermore, by $p_{k,\alpha,\beta} \in \mathcal{P}$ we denote the function such that

$$p_{k,\alpha,\beta}(\mathbb{U}) = \Omega_{k,\alpha,\beta}.$$

We denote also by $\mathcal{P}(p_{k,\alpha,\beta})$ the following function class:

$$\begin{aligned}\mathcal{P}(p_{k,\alpha,\beta}) &:= \{p : p \in \mathcal{P} \quad \text{and} \quad p(\mathbb{U}) \subset \Omega_{k,\alpha,\beta}\} \\ &= \{p : p \in \mathcal{P} \quad \text{and} \quad p(z) \prec p_{k,\alpha,\beta}(z) \qquad (z \in \mathbb{U})\}.\end{aligned}$$

We thus find the following equivalence conditions for the function classes k-$\mathcal{SP}$ and k-$\mathcal{UCV}$:

$$f \in k\text{-}\mathcal{SP}(\alpha, \beta) \iff \frac{zf'(z)}{f(z)} \prec p_{k,\alpha,\beta}(z) \qquad (z \in \mathbb{U})$$

and

$$f \in k\text{-}\mathcal{UCV}(\alpha, \beta) \iff 1 + \frac{zf''(z)}{f'(z)} \prec p_{k,\alpha,\beta}(z) \qquad (z \in \mathbb{U}),$$

respectively.

We shall specify the functions $p_{k,\alpha,\beta}$, which are extremal for the class $\mathcal{P}(p_{k,\alpha,\beta})$ for nonnegative real numbers k, α and β. First of all, we take the function $p_{k,\alpha,\beta}$ for $k = 0$ as follows:

$$p_{0,\alpha,\beta}(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \qquad (z \in \mathbb{U})$$

Now, for $k = 1$, we shall give an explicit form of the function $p_{k,\alpha,\beta}$ which maps $\mathbb{U}$ onto the parabolic region:

$$\Omega_{1,\alpha,\beta} = \left\{w : w = u + iv \quad \text{and} \quad v^2 < (\alpha - \beta)(2u - (\alpha + \beta))\right\}.$$

It is well known from [16] that

$$w_1(z) = -\tan^2\left(\frac{\pi}{2\sqrt{2(\alpha - \beta)}}\sqrt{z}\right)$$

maps

$$\mathbb{D} := \{w : w = u + iv \quad \text{and} \quad v^2 < (\alpha - \beta)(\alpha - \beta - 2u)\}$$

conformally onto the open unit disk $\mathbb{U}$. And the mapping

$$w_2(w_1) = \alpha - w_1$$

maps the domain $\Omega_{1,\alpha,\beta}$ onto the above-defined domain $\mathbb{D}$. By composing the mappings, we can obtain the following transformation:

$$w(z) = -\tan^2\left(\frac{\pi}{2\sqrt{2(\alpha-\beta)}}\sqrt{\alpha - z}\right),$$

which maps $\Omega_{1,\alpha,\beta}$ onto the open unit disk $\mathbb{U}$. If we let $q(z)$ be the inverse function of the function $w(z)$, then we find that

$$q(z) = \alpha + \frac{2(\alpha-\beta)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2.$$

In order to obtain $p_{1,\alpha,\beta}(z)$ which satisfies the following normalization condition:

$$p_{1,\alpha,\beta}(0) = 1,$$

we have to solve the equation

$$p_{1,\alpha,\beta}(z) = q(u_1(z)) = \alpha + \frac{2(\alpha-\beta)}{\pi^2}\left(\log\frac{1+\sqrt{u_1(0)}}{1-\sqrt{u_1(0)}}\right)^2 = 1,$$

where u_1 is the Möbius transformation of the unit disk $\mathbb{U}$ onto itself. Solving the above equation, we get

$$u_1(z) = \frac{z+\rho_1}{1+\rho_1 z},$$

where, for convenience,

$$\rho_1 = \left(\frac{e^A - 1}{e^A + 1}\right)^2 \quad \text{and} \quad A = \sqrt{\frac{1-\alpha}{2(\alpha-\beta)}}\,\pi.$$

Clearly, $u_1(z)$ provides the required self-mapping of open unit disk $\mathbb{U}$. Consequently,

$$p_{1,\alpha,\beta}(z) = \alpha + \frac{2(\alpha-\beta)}{\pi^2}\left(\log\frac{1+\sqrt{u_1(z)}}{1-\sqrt{u_1(z)}}\right)^2$$

is the desired mapping of the unit disk $\mathbb{U}$ onto the parabolic region $\Omega_{1,\alpha,\beta}$, with the following normalization:

$$p_{1,\alpha,\beta}(0) = 1.$$

Next, for the case when $0 < k < 1$, we shall give an explicit form of the function which maps the unit disk $\mathbb{U}$ onto the hyperbolic region:

$$\Omega_{k,\alpha,\beta} = \left\{ w : w = u+iv \text{ and } \frac{\left(u+\frac{\alpha k^2-\beta}{1-k^2}\right)^2}{\left(\frac{k^2(\alpha-\beta)^2}{(1-k^2)^2}\right)} - \frac{v^2}{\left(\frac{(\alpha-\beta)^2}{1-k^2}\right)} > 1 \quad (u>0) \right\}. \tag{4}$$

It is known from [13] that the transformation:

$$w_1(z) = \frac{1}{2}\left[\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{\mathfrak{A}(k)} + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\mathfrak{A}(k)}\right] \qquad (z \in \mathbb{U}),$$

where

$$\mathfrak{A}(k) = \frac{2}{\pi} \operatorname{Arccos} k, \tag{5}$$

maps $\mathbb{U}$ onto the domain $\mathbb{G}$ which is the interior of the right branch of the hyperbola whose vertex is at the point k. We next observe that the mapping

$$w_2(z) = \frac{\alpha-\beta}{1-k^2} z + \frac{\beta-\alpha k^2}{1-k^2}$$

transforms the domain $\mathbb{G}$ onto the domain $\Omega_{k,\alpha,\beta}$ given by (4). Hence the function $h(z)$ defined by

$$\begin{aligned} h(z) &:= (w_2 \circ w_1)(z) \\ &= \frac{\alpha-\beta}{2(1-k^2)}\left[\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{\mathfrak{A}(k)} + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\mathfrak{A}(k)}\right] + \frac{\beta-\alpha k^2}{1-k^2} \\ &= \frac{\alpha-\beta}{1-k^2} \cosh\left(\mathfrak{A}(k) \log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right) + \frac{\beta-\alpha k^2}{1-k^2} \end{aligned}$$

where $\mathfrak{A}(k)$ is given by (5), gives the desired mapping. Thus, in order to obtain the function $p_{k,\alpha,\beta}$ with the desired normalization:

$$p_{k,\alpha,\beta}(0) = 1,$$

we need to solve the following equation:

$$p_{k,\alpha,\beta}(0) = h\Big(u_k(0)\Big) = 1,$$

where u_k is the Möbius transformation of the unit disk $\mathbb{U}$ onto itself. Hence we get

$$\left(\frac{\alpha-\beta}{1-k^2}\right)\cosh\left(\mathfrak{A}(k)\log\frac{1+\sqrt{u_k(0)}}{1-\sqrt{u_k(0)}}\right)+\frac{\beta-\alpha k^2}{1-k^2}=1,$$

which yields

$$u_k(0)=:\rho_k=\left(\frac{\exp\left(\frac{1}{\mathfrak{A}(k)}\text{Arccosh } B\right)-1}{\exp\left(\frac{1}{\mathfrak{A}(k)}\text{Arccosh } B\right)+1}\right)^2,$$

where $\mathfrak{A}(k)$ is given by (5) and

$$B=\frac{1}{\alpha-\beta}(1-k^2-\beta+\alpha k^2).$$

The automorphism $u_k(z)$ given by

$$u_k(z)=\frac{z+\rho_k}{1+\rho_k z}$$

provides the required self-mapping of the unit disk $\mathbb{U}$. Therefore, the mapping $p_{k,\alpha,\beta}(z)$ given by

$$p_{k,\alpha,\beta}(z)=\left(\frac{\alpha-\beta}{1-k^2}\right)\cosh\left(\mathfrak{A}(k)\log\frac{1+\sqrt{u_k(z)}}{1-\sqrt{u_k(z)}}\right)+\frac{\beta-\alpha k^2}{1-k^2}\qquad(z\in\mathbb{U})$$

is the desired mapping of the unit disk $\mathbb{U}$ onto the hyperbolic domain $\Omega_{k,\alpha,\beta}$, given by (4), with the following normalization:

$$p_{k,\alpha,\beta}(0)=1.$$

Finally, we shall give the explicit representations for $p_{k,\alpha,\beta}(z)$ for $1<k<\infty$. The conformal mapping of the unit disk $\mathbb{U}$ onto the interior of the ellipse:

$$(6)\quad \Omega_{k,\alpha,\beta}=\left\{w:w=u+iv \text{ and } \frac{\left(u-\frac{\alpha k^2-\beta}{k^2-1}\right)^2}{\left(\frac{k^2(\alpha-\beta)^2}{(k^2-1)^2}\right)}+\frac{v^2}{\left(\frac{(\alpha-\beta)^2}{k^2-1}\right)}<1\ (1<k<\infty)\right\}$$

requires the use of the Jacobian elliptic functions. It is known from [21, p. 280] that the Jacobian elliptic function $\operatorname{sn}(s,\kappa)$ transforms the upper half-plane and the upper semidisk of

$$\left\{s:|s|\leqq\frac{1}{\sqrt{\kappa}}\right\}$$

onto the interior of a rectangle with vertices at

$$\pm K(\kappa) \quad \text{and} \quad \pm K(\kappa) + iK'(\kappa).$$

Here, and in what follows, $K(\kappa)$ $(0 < \kappa < 1)$ is Legendre's complete elliptic integral of the first kind defined by

$$K(\kappa) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}} \qquad (0 < \kappa < 1) \tag{7}$$

and

$$K'(\kappa) = K\left(\sqrt{1-\kappa^2}\right) \qquad (0 < \kappa < 1) \tag{8}$$

is the complementary integral of $K(\kappa)$. The mapping

$$z(w_1) = \sqrt{\kappa}\,\mathrm{sn}\left(\frac{2K(\kappa)}{\pi}\mathrm{Arcsin}\,\frac{w_1}{c}\right) \qquad \left(c := \frac{\alpha-\beta}{k^2-1}\right)$$

maps conformally the elliptic domain:

$$\mathbb{E} = \left\{ w : w = u + iv \quad \text{and} \quad \frac{u^2}{\left(\frac{k^2(\alpha-\beta)^2}{(k^2-1)^2}\right)} + \frac{v^2}{\left(\frac{(\alpha-\beta)^2}{k^2-1}\right)} < 1 \right\}$$

onto the unit disk $\mathbb{U}$. Its inverse

$$w_1(z) = \left(\frac{\alpha-\beta}{k^2-1}\right)\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{z}{\sqrt{\kappa}}} \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2t^2)}}\right),$$

where $\kappa \in (0,1)$ is so chosen that

$$k = \cosh\left(\frac{\pi K'(\kappa)}{4K(\kappa)}\right), \tag{9}$$

maps the unit disk $\mathbb{U}$ onto the elliptic domain $\mathbb{E}$ such that $w_1(0) = 0$. The shift through the distance

$$\frac{\alpha k^2 - \beta}{k^2-1}$$

to the right, that is,

$$w(w_1) = w_1 + \frac{\alpha k^2 - \beta}{k^2-1}$$

maps $\mathbb{E}$ onto $\Omega_{k,\alpha,\beta}$. Hence

$$h(z) = \left(\frac{\alpha-\beta}{k^2-1}\right)\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{z}{\sqrt{\kappa}}} \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2t^2)}}\right) + \frac{\alpha k^2-\beta}{k^2-1}$$

maps the unit disk $\mathbb{U}$ onto the domain $\Omega_{k,\alpha,\beta}$ given by (6), but $h(z)$ is not normalized. By the above method using the Möbius transformation, we can easily show that $p_{k,\alpha,\beta}(z)$ given by

$$p_{k,\alpha,\beta}(z) = \left(\frac{\alpha-\beta}{k^2-1}\right) \sin\left(\frac{\pi}{2K(\kappa)} \int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}\right) + \frac{\alpha k^2-\beta}{k^2-1},$$

where

$$u_k(z) = \frac{z+\rho_k}{1+\rho_k z}$$

and

$$\rho_k = \sqrt{\kappa}\,\mathrm{sn}\left[\frac{2K(\kappa)}{\pi} \mathrm{Arcsin}\left(\frac{k^2-1-\alpha k^2+\beta}{\alpha-\beta}\right)\right],$$

is the desired mapping.

Thus the functions which play the rôle of extremal functions for the class $\mathcal{P}(p_{k,\alpha,\beta})$ are obtained as in the following theorem.

Theorem 1. *Let α, β and k be nonnegative numbers satisfying the following inequalities*:

$$0 \leqq \beta < \alpha \leqq 1 \quad \textit{and} \quad k(1-\alpha) < 1-\beta.$$

If $p \in \mathcal{P}(p_{k,\alpha,\beta})$, then

$$p(z) \prec p_{k,\alpha,\beta}(z) \qquad (z \in \mathbb{U}),$$

where $p_{k,\alpha,\beta}$ is the conformal mapping of the unit disk $\mathbb{U}$ onto the domain $\Omega_{k,\alpha,\beta}$ such that

$$p_{k,\alpha,\beta}(z) = \begin{cases} \frac{1+(1-2\beta)z}{1-z} & (k=0) \\ \alpha + \frac{2(\alpha-\beta)}{\pi^2}\left(\log\frac{1+\sqrt{u_k(z)}}{1-\sqrt{u_k(z)}}\right)^2 & (k=1) \\ \left(\frac{\alpha-\beta}{1-k^2}\right)\cosh\left(\mathfrak{A}(k)\log\frac{1+\sqrt{u_k(z)}}{1-\sqrt{u_k(z)}}\right) + \frac{\beta-\alpha k^2}{1-k^2} & (0<k<1) \\ \frac{\alpha-\beta}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{u_k(z)}{\sqrt{\kappa}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right) + \frac{\alpha k^2-\beta}{k^2-1} & (k>1), \end{cases}$$

where $\mathfrak{A}(k)$ is given in (5),

$$u_k(z) = \frac{z+\rho_k}{1+\rho_k z}$$

and

$$\rho_k = \begin{cases} \left(\frac{e^A-1}{e^A+1}\right)^2 & (k=1) \\ \left(\frac{\exp\left(\frac{1}{\mathfrak{A}(k)}\operatorname{Arccosh} B\right)-1}{\exp\left(\frac{1}{\mathfrak{A}(k)}\operatorname{Arccosh} B\right)+1}\right)^2 & (0<k<1) \\ \sqrt{\kappa}\ \operatorname{sn}\left[\frac{2K(\kappa)}{\pi}\operatorname{Arcsin} C\right] & (k>1), \end{cases}$$

with

$$A = \sqrt{\frac{1-\alpha}{2(\alpha-\beta)}}\ \pi, \quad B = \frac{1}{\alpha-\beta}(1-k^2-\beta+\alpha k^2)$$

and

$$C = \frac{1}{\alpha-\beta}(k^2-1-\alpha k^2+\beta),$$

and $K(\kappa)$, $K'(\kappa)$ *and* κ *are defined by* (7), (8) *and* (9), *respectively.*

3. The Fekete-Szegö Problem for the Function Class k-$\mathcal{SP}(\alpha,\beta)$

In this section, we obtain the Fekete-Szegö inequality for functions in the class k-$\mathcal{SP}(\alpha,\beta)$ for $k\in[0,1]$. In order to solve the Fekete-Szegö problem for the class k-$\mathcal{SP}(\alpha,\beta)$ $(0 \leqq k \leqq 1)$, we need some coefficients of $p_{k,\alpha,\beta}(z)$, which would play the rôle of maximal function in that class. Obviously, if $k=0$, we have

$$p_{0,\alpha,\beta}(z) = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots .$$

For $k=1$, using the following equation (see [24]):

$$\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 = 4\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{1}{4m-4m^2+4mn-1-2n}\right)z^n,$$

we can obtain another expression for $p_{1,\alpha,\beta}(z)$, that is,

$$p_{1,\alpha,\beta}(z) = \alpha + \frac{8(\alpha-\beta)}{\pi^2}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{1}{4m-4m^2+4mn-1-2n}\right)\left(\frac{z+\rho_1}{1+\rho_1 z}\right)^n.$$

By a simple calculation, we can find that

$$p_{1,\alpha,\beta}(z) = 1 + P_1 z + P_2 z^2 + \cdots ,$$

where

$$P_1 = \frac{8}{\pi^2}(\alpha-\beta)(1-\rho_1^2)\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{n\rho_1^{n-1}}{4m-4m^2+4mn-1-2n}\right)$$

and

$$P_2 = \frac{4}{\pi^2}(\alpha-\beta)(1-\rho_1^2)\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{n\rho_1^{n-2}(n(1-\rho_1^2)-(1+\rho_1^2))}{4m-4m^2+4mn-1-2n}\right).$$

Finally, in the case when $0<k<1$, we need the expansion of the function

$$\cosh\left(\mathfrak{A}(k)\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right).$$

In fact, Kanas (see [12] and [13]) found that the coefficients of this function are as follows:

$$\cosh\left(\mathfrak{A}(k)\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) = 1+\sum_{n=1}^{\infty}\left[\sum_{l=1}^{2n}2^l\binom{\mathfrak{A}(k)}{l}\binom{2n-1}{2n-l}\right]z^n.$$

Using the above method for finding the coefficients for the case when $k=1$, we also find that

$$p_{1,\alpha,\beta}(z) = 1+P_1z+P_2z^2+\cdots,$$

where

$$P_1 = \frac{(\alpha-\beta)(1-\rho_k^2)}{1-k^2}\sum_{n=1}^{\infty}\left[n\rho_k^{n-1}\sum_{l=1}^{2n}2^l\binom{\mathfrak{A}(k)}{l}\binom{2n-1}{2n-l}\right]$$

and

$$\begin{aligned}P_2 &= \frac{(\alpha-\beta)(1-\rho_k^2)}{2(1-k^2)}\\ &\quad\cdot\sum_{n=1}^{\infty}\left[n\rho_k^{n-2}\left(n(1-\rho_k^2)-(1+\rho_k^2)\right)\sum_{l=1}^{2n}2^l\binom{\mathfrak{A}(k)}{l}\binom{2n-1}{2n-l}\right].\end{aligned}$$

The proof of our main result in this section is based upon the following lemma given by Keogh and Merkes [15] (see also Ma and Minda [18]).

Lemma. (see [15] and [18]). *Let*

$$p(z) = 1+c_1z+c_2z^2+\cdots$$

be a function with positive real part in $\mathbb{U}$. *Then*, *for any complex number* v,

$$|c_2-vc_1^2| \leqq 2\max\{1,|1-2v|\}.$$

In particular, *if* v *is a real parameter*, *then*

$$|c_2 - vc_1^2| \leqq \begin{cases} -4v+2 & (v \leqq 0) \\ 2 & (0 \leqq v \leqq 1) \\ 4v-2 & (v \geqq 1). \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds true if and only if

$$p(z) = \frac{1+z}{1-z}$$

or one of its rotations. If $0 < v < 1$, then the equality holds true if and only if

$$p(z) = \frac{1+z^2}{1-z^2}$$

or one of its rotations. If $v = 0$, then the equality holds true if and only if

$$p(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z} \qquad (0 \leqq \lambda \leqq 1)$$

or one of its rotations. If $v = 1$, then the equality holds true if $p(z)$ is a reciprocal of one of the functions such that the equality holds true in the case when $v = 0$.

Theorem 2. *Let $0 \leqq k \leqq 1$ and let the function $f(z)$ given by (1) be in the class k-$\mathcal{SP}(\alpha, \beta)$. Then, for a complex number μ,*

$$|a_3 - \mu a_2^2| \leqq \frac{1}{2}\, P_1 \max\left\{1, \left|\frac{P_2}{P_1} - P_1 + 2\mu P_1\right|\right\}. \tag{10}$$

Furthermore, for a real parameter μ,

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{1}{2}P_2 + \frac{1}{2}P_1^2 - \mu P_1^2 & (\mu \leqq \sigma_1) \\ \frac{1}{2}P_1 & (\sigma_1 \leqq \mu \leqq \sigma_2) \\ -\frac{1}{2}P_2 - \frac{1}{2}P_1^2 + \mu P_1^2 & (\mu \geqq \sigma_2), \end{cases} \tag{11}$$

where

$$\sigma_1 = \frac{1}{2P_1^2}(P_1^2 - P_1 + P_2)$$

and

$$\sigma_2 = \frac{1}{2P_1^2}(P_1^2 + P_1 + P_2).$$

The result is sharp for a real parameter μ.

Proof. We begin by showing that the inequalities (10) and (11) hold true for $f \in k\text{-}\mathcal{SP}(\alpha, \beta)$. Let us consider a function $q(z)$ given by

$$q(z) = \frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U}).$$

Then, since $f \in k\text{-}SP(\alpha, \beta)$, we have the following subordination:

$$q(z) \prec p_{k,\alpha,\beta}(z) \qquad (z \in \mathbb{U}), \tag{12}$$

where

$$p_{k,\alpha,\beta}(z) = 1 + P_1 z + P_2 z^2 + \cdots .$$

Using the subordination relation (12), we see that the function $h(z)$ given by

$$h(z) = \frac{1 + p^{-1}_{k,\alpha,\beta}\big(q(z)\big)}{1 - p^{-1}_{k,\alpha,\beta}\big(q(z)\big)} = 1 + h_1 z + h_2 z^2 + \cdots \qquad (z \in \mathbb{U})$$

is analytic and has positive real part in the open unit disk $\mathbb{U}$. We also have

$$q(z) = p_{k,\alpha,\beta}\left(\frac{h(z) - 1}{h(z) + 1}\right) \qquad (z \in \mathbb{U}). \tag{13}$$

We find from the equation (13) that

$$a_2 = \frac{1}{2} P_1 h_1$$

and

$$a_3 = \frac{1}{4} P_1 h_2 - \frac{1}{8} P_1 h_1^2 + \frac{1}{8} P_2 h_1^2 + \frac{1}{8} P_1^2 h_1^2,$$

which, together, imply that

$$a_3 - \mu a_2^2 = \frac{1}{4} P_1 (h_2 - v h_1^2),$$

where

$$v = \frac{1}{2}\left(1 - \frac{P_2}{P_1} - P_1 + 2\mu P_1\right).$$

Therefore, our result follows immediately as an application of the above Lemma.

We will next show that the inequality (11) is sharp. By applying the above Lemma again, for the case when $\mu < \sigma_1$ or $\mu > \sigma_2$, the equality holds true if and only if

$$h(z) = \frac{1 + z}{1 - z}$$

or one of its rotations. Hence $f(z)$, which is the sharp function for this case, must satisfy the following equation:

$$\frac{zf'(z)}{f(z)} = q(z) = p_{k,\alpha,\beta}\left(\frac{h(z)-1}{h(z)+1}\right) = p_{k,\alpha,\beta}(z).$$

By means of the above method, we can obtain the sharp functions for each of the cases as follows:

$$f(z) = z\exp\left(\int_0^z \frac{p_{k,\alpha,\beta}\big(g(t)\big)-1}{t}\,dt\right),$$

where

$$g(t) = \begin{cases} t & (\mu < \sigma_1 \quad \text{or} \quad \mu > \sigma_2) \\ t^2 & (\sigma_1 < \mu < \sigma_2) \\ \frac{t(t+\lambda)}{1+\lambda z} & (\mu = \sigma_1) \\ -\frac{t(t+\lambda)}{1+\lambda z} & (\mu = \sigma_2) \end{cases}$$

or one of their rotations. Hence the proof of Theorem 2 is completed. ■

Remark 1. The Fekete-Szegö problem for the functions in the class k-$\mathcal{SP}(\alpha,\beta)$, which is restricted to the case which $k > 1$, remains still open.

4. Further Results for the Function Classes k-$\mathcal{SP}(\alpha,\beta)$ and k-$\mathcal{UCV}(\alpha,\beta)$

In this section, we shall derive some results on the function classes k-$\mathcal{SP}(\alpha,\beta)$ and k-$\mathcal{UCV}(\alpha,\beta)$. In Theorem 3 below, we first find a radius of k-$\mathcal{UCV}(\alpha,\beta)$ for functions $f \in \mathcal{S}$.

Theorem 3. *Let $f \in \mathcal{S}$. Then $f \in k$-$\mathcal{UCV}(\alpha,\beta)$ for all $r \leqq r_0$, where*

$$r_0 = \frac{2(k+1) - \sqrt{(3+\alpha^2)k^2 + (6+2\alpha\beta)k + 3 + \beta^2}}{\alpha k + \beta + k + 1}. \tag{14}$$

Proof. We know that, for $f \in \mathcal{S}$ and $|z| = r < 1$, the following sharp inequaility holds true ([5, p. 15]):

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leqq \frac{4r}{1-r^2} \tag{15}$$

or, equivalently,

$$\left|\left(1 + \frac{zf''(z)}{f'(z)}\right) - \frac{1+r^2}{1-r^2}\right| \leqq \frac{4r}{1-r^2}. \tag{16}$$

The above condition (15) or (16) represents a disk intersecting the real axis at the points

$$\left(\frac{1+r^2-4r}{1-r^2}, 0\right) \quad \text{and} \quad \left(\frac{1+r^2+4r}{1-r^2}, 0\right).$$

According to the developments presented in Section 2, we have to search for the largest value of $r = |z|$ such that the disk (16) lies completely inside the conic domain $\Omega_{k,\alpha,\beta}$. Since all conic domains have one vertex at the point

$$\left(\frac{\alpha k+\beta}{k+1}, 0\right),$$

it is necessary to fulfill the following condition:

$$\frac{1+r^2-4r}{1-r^2} \geqq \frac{\alpha k+\beta}{k+1}.$$

This last inequality is satisfied for $0 \leqq r \leqq r_0$, with r_0 given by (14). It suffices to check that, for r_0 given by (14), the disk (16) and the conic section $\partial\Omega_{k,\alpha,\beta}$ have only one common point $(u_1, 0)$, where

$$u_1 = \frac{1+r^2-4r}{1-r^2} = \frac{\alpha k+\beta}{k+1}. \tag{17}$$

In fact, for the value of k determined by (17), one can see that the following system of equations:

$$\left(u - \frac{1+r^2}{1-r^2}\right)^2 + v^2 = \frac{16r^2}{(1-r^2)^2}$$

and

$$k^2(u-\alpha)^2 + k^2v^2 = (u-\beta)^2$$

has only one solution for $u > 0$. Thus, for $r \leqq r_0$, the disk (16) lies completely inside the domain $\Omega_{k,\alpha,\beta}$. ■

For the analytic functions

$$f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad f_2(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $(f_1 * f_2)(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

In the following theorem, by using the Hadamard product (or convolution), we present a necessary and sufficient condition for a function $f \in \mathcal{S}$ to be in the class k-$\mathcal{UCV}(\alpha, \beta)$.

Theorem 4. *Let $0 < k < \infty$. A function $f \in \mathcal{S}$ is in the class k-$\mathcal{UCV}(\alpha, \beta)$ if and only if*

$$\frac{1}{z}\,(f * G_t)(z) \neq 0 \qquad (z \in \mathbb{U})$$

for all

$$t \geqq \frac{k(\alpha - \beta)}{1 + k}$$

such that

$$\left(\frac{t}{k}\right)^2 - (t + \beta - \alpha)^2 \geqq 0,$$

where

$$G_t(z) = \frac{z}{[1 - C(t)](1 - z)^2}\left(\frac{1 + z}{1 - z} - C(t)\right) \qquad (z \in \mathbb{U})$$

and

$$C(t) = t + \beta \pm i\sqrt{\left(\frac{t}{k}\right)^2 - (t + \beta - \alpha)^2}.$$

Proof. Let $0 < k < \infty$. Assume that $f \in \mathcal{S}$ and

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \qquad (z \in \mathbb{U}).$$

Since $p(0) = 1$, it follows that

$$f \in k\text{-}\mathcal{UCV}(\alpha, \beta) \Longleftrightarrow p(z) \notin \partial\Omega_{k,\alpha,\beta} \qquad (z \in \mathbb{U}),$$

where, as before,

$$\partial\Omega_{k,\alpha,\beta} = \left\{w : w = u + iv \quad \text{and} \quad k^2(u - \alpha)^2 + k^2v^2 = (u - \beta)^2\right\}.$$

We note for $\partial\Omega_{k,\alpha,\beta}$ that

$$\partial\Omega_{k,\alpha,\beta} = C(t) = t + \beta \pm i\sqrt{\left(\frac{t}{k}\right)^2 - (t + \beta - \alpha)^2}\,,$$

where

$$\left(\frac{t}{k}\right)^2 - (t + \beta - \alpha)^2 \geqq 0.$$

Since

$$\frac{z}{(1 - z)^2} * f(z) = zf'(z) \quad \text{and} \quad \frac{z(1 + z)}{(1 - z)^3} * f(z) = zf'(z) + z^2f''(z),$$

we have
$$\frac{1}{z}\,(f * G_t)(z) = \frac{f'(z)}{1 - C(t)}\left(1 + \frac{zf''(z)}{f'(z)} - C(t)\right).$$
Thus, finally, we obtain
$$\begin{aligned}\frac{1}{z}\,(f * G_t)(z) \neq 0 \;&\Longleftrightarrow\; p(z) \notin \partial\Omega_{k,\alpha,\beta}\\ &\Longleftrightarrow\; f(z) \in k\text{-}\mathcal{UCV}(\alpha,\beta),\end{aligned}$$
as asserted by Theorem 4. ■

Theorem 5. *The function*
$$\mathfrak{k}(z) = \frac{z}{1 - \lambda z} \qquad (z \in \mathbb{U})$$
is in the class k-$\mathcal{UCV}(\alpha,\beta)$ *if and only if*
$$|\lambda| \leqq \frac{-(k+\beta) + \sqrt{(k+\beta)^2 - [k(1+\alpha) + (1+\beta)][k(1-\alpha) - (1-\beta)]}}{k(1+\alpha) + \beta + 1}.$$

Proof. Simple calculation and the definition of the function class k-$\mathcal{UCV}(\alpha,\beta)$ would show that the condition $\mathfrak{k}(z) \in k$-$\mathcal{UCV}(\alpha,\beta)$ is equivalent to the following inequality:
$$k\left|\frac{1 - \alpha + (1+\alpha)\lambda z}{1 - \lambda z}\right| < \Re\left(\frac{1+\lambda z}{1 - \lambda z}\right) - \beta \qquad (z \in \mathbb{U}).$$
Since each of the following inequalities:
$$\left|\frac{1 - \alpha + (1+\alpha)\lambda z}{1 - \lambda z}\right| \leqq \frac{1 - \alpha + (1+\alpha)|\lambda z|}{1 - |\lambda z|} \qquad (z \in \mathbb{U})$$
and
$$\frac{1 - \beta - (1+\beta)|\lambda z|}{1 + |\lambda z|} \leqq \Re\left(\frac{1+\lambda z}{1 - \lambda z}\right) - \beta \qquad (z \in \mathbb{U})$$
holds true, it suffices to show that

(18) $[k(1+\alpha) + (\beta+1)]\,|\lambda z| + 2(k+\beta)|\lambda z| + k(1-\alpha) - (1-\beta) < 0 \qquad (z \in \mathbb{U}).$

Moreover, the inequality (18) is satisfied if
$$[k(1+\alpha) + (\beta+1)]\,|\lambda| + 2(k+\beta)|\lambda| + k(1-\alpha) - (1-\beta) \leqq 0.$$

Hence we can obtain the bound of the modulus of λ as follows:

$$|\lambda| \leqq \frac{-(k+\beta)+\sqrt{(k+\beta)^2-[k(1+\alpha)+(1+\beta)]\,[k(1-\alpha)-(1-\beta)]}}{k(1+\alpha)+\beta+1}. \tag{19}$$

Conversely, if we assume that $\mathfrak{k}(z) \in k\text{-}\mathcal{UCV}(\alpha,\beta)$, then

$$k\left|\frac{1-\alpha+(1+\alpha)\lambda z}{1-\lambda z}\right| < \Re\left(\frac{1+\lambda z}{1-\lambda z}\right) - \beta \qquad (z \in \mathbb{U}).$$

Upon letting $z \to 1-$, we can obtain the inequality (19) by simple calculations. Our proof of Theorem 5 is thus completed. ∎

Corollary 1. *The function*

$$\mathfrak{k}(z) = \frac{z}{(1-\lambda z)^2}$$

is in the class $k\text{-}\mathcal{SP}(\alpha,\beta)$ *if and only if*

$$|\lambda| \leqq \frac{-(k+\beta)+\sqrt{(k+\beta)^2-[k(1+\alpha)+(1+\beta)]\,[k(1-\alpha)-(1-\beta)]}}{k(1+\alpha)+\beta+1}.$$

Remark 2. If, in Corollary 1 above, we put

$$k=1, \quad \alpha=1 \quad \text{and} \quad \beta=0,$$

then we can obtain the result by Rønning [24].

Theorem 6. *Let* $k \geqq 1$ *and let* α *and* β *be nonnegative real numbers satisfying the following inequality*:

$$2k^2(1-\alpha) \leqq (1-\beta)(2-\beta). \tag{20}$$

Then the function

$$f(z) = z + a_n z^n \qquad (n \in \mathbb{N}^* := \mathbb{N} \setminus \{1\};\ \mathbb{N} = \{1,2,3,\cdots\})$$

is in the class $k\text{-}\mathcal{SP}(\alpha,\beta)$ *if and only if*

$$|a_n| \leqq \zeta_0 \qquad (n \in \mathbb{N}^*),$$

where $\zeta_0 \in (0,1)$ *is the smallest root of the following quadratic equation in* x :

$$\left[k^2(n-\alpha)^2-(n-\beta)^2\right]x^2 + 2\left[(n-\beta)(1-\beta)-k^2(1-\alpha)(n-\alpha)\right]x + k^2(1-\alpha)^2-(1-\beta)^2 = 0 \qquad (n \in \mathbb{N}^*).$$

Proof. It suffices to show that the following inequality:

$$k\left|\frac{zf'(z)}{f(z)}-\alpha\right| \leqq \Re\left(\frac{zf'(z)}{f(z)}\right)-\beta \qquad (z\in\mathbb{U}) \tag{21}$$

holds true for $|z|=1$. Let us set

$$|a_n| = r \quad \text{and} \quad a_n z^{n-1} = re^{i\xi}.$$

Then (21) for this function $f(z)$ would yield

$$\begin{aligned}
&k\left|\frac{1+nre^{i\xi}}{1+re^{i\xi}}-\alpha\right| < \Re\left(\frac{1+nre^{i\xi}}{1+re^{i\xi}}\right)-\beta \\
&\iff k\left|\frac{1-\alpha+(n-\alpha)re^{i\xi}}{1+re^{i\xi}}\right| < \frac{1}{|1+re^{i\xi}|^2}\left[1+(n+1)r\cos\xi+nr^2\right]-\beta \\
&\iff k^2(1-\alpha)^2+2(1-\alpha)(n-\alpha)r\cos\xi+(n-\alpha)^2r^2 \\
&\quad < \frac{\left[1+(n+1)r\cos\xi+nr^2\right]^2}{1+2r\cos\xi+r^2}-2\beta(1+(n+1)r\cos\xi+nr^2) \\
&\qquad +\beta^2(1+2r\cos\xi+r^2) \\
&\iff k^2(1-\alpha)^2+k^2(n-\alpha)^2r^2 < g(\xi),
\end{aligned}$$

where the function $g(\xi)$ is defined by

$$\begin{aligned}
g(\xi) := {} & \frac{\left[1+(n+1)r\cos\xi+nr^2\right]^2}{1+2r\cos\xi+r^2}-2\beta\left[1+(n+1)r\cos\xi+nr^2\right] \\
&+\beta^2(1+2r\cos\xi+r^2)-2k^2(1-\alpha)(n-\alpha)r\cos\xi.
\end{aligned}$$

By using the hypothesis of Theorem 4, we can show that the minimal value $g(\pi)$ is given by

$$\begin{aligned}
g(\pi) &= \frac{\left[1-(n+1)r+nr^2\right]^2}{(1-r)^2}+\beta^2(1-r)^2-2\beta\left[1-(n+1)r+nr^2\right] \\
&\qquad +2k^2(1-\alpha)(n-\alpha)r \\
&= (1-nr)^2+\beta^2(1-r)^2-2\beta\left[1-(n+1)r+nr^2\right]+2k^2(1-\alpha)(n-\alpha)r.
\end{aligned}$$

Hence we have to solve the following inequality:

$$\begin{aligned}
k^2(1-\alpha)^2+k^2(n-\alpha)^2r^2 &< (1-nr)^2+\beta^2(1-r)^2 \\
&\quad -2\beta\left[1-(n+1)r+nr^2\right]+2k^2(1-\alpha)(n-\alpha)r.
\end{aligned} \tag{22}$$

We now show the existence of $r \in (0,1)$ which satisfies the inequality (22). For this purpose, we define a real-valued function $G(x)$ on $[0,1]$ by

$$\begin{aligned} G(x) = & \left[k^2(n-\alpha)^2 - n^2 - \beta^2 + 2\beta n\right] x^2 \\ & + 2\left[n + \beta^2 - \beta(n+1) - k^2(1-\alpha)(n-\alpha)\right] x \\ & + k^2(1-\alpha)^2 - 1 - \beta^2 + 2\beta. \end{aligned}$$

Then

$$G(0) = k^2(1-\alpha)^2 - (1-\beta)^2 < 0$$

under the constraints on α, β and k as specified in Definition 1, and

$$G(1) = (k^2-1)(n-1)^2 > 0.$$

Hence there exists $\zeta_0 \in (0,1)$ such that

$$G(\zeta_0) = 0 \qquad (0 < \zeta_0 < 1).$$

The assertion of Theorem 6 now follows easily. ■

Corollary 2. *Let the real numbers* k, α *and* β *and the function* $f(z)$ *be defined as in Theorem* 6. *Then* $f(z) \in k\text{-}\mathcal{UCV}(\alpha,\beta)$ *if and only if*

$$|a_n| \leqq \frac{1}{n\zeta_0} \qquad (n \in \mathbb{N}^*;\ 0 < \zeta_0 < 1).$$

Proof. We can easily deduce Corrollary 2 by using the Alexander transformation in Theorem 6. ■

Remark 3. If, in Corollary 2, we put

$$k = 1, \quad \alpha = 1 \quad \text{and} \quad \beta = 0,$$

then we can obtain the result given earlier by Rønning [24].

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