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CONDITIONING OF STATE FEEDBACK POLE ASSIGNMENT PROBLEMS

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Abstract. In [26, 27, 35], condition numbers and perturbation bounds were produced for the state feedback pole assignment problem (SFPAP), for the single- and multi-input cases with simple closed-loop eigenvalues. In this paper, we consider the same problem in a different approach with weaker assumptions, producing simpler condition numbers and perturbation results. For the SFPAP, we shall show that the absolute condition number $\kappa \leq c_0\|B^\dagger\| \left[\kappa_X + \left(1+\|F\|^2\right)^{1/2}\right]$, where the closed-loop system matrix $A+BF = X\Lambda X^{-1}$, the closed-loop spectrum in Λ is pre-determined, $\kappa_X \equiv \|X\|\|X^{-1}\|$, the operators $P_c(\cdot) \equiv (A+BF)(\cdot) - (\cdot)\Lambda$ and $\mathcal{N}(\cdot) \equiv (I-BB^\dagger)P_c(\cdot)$, and $c_0 \equiv \|I(\cdot) - P_c\left[\mathcal{N}^\dagger(I-BB^\dagger)(\cdot)\right]\|$. With $c_B \equiv \|B\|\|B^\dagger\|$ and $c_1 \equiv (\|B\|\|F\|)^{-1}$, the relative condition number $\kappa_r \leq c_0 c_B \left[c_1\kappa_X\|\Lambda\| + \left(c_1^2\|A\|^2+1\right)^{1/2}\right]$. With B well-conditioned and Λ well chosen, κ and κ_r can be small even when Λ (not necessary in Jordan form) possesses defective eigenvalues, depending on c_0. Consequently, the SFPAP is not intrinsically ill-conditioned. Similar results were obtained in [23], although differentiability was not established for its local perturbation analysis. Simple as well as general multiple closed-loop eigenvalues are treated.

1. INTRODUCTION

Let (A, B) denote the control system

$$\dot{x} = Ax + Bu \tag{1}$$

with the open-loop system matrix $A \in \mathcal{R}^{n\times n}$ and the input matrix $B \in \mathcal{R}^{n\times m}$. Without loss of generality, let us assume that $B \equiv [b_1, \cdots, b_m]$ is full-ranked. The

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state feedback pole assignment problem (SFPAP) seeks a control matrix $F \in \mathcal{R}^{m\times n}$ such that the closed-loop system matrix $A_c = A + BF$ has a prescribed eigenvalues or poles. Equivalently, we are seeking the control matrix F such that

$$(A + BF)X = X\Lambda \tag{2}$$

for some given Λ with desirable poles and nonsingular matrix X. Notice that Λ does not have to be in Jordan form, as in [26, 27, 35], and X can be well-conditioned even with defective multiple eigenvalues in some well-chosen Λ. This is similar in spirit to the "synthesis problem" in [23]. The choice of Λ will be important in our discussion. (In [3, 5, 36], parts of Λ in Schur form are chosen within the algorithms.) The SFPAP is solvable for *arbitrary* closed-loop spectrum when (A, B) is controllable, i.e., when $[sI - A, B]$ $(\forall s \in \mathcal{C})$ or $[B, AB, \cdots, A^{n-1}B]$ are full-ranked [38].

The SFPAP is a much investigated problem in control system design and everyone have their own favourite approach of solution; (see, e.g., [7, 29, 38] or any standard textbook in control theory and the references therein). It is well known that the single-input case ($m = 1$) has a unique solution, while the multi-input case has some degrees of freedom left in the problem. A notable effort in utilizing these degrees of freedom sensibly was made by Kautsky *et al* in [19], with the conditioning of the closed-loop spectrum being optimized.

Interestingly, relatively few results ([1, 17, 20, 21, 22, 23, 26, 27, 35]) are known on the conditioning of the SFPAP, although error analysis was carried out for various algorithms (e.g., [14, 15, 29]). The main results on the conditioning of the SFPAP are from Sun (1996), Mehrmann and Xu (1996, 97) and Konstantinov *et al* (1997) (see details in §2). In [35], Sun analyzed the SFPAP using the implicit function theorem, producing various power series expansions. Building on the differentiability of simple closed-loop eigenvalues, systems with multiple eigenvalues were not considered. In [26, 27], perturbation results were produced for systems with simple closed-loop eigenvalues. Multiple eigenvalues were ignored because of the associated ill-conditioning for the closed-loop eigenvalue problem. Nonetheless, many interesting results were produced for the SFPAP with more general eigen-structures. In [23], the conditioning of the more general feedback synthesis problem, including the state and output feedback pole assignment problems, were considered. Analysis were performed with Λ being a general matrix with prescribed spectrum. Local perturbation analysis was done using Fréchet derivatives, without proving the associated differentiability. Nonlocal perturbation analysis was performed using the Schauder fixed point principle and the splitting technique. Our technique shares similar spirit as Konstantinov *et al*, plus the possibility of Λ being non-Jordan and the irrelevance of controllability. We shall generalize Sun's approach to prove differentiability but obtain derivatives and perturbation results not through the implicit function theorem.

Nonlocal perturbation is also possible via perturbation expansions, with details for the SFPAP in §3.3.

Is Pole Assignment Intrinsically Ill-Conditioned

Simple linear algebra shows that (2) is equivalent to [7, 8, 11, 19]

$$(I - BB^{\dagger})(AX - X\Lambda) = 0 \tag{3}$$

and

$$F = B^{\dagger}(X\Lambda X^{-1} - A) \tag{4}$$

Equation (3) then defines the subspaces from which the right-(generalized) eigenvectors in X can be chosen (using a Kronecker product if Λ is not diagonal; see §3.8), and (4) produces the feedback matrix F explicitly once an invertible X is chosen. It is easy to see that this formulation does not involve the spurious Sylvester operator $P(\cdot) \equiv A(\cdot) - (\cdot)\Lambda$, whose invertibility requires that no open-loop poles are re-assigned. Similarly, no Cauchy type matrix $C = [c_{ij}^{-1}]$, where $c_{ij} = \gamma_i - \lambda_j$ denotes the difference between the open-loop pole γ_i and the closed-loop pole λ_j, is involved. One may argue that ill-conditioning may be hiding behind the selection of X. When Λ is in Jordan form, the selection of eigenvectors in X is universal, implicitly or explicitly, in most pole assignment algorithms. A better question to ask is the relationship between the conditioning of the eigenvector matrix X and various robustness measures in control system design (see [19] for some related results). With Λ *not* in Jordan form, the situation is totally different. There is no reason why X should be ill-conditioned even when Λ possesses defective multiple closed-loop eigenvalues. As an extreme example, we may choose $A_c \equiv A + BF$ if it is known and X degenerates into the identity matrix with perfect conditioning. Notice that we have bypassed the ill-conditioning of the closed-loop eigenvalues, delaying it until these eigenvalues are sought. This is nevertheless allowed in the SFPAP, when a well-chosen Λ containing the desirable poles is available. The subsequent ill-conditioning is someone else's problem! Notice that this approach is not totally new and has been applied in, e.g., [3, 5, 36], with Λ chosen to be in triangular or Schur form. For examples with reassigned open-loop poles and multiple poles which minimize sensitivity, see [23].

In this paper, we first show that the power series expansion for Λ, when chosen in Jordan form with general Jordan structure, exists [6] and is thus differentiable. Similar differentiability for a general Λ follows easily. We then generalized the approach in [35] to show that the feedback matrix F is differentiable with respect to components in the data A, B and Λ, even though the eigenvalues of Λ (with *fixed* Jordan structures) may be multiple and may not be differentiable. Without obtaining the condition numbers and error bounds directly from this implicit function theorem approach, we shall differentiate the simple explicit formula for F in (4) directly, with respect to the components in the data matrices A, B and Λ. Condition numbers and first-order perturbation expansions are then produced, similar to those in [23, 26, 27, 35]. The results are simpler and easier to interpret and apply. Note that most condition numbers are not computed but used to provide qualitative guidelines so as to avoid ill-conditioning. As such, computable but difficult-to-interpret condition numbers are not all that useful.

When Λ is in Jordan form, we shall show that the conditioning of pole assignment (as an inverse eigenvalue problem) is equivalent to the conditioning of the closed-loop eigenvector matrix for *any closed-loop eigen-structure*. This contrasts with the ill-conditioning associated with multiple eigenvalues for eigenvalue problems. Notice that the pole assignment problem and the closed-loop eigenvalue problem are inverse to each other. Indeed, when A and B are fixed, the condition numbers for the two problems are essentially sizes of the derivatives of two functions which are reciprocal to each other. This observation means that the ill-conditioning of defective closed-loop eigenvalues should imply good conditioning for the associated pole assignment problem! When Λ is well-chosen and not in Jordan form (as in [3, 5, 36]), the condition numbers κ and κ_r of the SFPAP (see the bounds in (26) and (28)) contain no dangerous terms, when B and X are well-conditioned and when $\mathcal{M}$ or c_0 are moderate in size (reflecting solvability). Again, X no longer contains eigenvectors and there is no reason why X should be ill-conditioned. Choosing Λ to be a Schur form is one such possibility with X being unitary and $\kappa_X \equiv \|X\| \, \|X^{-1}\| = 1$ (as in [23]). This contrasts with the case when (Λ, X) is in Jordan form and κ_X can be made arbitrarily large by scaling of eigenvectors.

Finally, we would like to point out that controllability ought to have no direct impact on the solvability or conditioning of pole assignment. For a given system (1), we can select a random control matrix F, giving rise to some closed-loop poles in Λ. If we then solve the pole assignment problem associated with Λ for (1), solvability is guaranteed, irrespective of controllability. Note also in practice that controllability should usually be replaced by stabilizability, with selected unstable poles reassigned (as in *partial* pole assignment [31]). Afterall, controllability is the necessary and sufficient condition for the solvability of the SFPAP for an *arbitrary* closed-loop spectrum, but usually one is only interested in solving the problem for a *particular* closed-loop spectrum. How Λ is chosen depends on the algorithm used and is independent of the conditioning of the SFPAP. We shall show later that controllability or stabilizability are not essential for the investigation of conditioning of pole assignment. Clearly, the sizes of $\mathcal{M}$ and c_0 in (26) and (28) is bounded from above by an expression involving $\|\mathcal{N}^{\dagger}\|$ which reflects the sensitivity of of the selection of X and the solvability of our problem. What is required is the solvability for a specific closed-loop spectrum, not the more restrictive controllability or stabilizability.

In summary, we shall show that the SFPAP is not *intrinsically* ill-conditioned, when well-conditioned X can be chosen in the sense that κ_X is acceptably bounded (the corresponding non-Jordan Λ is then referred to as "well-chosen"), irrespective of controllability/stabilizability, or locations or structure of open/closed-loop poles. This view is partly consistent with that in [18], which stated that [18, p. 14] "In general, the computation of K does not dependent substantially on the desired spectrum" (here $K = F$). We shall show that the changes of the feedback matrix F is mainly controlled by $c_0 = \|\mathcal{M}\| = \|I - P_c\mathcal{N}^\dagger(I - BB^\dagger)\|$, which is bounded from above by an expression involving the solvability measure $\|\mathcal{N}^\dagger\|$.

Plan of Paper

In §2, results on the conditioning of the SFPAP by Sun [35], Mehrmann and Xu [26, 27] and Konstantinov *et al* [23] are quoted for reference and comparison. The conditioning for the SFPAP is revisited in §3. Section 4 contains a discussion of the Rice condition number. The paper is concluded in §5.

2. Sun's Implicit Functions

We first quote the result by Sun [35] on the conditioning of the SFPAP. We shall apply and generalize elements of this approach later.

Let $Y \equiv X^{-T} = [y_1, y_2, \cdots, y_n]$ with y_i being the left-eigenvector corresponding to λ_i.

We quote [35, Theorem 2.1] for the single-input case ($m = 1$):

Theorem 2.1. *Let a controllable system* (A, b) *and a set of self-conjugate complex numbers* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *be given, where* $\lambda_i \neq \lambda_j$, $i \neq j$. *Let* $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$, $a =$ *vec*(A). *Assume that* $f \in \mathcal{R}^n$ *and* $X \in \mathcal{C}^{n\times n}$ *satisfy*

$$A + bf^T = X\Lambda X^{-1}$$

Then there is a differential relation

$$df = \Phi da + \Psi db + Z d\lambda \tag{5}$$

where

$$\begin{aligned}
Z &= Y\mathrm{diag}\left(\frac{1}{y_1^T b}, \frac{1}{y_2^T b}, \cdots, \frac{1}{y_n^T b}\right) \in \mathcal{C}^{n\times n} \\
\Phi &= -Z\left[D_1(X)X^{-1}, D_2(X)X^{-1}, \cdots, D_n(X)X^{-1}\right] \in \mathcal{C}^{n\times n^2} \\
\Psi &= -Z\mathrm{diag}(f^T x_1, f^T x_2, \cdots, f^T x_n)X^{-1} \in \mathcal{C}^{n\times n}
\end{aligned}$$

with

$$D_i(X) = \mathrm{diag}(x_{i1}, x_{i2}, \cdots, x_{in}) \quad (i = 1, 2, \cdots, n)$$

From Theorem 2.1 or [35, Remark 2.2], we have

$$\|df\| \le \|\Phi\|\|da\| + \|\Psi\|\|db\| + \|Z\|\|d\lambda\| \tag{6}$$

with the condition numbers of the state feedback f with respect to, respectively, A, b and λ defined as

$$\kappa_A(f) = \|\Phi\| \ , \quad \kappa_b(f) = \|\Psi\| \ , \quad \kappa_\lambda(f) = \|Z\|$$

The absolute condition number of f is then defined as

$$\kappa(f) = \sqrt{\kappa_A^2(f) + \kappa_b^2(f) + \kappa_\lambda^2(f)}$$

We also have the following bounds [35, Remark 2.4]:

$$\kappa_A(f) \le \|Z\|_2 \|X^{-1}\|_2 \ , \quad \kappa_b(f) \le \|f\|_2 \|X\|_2 \|X^{-1}\|_2 \ , \quad \kappa_\lambda(f) = \|Z\|_2$$

First order error bounds can then be produced using the above derivatives and condition numbers (see [35, Corollary 2.5]). Relative condition numbers ([35, §2.2]; ignored here) can also be obtained in a similar fashion.

Similar to Theorem 2.1, we have the following theorem for the multi-input case [35, Theorem 3.1]:

Theorem 2.2. *Let a controllable system* (A, B) *and a set of self-conjugate complex numbers* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *be given, where* $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ $(m > 1)$, *and* $\lambda_i \ne \lambda_j$, $i \ne j$. *Let* $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$, $a =$ *vec*(A), $b =$ *vec*(B). *Assume that* $F \in \mathcal{R}^{n\times m}$ *and* $X \in \mathcal{C}^{n\times n}$ *satisfy*

$$A + BF = X\Lambda X^{-1}$$

and let $f =$ *vec*(F). *Then there is a differential relation*

$$df = \Phi da + \Psi db + Z d\lambda$$

where

$$Z = -W_f^{\dagger} \in \mathcal{C}^{mn\times n}, \ \Phi = -ZW_a \in \mathcal{C}^{mn\times n^2}, \ \Psi = -ZW_b \in \mathcal{C}^{mn\times mn}$$

with

$$\begin{aligned} W_f &\equiv \left[S_1 X^T, S_2 X^T, \cdots, S_m X^T\right] \in \mathcal{C}^{n\times mn} \\ W_a &= \left[D_1(X)X^{-1}, D_2(X)X^{-1}, \cdots, D_n(X)X^{-1}\right] \in \mathcal{C}^{n\times n^2} \\ W_b &= \mathrm{diag}(T_1 X^{-1}, T_2 X^{-1}, \cdots, T_m X^{-1}) \in \mathcal{C}^{n\times mn} \\ W_\lambda &= -I_n \end{aligned}$$

and

$$\begin{aligned} S_j &= \mathrm{diag}(y_1^T b_j, y_2^T b_j, \cdots, y_n^T b_j) \quad (i = 1, 2, \cdots, n) \\ D_i(X) &= \mathrm{diag}(x_{i1}, x_{i2}, \cdots, x_{in}) \quad (i = 1, 2, \cdots, n) \\ T_j &= \mathrm{diag}(f_j^T x_1, f_j^T x_2, \cdots, f_j^T x_n) \quad (j = 1, 2, \cdots, m) \end{aligned}$$

Similar differential relations, condition numbers and error bounds can be derived as in the single-input case.

3. State Feedback Pole Assignment Problem

We shall first consider the general multi-input case with a general Jordan structure in Λ in (2), where $\lambda = \mathrm{diag}\{\Lambda\}$. We first consider the existence of the power series expansion of F in terms of changes in the data A, B and Λ, generalizing the implicit function theorem approach by Sun [35]. With differentiability guaranteed, derivatives, condition numbers, error expansions and error bounds are then derived using the explicit formula (4). Equivalent formulae can also be obtained from the implicit function theorem but will not be attempted here as they are less user-friendly. Controllability of (1) is assumed first but will be shown to be unnecessary later.

3.1. Differentiability when Λ is in Jordan Form

Let $p \equiv [p_1, \cdots, p_N]^T \in \mathcal{B}(0) \subset \mathcal{C}^N$ be a vector of perturbation parameters selected from some neighbourhood of the origin $\mathcal{B}(0)$. Consider the eigenvalue problem

$$A_c(p)x(p) = \lambda(p)x(p) \ , \quad y(p)^T A_c(p) = \lambda(p)y(p)^T$$

where $A_c(p) = A_c(0) + E(p)$, $E(p) = [\epsilon_{ij}(p)]$ with

$$\epsilon_{ij}(p) = \sum_{r=1}^{\infty} \sum_{\Sigma(t)=r} \alpha^{(i,j)}_{t_1,\cdots,t_N} p_1^{t_1} \cdots p_N^{t_N} \ , \quad \Sigma(t) \equiv \sum_{k=1}^{N} t_k$$

When $p = 0$, the above eigenvalue problem reverts back to the unperturbed closed-loop eigenvalue problem. In [6], the existence of the power series for the average eigenvalues of an group of multiple eigenvalues, as well as the associated (generalized) eigenvectors, was proven. More relevant to our application here in pole assignment, the paper has also shown the existence of the power series for the generalized Rayleigh quotient

$$Q_i(p) \equiv Y_i(p)^T [A(p) + B(p)F(p)] X_i(p) \tag{7}$$

where $X_i(p)$ and $Y_i(p)$, respectively, span the right- and left-invariant subspaces corresponding to the eigenvalues in $Q_i(p)$, and the unperturbed subsystem

$$Y_i^T (A + BF) X_i = J_i \ , \quad Y_i^T X_i = I \tag{8}$$

with the $n_i \times n_i$ J_i contains the Jordan blocks associated with one of the closed-loop eigenvalues λ_i. Let

$$\Lambda = \mathrm{diag}\{J_1, \cdots, J_s\} \ , \quad \lambda \equiv \mathrm{diag}\Lambda \tag{9}$$

Applying the above results in [6] and using an approach similar to that in Sun [35], we next select the analytic function ω to which we can apply the implicit function theorem.

From (3) and (4), it is easy to see that the feedback matrix F, as a solution of the SFPAP, varies continuously with respect to changes in the data (A, B, Λ) of the problem, exactly the same way how the null-spaces in (3) change. With Kronecker products, write (3) generically as $Nv = 0$, with the corresponding perturbed system $(N + \delta N)(v + \delta v) = 0$. The perturbation in the null vector δv then satisfies, for some arbitrary z,

$$\delta v = -N^{\dagger}\delta N(v + \delta v) + (I - N^{\dagger}N)z \tag{10}$$

With $z = 0$, small changes in N produce small changes in δv, implying continuity. When $z \neq 0$, the last term $(I - N^{\dagger}N)z$ on the right-hand-side of (10) corresponds to the non-uniqueness of F when $m > 1$ and this alternative F can be achieved with the original v replaced by $\tilde{v} = v + (I - N^{\dagger}N)z$. In the following perturbation analysis we shall assume that $z = 0$. (In [23], a similar approach was taken, with the nearest perturbed system considered in the perturbation results.)

We now consider the differentiability of F with respect to the data (A, B, Λ). Let $(\widetilde{A}, \widetilde{B})$ be a neighbouring system to (A, B), and let $\widetilde{\lambda}(\approx \lambda)$ contain the perturbed closed-loop poles, with the Jordan structure in Λ fixed. For $t \in [-1, 1]$, denote

$$A(t) \equiv A + t(\widetilde{A} - A) \ , \quad B(t) \equiv B + t(\widetilde{B} - B) \ , \quad \Lambda(t) \equiv \Lambda + t(\widetilde{\Lambda} - \Lambda)$$

With the perturbed system $(\widetilde{A}, \widetilde{B})$ sufficiently near the original system (A, B) and $\widetilde{\Lambda}$ sufficiently near Λ, $(A(t), B(t))$ inherits any controllability or similar properties from (A, B), with the eigenvalues in $\Lambda(t)$ closed under complex conjugation. We can thus find a feedback matrix $F(t)$ solving the SFPAP for the data $(A(t), B(t), \Lambda(t))$. For the investigation of how $F(t) - F$ is dependent on $A(t) - A$, $B(t) - B$ and $\Lambda(t) - \Lambda$ when $t \to 0$, we can embed in $\mathcal{C}^n$, as in [35]. Let

$$\widehat{A} = A + \delta A \ , \quad \widehat{B} = B + \delta B \ , \quad \widehat{F} = F + \delta F \ , \quad \widehat{\lambda} = \lambda + \delta\lambda$$

Similar to [6], we produces the following results for the generalized Rayleigh quotients $\widehat{Q}_i \equiv \widehat{Y}_i^T(\widehat{A} + \widehat{B}\widehat{F})\widehat{X}_i$:

(i) The generalized Rayleigh quotients $\widehat{Q}_i$ are analytic functions of the elements of $(\widehat{A}, \widehat{B}, \widehat{\lambda})$ in some small neighbourhood $\mathcal{B}$ of (A, B, λ).

(ii) The matrices $\widehat{X}_i$ and $\widehat{Y}_i$, containing vectors spanning the associated right and left invariant subspaces, may be defined to be analytic functions of $(\widehat{A}, \widehat{B}, \widehat{\lambda})$ in $\mathcal{B}$.

(iii) The Rayleigh quotient $\widehat{Q}_i$ and associated (generalized) eigenvectors of $\widehat{A}+\widehat{B}\widehat{F}$ become that of $A + BF$ when $(\widehat{A}, \widehat{B}, \widehat{F}) = (A, B, F)$.

We are now in position to define the analytic function ω (to which the implicit function theorem is apply). Let

$$\Omega \equiv \mathrm{diag}\{\widehat{Q}_j\} - \widehat{\Lambda} = \mathrm{diag}\{\widehat{Y}_i^T(\widehat{A} + \widehat{B}\widehat{F})\widehat{X}_i\} - \widehat{\Lambda}$$

Construct $\omega(\widehat{F}; \widehat{A}, \widehat{B}, \widehat{\lambda}) : \mathcal{C}^{mn+(n^2+mn+s)} \to \mathcal{C}^s$ by selecting only the averages of the eigenvalues (or traces divided by n_i) of the diagonal blocks of Ω involving the Jordan block J_i in (8). Notice that there are only s distinct eigenvalues in λ and $\widehat{\lambda}$. The analyticity of $\widehat{Q}_i$ proven in [6] guarantees the analyticity of Ω and ω. Note that the eigen-structure in J_i or Λ is fixed, with the super-diagonal containing various zeroes and ones and the only variables being the given data λ_i $(i = 1, \cdots, s)$ on the diagonal. Expressing $\widehat{F}$ as an analytic function in the other variables, we need the number of unknowns in F (nm) to be greater than or equal to the number of equations in ω (s), as required by [35, Theorems 1.2 and 2.1]. In the worst case, equality holds with $mn = s$, when $m = 1$ and the closed-loop poles are distinct.

All steps in the development of [35, Theorem 2.1] go through in similar fashion, except we have to show that the Jacobian matrix W_F, the derivatives of ω with respect to the elements in $\widehat{F}$, is full-ranked. The components of ω equals

$$\omega_i \equiv \frac{1}{n_i}\sum_{k=1}^{n_i}\left[\widehat{y}_{ik}^T(\widehat{A} + \widehat{B}\widehat{F})\widehat{x}_{ik} - \widehat{\lambda}_i\right] \quad , \quad i = 1, \cdots, s$$

with $\widehat{x}_{ik}$ and $\widehat{y}_{ik}$ being the vectors in X_i and Y_i respectively.

At (A, B, F, λ), differentiation of these entries with respect to elements in $\widehat{F}$ yields the Jacobian matrix

$$W_F \equiv \left[\ S_1X^T, \quad S_2X^T, \quad \cdots, \quad S_mX^T\ \right] \in \mathcal{C}^{s\times mn}$$

where

$$S_i \equiv \mathrm{diag}\left\{\frac{1}{n_1}b_i^TY_1, \cdots, \frac{1}{n_s}b_i^TY_s\right\} \in \mathcal{C}^{s\times n}$$

As X is selected to be nonsingular, the rank of W_F equals that of $S \equiv [S_1, \cdots, S_m]$. With the assumption of controllability, as in [35, §3], there are indices j' such that $y_j^Tb_{j'} \neq 0$ and $\mathrm{rank}S = s$. Consequently, W_F is full-ranked and the analyticity of F are proven.

Finally, recall from [35] that the power series expansion of F is not unique when $m > 1$, reflecting the degrees of freedom in X

3.2. Differentiability for General Λ

When Λ is not in Jordan form, there exists nonsingular Z such that $\Lambda = ZJZ^{-1}$ for some Jordan form J. The differentiability of $\widehat{Q}_i$ for Λ is can then be proven using J, with X replaced by XZ. Notice that the conditioning of Z or XZ (the closed-loop eigenvector matrix) may well be bad, but only X play a part in the SFPAP. Consequently, the analyticity of F can be proven similarly.

3.3. Differentiability for Λ in Schur Form

When Λ is in the Schur form and the corresponding Jordan structure is unknown, it follows from the previous development that the unitary Schur vector matrix X and Λ are differentiable, as $\Lambda = X^H AX$ can be consider to be a Rayleigh quotient of A.

Alternatively, consider the Schur decomposition of A (when $\epsilon = 0$):

$$AX = X\Lambda$$

with $X^H X = I$ and Λ in Schur form. With $\epsilon \neq 0$, let the perturbed equation become

$$A(\epsilon)X(\epsilon) = X(\epsilon)\Lambda(\epsilon)$$

We know that $X(\epsilon)$, whose columns span the invariant subspace for the whole spectrum of $A(\epsilon)$, is differentiable with respect to ϵ in some small neighbourhood $\mathcal{B}(0)$ of the origin. The perturbed equation can then be re-written as

$$A(\epsilon)\tilde{X}(\epsilon) = \tilde{X}(\epsilon)\tilde{\Lambda}(\epsilon) \ , \quad \tilde{X}(\epsilon)^H \tilde{X}(\epsilon) = I$$

with

$$T(\epsilon) \equiv \left[X(\epsilon)^H X(\epsilon)\right]^{-1/2} \ , \quad \tilde{X}(\epsilon) \equiv X(\epsilon)T(\epsilon) \ , \ \ \tilde{\Lambda}(\epsilon) = T(\epsilon)^{-1}\Lambda(\epsilon)T(\epsilon)$$

It is obvious that $\tilde{X}(\epsilon)$ and $\tilde{\Lambda}(\epsilon)$ are differentiable with respect to ϵ in some small neighbourhood $\mathcal{B}(0)$. Consequently, the analyticity of F can be proven.

3.4. Removal of Controllability Assumption

From the above proof of analyticity of F, we can see that the controllability assumption can be replaced by stabilizability when only the unstable poles are reassigned. The conditioning we really required is that W_F or S are full-ranked, which is necessary but not sufficient for the controllability of (A, B). Solvability of the SFPAP is later reflected by $\|\mathcal{N}^\dagger\|$ in (26) and (28).

3.5. Derivatives and Bounds of Derivatives

With the analyticity of F proven, we can now differentiate F with respect to components in A, B and Λ using (4):

$$F = B^\dagger (X\Lambda X^{-1} - A)$$

Let $A \equiv [a_{pq}]$, $B = [b_{pq}]$ and e_i be the ith column of I_n. We shall differentiate f_{ij}, the (i, j) component of F, with respect to the λ_k, a_{pq} and b_{pq}.

We first consider the derivative of $B^\dagger$ with respect to some parameter ρ:

$$\frac{\partial\{B^\dagger\}}{\partial\rho} = \frac{\partial\left\{(B^TB)^{-1}B^T\right\}}{\partial\rho} = \frac{\partial\left\{(B^TB)^{-1}\right\}}{\partial\rho}B^T + (B^TB)^{-1}\frac{\partial\{B^T\}}{\partial\rho}$$

$$= -\left[(B^TB)^{-1}\left(\frac{\partial\{B^T\}}{\partial\rho}B + B^T\frac{\partial B}{\partial\rho}\right)(B^TB)^{-1}\right]B^T + (B^TB)^{-1}\frac{\partial\{B^T\}}{\partial\rho}$$

We arrive at

$$\frac{\partial\{B^\dagger\}}{\partial\rho} = -B^\dagger\frac{\partial B}{\partial\rho}B^\dagger + (B^TB)^{-1}\frac{\partial\{B^T\}}{\partial\rho}\left(I - BB^\dagger\right) \tag{11}$$

Obviously, (11) generalizes the well-known formula $\frac{\partial\left\{B^{-1}\right\}}{\partial\rho} = -B^{-1}\frac{\partial B}{\partial\rho}B^{-1}$.

We then consider the derivatives of X with respect to some arbitrary parameter ρ. Rewrite (3) using the operators P and $\mathcal{N}$ as

$$\mathcal{N}(X) \equiv (I - BB^\dagger)(AX - X\Lambda) = (I - BB^\dagger)P(X) = 0$$

Similar to the development of (10) with $z = 0$, we have

$$\mathcal{N}\left(\frac{\partial X}{\partial\rho}\right) = -\left(-\frac{\partial B}{\partial\rho}B^\dagger - B\frac{\partial\{B^\dagger\}}{\partial\rho}\right)(AX - X\Lambda) - (I - BB^\dagger)\left(\frac{\partial A}{\partial\rho}X - X\frac{\partial\Lambda}{\partial\rho}\right)$$
$$= -(I - BB^\dagger)\left(\frac{\partial A}{\partial\rho}X - X\frac{\partial\Lambda}{\partial\rho} + \frac{\partial B}{\partial\rho}FX\right)$$

and

$$\frac{\partial X}{\partial\rho} = -\mathcal{N}^\dagger\left\{(I - BB^\dagger)\left[\left(\frac{\partial A}{\partial\rho} + \frac{\partial B}{\partial\rho}F\right)X - X\frac{\partial\Lambda}{\partial\rho}\right]\right\} \tag{12}$$

Here, the above linear equations can be rewritten with the Sylvester operator P expanded using Kronecker products.

Define the operator $P_c(\cdot) \equiv A_c(\cdot) - (\cdot)\Lambda$, and notice that

$$\mathcal{N}(\cdot) = (I - BB^\dagger)P(\cdot) = (I - BB^\dagger)P_c(\cdot)$$

We then consider, with respect to an arbitrary parameter ρ, the derivative

$$\begin{aligned}
\frac{\partial f_{ij}}{\partial \rho} &= \frac{\partial}{\partial \rho}\left\{ e_i^T B^\dagger (X\Lambda X^{-1} - A) e_j \right\} \\
&= e_i^T \left[\frac{\partial\left\{B^\dagger\right\}}{\partial \rho}(X\Lambda X^{-1} - A) \right. \\
&\qquad \left. + B^\dagger \left(\frac{\partial X}{\partial \rho}\Lambda X^{-1} + X\frac{\partial \Lambda}{\partial \rho}X^{-1} - X\Lambda X^{-1}\frac{\partial X}{\partial \rho}X^{-1} - \frac{\partial A}{\partial \rho} \right) \right] e_j \\
&= e_i^T \left\{ \frac{\partial\left\{B^\dagger\right\}}{\partial \rho} BF + B^\dagger \left[-P_c\left(\frac{\partial X}{\partial \rho}\right) X^{-1} + X\frac{\partial \Lambda}{\partial \rho}X^{-1} - \frac{\partial A}{\partial \rho} \right] \right\} e_j
\end{aligned}$$

Using (11) and (4), we obtain

$$\frac{\partial f_{ij}}{\partial \rho} = e_i^T \left\{ -B^\dagger \frac{\partial B}{\partial \rho} F + B^\dagger \left[-P_c\left(\frac{\partial X}{\partial \rho}\right) X^{-1} + X\frac{\partial \Lambda}{\partial \rho}X^{-1} - \frac{\partial A}{\partial \rho} \right] \right\} e_j \tag{13}$$

Substitute in (12), we obtain

$$\frac{\partial f_{ij}}{\partial \rho} = -e_i^T \left[B^\dagger \mathcal{M}\left(\frac{\partial B}{\partial \rho}F - X\frac{\partial \Lambda}{\partial \rho}X^{-1} + \frac{\partial A}{\partial \rho} \right) \right] e_j \tag{14}$$

with the operator

$$\mathcal{M}(\cdot) \equiv I(\cdot) - P_c\left\{ \mathcal{N}^\dagger \left[(I - BB^\dagger)(\cdot) \right] \right\}$$

Let $E_k = e_k$ when all λ_k are distinct and, in general, let E_k contain the columns of I_n which pick up the (generalized) eigenvectors in X and Y corresponding to λ_k. Standard calculations from (14) with $\rho = \lambda_k$ then yield

$$\frac{\partial f_{ij}}{\partial \lambda_k} = e_i^T \left[B^\dagger \mathcal{M}\left(X\frac{\partial \Lambda}{\partial \lambda_k}X^{-1} \right) \right] e_j = e_i^T \left[B^\dagger \mathcal{M}(XE_kE_k^TX^{-1}) \right] e_j$$

Consequently, we obtain

$$\frac{\partial f_{ij}}{\partial \lambda_k} = \left[B^\dagger \mathcal{M}\left(X_k Y_k^T \right) \right]_{ij} \tag{15}$$

With respect to $\rho = a_{pq}$ in (14), and with δ_{qj} denoting the Kronecker delta, we have

$$\frac{\partial f_{ij}}{\partial a_{pq}} = -e_i^T \left[B^\dagger \mathcal{M}\left(\frac{\partial A}{\partial a_{pq}} \right) \right] e_j = -e_i^T \left[B^\dagger \mathcal{M}(e_p \cdot e_q^T) \right] e_j$$

or, abusing the notation,

$$\frac{\partial f_{ij}}{\partial a_{pq}} = -\left[B^\dagger \mathcal{M} \right]_{ip} \cdot \delta_{qj} \tag{16}$$

With respect to $\rho = b_{pq}$ in (14), we deduce that

$$\frac{\partial f_{ij}}{\partial b_{pq}} = -e_i^T \left[B^\dagger \mathcal{M} \left(\frac{\partial B}{\partial b_{pq}} F \right) \right] e_j = -e_i^T \left[B^\dagger \mathcal{M}(e_p \cdot e_q^T F) \right] e_j$$

We thus obtain

$$\frac{\partial f_{ij}}{\partial b_{pq}} = -\left[B^\dagger \mathcal{M} \right]_{ip} \cdot f_{qj} \tag{17}$$

The derivatives in (15), (16) and (17) are much simpler than the those in §2.

Similar to (5), we have the differential relationship:

$$\begin{aligned} df_{ij} &= \sum_{k=1}^{s} \frac{\partial f_{ij}}{\partial \lambda_k} d\lambda_k + \sum_{p=1}^{n} \left[\sum_{q=1}^{n} \frac{\partial f_{ij}}{\partial a_{pq}} da_{pq} + \sum_{q=1}^{m} \frac{\partial f_{ij}}{\partial b_{pq}} db_{pq} \right] \\ &= \frac{\partial f_{ij}}{\partial \lambda} \cdot d\lambda + \sum_{p=1}^{n} \left(\frac{\partial f_{ij}}{\partial a_{p\bullet}} \cdot da_{p\bullet} + \frac{\partial f_{ij}}{\partial b_{p\bullet}} \cdot db_{p\bullet} \right) \end{aligned}$$

where $a_{p\bullet}$ and $b_{p\bullet}$ are the pth row of A and B, respectively, and $\lambda \equiv [\lambda_1, \cdots, \lambda_s]^T$. Substitute in (15), (16) and (17), we have

$$df_{ij} = \begin{bmatrix} \left[B^\dagger \mathcal{M} \left(X_1 Y_1^T \right) \right]_{ij} \\ \vdots \\ \left[B^\dagger \mathcal{M} \left(X_s Y_s^T \right) \right]_{ij} \end{bmatrix} \cdot d\lambda - \sum_{p=1}^{n} \left[B^\dagger \mathcal{M} \right]_{ip} (da_{p\bullet} \cdot e_j + db_{p\bullet} \cdot F_{\bullet j})$$

or, putting into matrix form,

$$dF = B^\dagger \mathcal{M}(X \; d\Lambda \; X^{-1} - dA - dB \; F) \tag{18}$$

(Putting the differentials in long vectors and using the implicit function theorem produce the more complicated but equivalent expressions in §2.1 and [35]. See also the analogous expansion in (32).)

Let $\|d\lambda\| \le \epsilon_\lambda$, $\|dA\| \le \epsilon_A$ and $\|dB\| \le \epsilon_B$. With consistent norms, we have the error bound (c.f. (6))

$$\|dF\| \le \tilde{\kappa}_\lambda \epsilon_\lambda + \tilde{\kappa}_A \epsilon_A + \tilde{\kappa}_B \epsilon_B \tag{19}$$

with the bounds of derivatives

$$\begin{aligned} \tilde{\kappa}_\lambda &\equiv \|B^\dagger \mathcal{M}[X(\cdot)X^{-1}]\| \\ \tilde{\kappa}_A &\equiv \|B^\dagger \mathcal{M}(\cdot)\| \\ \tilde{\kappa}_B &\equiv \|B^\dagger \mathcal{M}[(\cdot)F]\| \end{aligned}$$

Note that

$$\tilde{\kappa}_\lambda \leq \|B^\dagger \mathcal{M}\| \, \kappa_X \ , \quad \kappa_X \equiv \|X\| \, \|X^{-1}\| \tag{20}$$

We have applied properties of norms in the above definitions of condition numbers (c.f. Theorems 2.1–2.5), which over-estimates errors. It will be far more accurate to use (18) directly in any error estimation as in §3.3. Sharper condition numbers are possible, with different application of the theory of norms to (18). It will not substantially improve the theory in quality and we shall not attempt that here.

Combining the individual $\tilde{\kappa}$'s above, we can define the overall absolute condition number for the SFPAP as

$$\tilde{\kappa} \equiv \sqrt{\tilde{\kappa}_\lambda^2 + \tilde{\kappa}_A^2 + \tilde{\kappa}_B^2} \tag{21}$$

Using (20), we have

$$\tilde{\kappa} \leq \|B^\dagger \mathcal{M}\| \sqrt{\kappa_X^2 + 1 + \|F\|^2} \tag{22}$$

The above result is consistent with those obtained in Theorem 2.4 [27, Theorem 4.1]. Condition numbers for relative errors can also be defined in similar fashion:

$$\tilde{\kappa}_r \equiv \sqrt{\tau_\lambda^2 + \tau_A^2 + \tau_B^2} \tag{23}$$

with

$$\begin{aligned}
\tau_\lambda &= \frac{\|B^\dagger \mathcal{M}[X(\cdot)X^{-1}]\| \, \|\Lambda\|}{\|F\|} \\
\tau_A &= \frac{\|B^\dagger \mathcal{M}(\cdot)\| \, \|A\|}{\|F\|} \\
\tau_B &= \frac{\|B^\dagger \mathcal{M}[(\cdot)F]\| \, \|B\|}{\|F\|}
\end{aligned}$$

The above condition number are defined from upper bounds of derivatives. We shall derived another set of condition numbers, using the theory of condition by Rice [30] next. All these condition numbers share similar qualities and have similar components. We shall interpret the proper condition numbers κ and κ_r in (24) and (27) respectively, instead of the $\tilde{\kappa}$ and $\tilde{\kappa}_r$, later. A more explicit formula for the expression $\|Z_0 \mathcal{M}[Z_1(\cdot)Z_2]\|$ will be presented in the next subsection.

4. Rice Condition Numbers

From [30], we can define the absolute condition number as

$$\kappa \equiv \lim_{\epsilon \to 0} \sup_{\|M\|=\epsilon} \frac{\|dF\|}{\|M\|} \ , \quad M \equiv (d\Lambda, dA, dB) \tag{24}$$

If Λ is chosen in Schur form with X being unitary, $\|d\Lambda\| = \|Xd\Lambda X^{-1}\|$ for the 2- and F-norms. It is then easy to show that

$$\kappa = \sup_{\|M\|=1} \left\| B^\dagger \mathcal{M} \left(M \begin{bmatrix} I \\ I \\ F \end{bmatrix} \right) \right\| = \|B^\dagger \mathcal{M}\| \sqrt{2 + \|F\|^2} \tag{25}$$

Note that the equality is exact.

When Λ is not in Schur form and X is not unitary, we can derive from (24) and (18) the bound

$$\kappa \le \lim_{\epsilon \to 0} \sup_{\|M\|=\epsilon} \frac{\|B^\dagger \mathcal{M}(X\ d\Lambda\ X^{-1})\| + \|B^\dagger \mathcal{M}(dA + dB\ F)\|}{\|M\|}$$

or

$$\begin{aligned} \kappa &\le \|B^\dagger \mathcal{M}[X(\cdot)X^{-1}]\| + \|B^\dagger \mathcal{M}\| \sqrt{1 + \|F\|^2} \\ &\le \|B^\dagger \mathcal{M}\| \ (\kappa_X + \sqrt{1 + \|F\|^2}) \end{aligned} \tag{26}$$

Notice the similar expressions in Theorem 2.4 [27, Theorem 3.2].

The more useful condition number for relative errors can be defined, similar to (24), as

$$\kappa_r \equiv \lim_{\epsilon \to 0} \sup_{\|M\|=\epsilon} \frac{\|B^\dagger \mathcal{M}(X\ d\Lambda\ X^{-1} - dA - \ dB\ F)\|}{\|F\|\ \|M\|} \tag{27}$$

with $M \equiv (\alpha\ d\Lambda, \beta\ dA, \gamma\ dB)$, $\alpha \equiv \|\Lambda\|^{-1}$, $\beta \equiv \|A\|^{-1}$ and $\gamma \equiv \|B\|^{-1}$.

When X is unitary and with $c_B \equiv \|B\|\ \|B^\dagger\|$ and $c_0 \equiv \|\mathcal{M}\|$, we have the exact relation similar to (25):

$$\kappa_r = \frac{\|B^\dagger \mathcal{M}\|}{\|F\|} \left(\|\Lambda\|^2 + \|A\|^2 + \|B\|^2\ \|F\|^2 \right)^{1/2} = c_0\ c_B\ \left(\frac{\|\Lambda\|^2 + \|A\|^2}{\|B\|^2\ \|F\|^2} + 1 \right)^{1/2}$$

When X is not unitary, we have the bound

$$\begin{aligned} \kappa_r &\le \frac{\|B^\dagger \mathcal{M}[X(\cdot)X^{-1}]\|\ \|\Lambda\|}{\|F\|} + \|B^\dagger \mathcal{M}\| \left\{ \frac{\|A\|^2}{\|F\|^2} + \|B\|^2 \right\}^{1/2} \\ &\le c_0\ c_B\ \left[\kappa_X \frac{\|\Lambda\|}{\|B\|\ \|F\|} + \left(\frac{\|A\|^2}{\|B\|^2\ \|F\|^2} + 1 \right)^{1/2} \right] \end{aligned} \tag{28}$$

We have tried not to use any inequality of norms unnecessarily and the above bounds should be reasonably sharp.

Finally, the *exact* condition numbers can be derived, using Kronecker products to rewrite (18) as an operator on $\mathcal{C}^{n+n^2+mn}$. Let $v(\cdot)$ be the operator which stacks the columns of a matrix into a long vector and let $\mathrm{mat}(\cdot)$ be its inverse. We first define the generic matrix operator

$$\begin{aligned} M(Z_0, Z_1, Z_2)v(Y) &\equiv v\{Z_0\mathcal{M}(Z_1YZ_2)\} \\ &= v\left\{Z_0\left[Z_1YZ_2 - P_c\mathcal{N}^\dagger\left((I_n - BB^\dagger)Z_1YZ_2\right)\right]\right\} \end{aligned}$$

With $B^\perp \equiv I_n - BB^\dagger$, M can be shown to satisfy

$$M(Z_0, Z_1, Z_2) = I_n \otimes Z_0 -$$

$$(I_n \otimes Z_0)(I_n \otimes A_c - \Lambda^T \otimes I_n)\left[I_n \otimes (B^\perp A_c) - \Lambda^T \otimes B^\perp\right]^\dagger \left[(Z_2^T \otimes (B^\perp Z_1)\right]$$

The absolute and relative condition numbers, on $\mathcal{C}^{n+n^2+mn}$, are

$$\hat{\kappa} \equiv \left\|\left[M(B^\dagger, X, X^{-1}),\ M(B^\dagger, I_n, I_n),\ M(B^\dagger, I_n, F)\right]\right\| \tag{29}$$

and

$$\hat{\kappa}_r \equiv \frac{\left\|\left[M\left(B^\dagger, \frac{\|\Lambda\|}{\|F\|}X, X^{-1}\right), M\left(B^\dagger, \frac{\|A\|}{\|F\|}I_n, I_n\right), M\left(B^\dagger, \frac{\|B\|}{\|F\|}I_n, F\right)\right]\right\|}{\|F\|} \tag{30}$$

These condition numbers are exact but are more expensive to calculate, and may only be of theoretical interest. They yield similar qualitative interpretation as other bounds previously derived. Notice that the presence of the operator $\mathcal{N}^\dagger$ in $\mathcal{M}$ which reflects the sensitivity of the selection of X and thus the solvability of our problem. The relationship between c_0 and $\|\mathcal{N}^\dagger\|$ is complicated as the latter appears only in an upper bound of the former. However, it is realistic to expect a large perturbation error when $\|\mathcal{N}^\dagger\|$ is large and the problem is nearly unsolvable.

For the bounds in (21) and (23), $v\{Z_0\mathcal{M}(Z_1YZ_2)\} = M(Z_0, Z_1, Z_2)v(Y)$ for various Z_i, and

$$\|Z_0\mathcal{M}[Z_1(\cdot)Z_2]\| = \|M(Z_0, Z_1, Z_2)\|$$

4.1. Interpretation of Results

We now attempt to interpret the inequality (28). The most important factor in the error bound is c_0. It is easy to obtain the upper bound:

$$c_0 = \left\|I_n(\cdot) - P_c\left\{\mathcal{N}^\dagger\left[B^\perp(\cdot)\right]\right\}\right\| \leq 1 + \frac{\|P_c\|}{\sigma_{\min}(N)} \leq 1 + \frac{\|A + BF\| + \|\Lambda\|}{\sigma_{\min}(N)}$$

where N (in $\mathcal{C}^{n^2}$) is the matrix representation of the operator $\mathcal{N}$ (in $\mathcal{C}^{n \times n}$) and

$$N \equiv I_n \otimes (B^{\perp} A) - \Lambda^T \otimes B^{\perp}$$

The minimum singular value of N is exactly the distance from uncontrollability. However, it is uncertain that a large upper bound implies much on c_0. The norm should be estimated in the structured subspaces corresponding to the problem under investigation, which may well be uncontrollable. From the origin of $\mathcal{N}$ in (3), $\|\mathcal{N}^{\dagger}\|$ clearly reflects the sensitivity of the selection of X from the corresponding invariant subspaces. However, $\mathcal{N}^{\dagger}$ is only part of a more complicated $\mathcal{M}$ (see §3.4 for more results) and c_0, and a large $\|\mathcal{N}^{\dagger}\|$ does not necessarily means a large c_0. Nonetheless, it is prudent to expect a large c_0 and a large perturbation error when the problem is nearly unsolvable.

The coefficient $c_B \equiv \|B\| \; \|B^{\dagger}\|$ relates to the conditioning of B and our assumption that it is full-ranked. When an input matrix B has columns which are nearly linearly dependent to other columns, we should not use the problematic columns in controlling the system, effectively deleting them from B. The resulting B will be well-conditioned with a small c_B. The factor $\|F\|$ is irrelevant, as it has disappeared from the numerators when relative errors are considered. Their presence in the denominators is not important, as they should be cancelled out with $\|\Lambda\|$ and $\|A\|$ in the numerators. It may not be appropriate to interpret that the larger the size of F the smaller the (bound of the) relative condition number. Consequently, this is no reason to suggest the maximization of feedback gain. Similarly, the factor $\|F\|$ in κ in (26) cannot be used to justify the minimization of F. Minimizing feedback gain is obviously a good practice, related to energy and other engineering argument, but (27) only reflects the unrelated but common phenomenon that absolute errors are proportional to the quantities being calculated.

The factor κ_X in (26) and (28) can be controlled by choosing Λ and X well. This, of course, may not be possible for ill-conditioned problems with Λ in Jordan form. For example in second-order systems or other highly structured problems, the restriction in structure shrinks the subspaces from which the eigenvectors x_j are chosen, possibly making κ_X large. Fortunately, X is easily calculated, or chosen explicitly [19] or implicitly when $m > 1$, for the SFPAP and its conditioning can usually be checked numerically. With Λ chosen alternatively, e.g., to be in Schur form as in [3, 5, 36], the ill-conditioning of X is passed on and κ_X should be much smaller. Note that the selection of X to satisfy (3) is algorithm-dependent and is irrelevant to the conditioning of the SFPAP. As mentioned before, any ill-conditioning in the eigen-system in A_c is delayed until more spectral information is sought. However, there is no reason why we have to use Jordan forms in Λ so long as the poles in Λ are assigned as required.

As Λ can be varied [3, 5, 36], the minimization of $\|\Lambda\|$ in (28) becomes important. This minimization was carried out in a least squares sense in [3, 5]. When Λ

is chosen to be in Schur form with its diagonal λ (the closed-loop poles) specified, it is the size of the upper triangular part of Λ we need to control. This quantity is well-known to be a measure of departure from normality and is closely related to the conditioning of the eigenvalues of Λ. Thus in this sense, controlling $\|\Lambda\|$ improves the conditioning of the closed-loop eigenvalues.

It is unavoidable to apply properties of norms and compare upper bounds of derivatives, but we have to do so with care. Large upper bounds may or may not imply ill-conditioning, but small upper bounds definitely indicate good conditioning, as in our case when c_B and κ_X are small. Equally, products of ill-conditioned matrices may well be well-conditioned. It may be far better to use the results in (18), rather than the related results after more applications of norm-inequalities, such as (26) or (28). Of course, exact condition numbers in (29) and (30) can also be used, as well as the analogous condition numbers in (21) and (23).

Finally, we conclude that the SFPAP is not intrinsically ill-conditioned, when B and Λ are well-chosen and when c_0 is not large. This is an interesting result, independent of controllability/stabilzability and open/closed-loop eigen-structures. Our approach in using non-Jordan Λ is somewhat similar to that for the "synthesis problem' 'in [23], where consistent results were obtained.

Notice that the above results are consistent with those in [26, 27, 17], where the possibility of Λ being non-Jordan somehow escapes attention. The conditioning of X affects the conditioning of the SFPAP and the numerical experiments in [17] confirm that. In addition, the increase in the size of κ and κ_X as n increases, for a fixed small value of m, is expected. After all, controlling a large number of state variables with a small number of inputs is well-known to be difficult. Still, with Λ chosen well and not in Jordan form, there is no reason why X (containing, e.g., approximate Schur vectors) cannot be well-conditioned. A naive search, given infinite time, will yield an acceptable closed-loop system matrix A_c, which can then be used as Λ with a perfectly conditioned $X = I$. More realistically, we need $\Lambda \approx ZA_cZ^{-1}$ (for some well-conditioned Z) and $\kappa_X \approx \kappa_Z$ will be small. Again, the choice of Λ or X is algorithm-dependent and is not our concern.

Ultimately, the SFPAP may be redefined (c.f. definition in [23]), with more restrictions put onto the data in Λ. This will remove the freedom in Λ which gives us the technical well-conditioning of the SFPAP. The analysis and condition number, in terms of a general Λ and X, still hold for these more restricted problems, and their conditioning is essentially equivalent to that of X.

4.2. Perturbation Expansions

Similar to (4), let us consider

$$\widetilde{F} = \tilde{B}^{\dagger}(\widetilde{X}\widetilde{\Lambda}\widetilde{X}^{-1} - \widetilde{A}) \tag{31}$$

with $\widetilde{\cdot}$ indicating perturbed or approximate quantities. Let $\delta A \equiv \widetilde{A} - A$, $\delta B \equiv \widetilde{B} - B$

and $\delta\Lambda \equiv \widetilde{\Lambda} - \Lambda$. Subtracting (4) from (31) produces, almost identical to (18),

$$\delta F \equiv \widetilde{F} - F = B^{\dagger}\mathcal{M}(X\ \delta\Lambda\ X^{-1} - \delta A - \delta B\ F) + \Delta \tag{32}$$

where Δ contains the higher order terms. Derivatives, condition numbers and error bounds can easily be obtained through (32). More usefully, (18) and (32) can be used to provide accurate first-order estimate of δF or F, when δA, δB and $\delta\Lambda$ are known or can be estimated.

4.3. Error in Terms of Residuals

For some approximate feedback matrix F, consider the residual

$$R \equiv (A + BF)X - X\Lambda \tag{33}$$

Let δF, δX and $\delta\Lambda$ be the refinements to F, X and Λ, respectively, which satisfy

$$[A + B(F + \delta F)]\,(X + \delta X) - (X + \delta X)(\Lambda + \delta\Lambda) = 0$$

Simple calculations produce

$$P_c(\delta X) + B\ \delta F\ X - X\ \delta\Lambda \approx -R$$

or

$$(I_n \otimes A_c - \Lambda^T \otimes I_n)\ v(\delta X) + (X^T \otimes B)\ v(\delta F) \tag{34}$$

$$-(I_n \otimes X)\ v(\delta\Lambda) \approx -v(R)$$

with $\Lambda + \delta\Lambda$ containing the desirable closed-loop spectrum and ignoring higher order terms. With Λ chosen and $\delta\Lambda = 0$, we can solve the resulting linear equation in a least squares sense to obtain

$$\begin{bmatrix} v(\delta X) \\ v(\delta F) \end{bmatrix} = -\begin{bmatrix} I_n \otimes A_c - \Lambda^T \otimes I_n, & X^T \otimes B \end{bmatrix}^{\dagger} v(R)$$

The constrained computation involving a full nonzero δX will be much more difficult and is ignored here. An intermediate situation involves an Λ in Schur form and a strictly upper triangular $\delta\Lambda$, for which an unconstrained linear equation similar to (34) can be obtained.

5. Conclusion

We have investigated the condition of the state-feedback pole assignment problem for given closed-loop poles with fixed structure. The usual controllability condition for solvability is relaxed. Condition numbers and error bounds have been derived, implying that the problem can be well-conditioned, even for systems with defective eigenvalues. We have included only the theoretical results in this paper. Related numerical experiments will be reported elsewhere. We hope the results in this paper will be useful to others interested in pole assignment in particular and control system design in general, providing new insights into and motivations for new algorithms.

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