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LEVITIN-POLYAK WELL-POSEDNESS FOR GENERALIZED QUASI-VARIATIONAL INCLUSION AND DISCLUSION PROBLEMS AND OPTIMIZATION PROBLEMS WITH CONSTRAINTS

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Abstract. In this paper, Levitin-Polyak well-posedness for generalized quasi-variational inclusion and disclusion problems are introduced and studied. Necessary and sufficient conditions for Levitin-Polyak well-posedness of these problems are proved. Moreover, Levitin-Polyak well-posedness for optimization problems with generalized quasi-variational inclusion problems, generalized quasi-variational disclusion problems and scalar generalized quasi-equilibrium problems as constraints are also given under some suitable conditions.

1. Introduction

It is well known that the well-posedness is very important for both optimization theory and numerical methods of optimization problems, which guarantees that, for every approximating solution sequence, there is a subsequence which converges to a solution. Well-posedness of unconstrained and constrained scalar optimization problems was first introduced and studied by Tykhonov [45] and Levitin and Polyak [21], respectively. Since then, various concepts of well-posedness have been introduced and extensively studied for minimization problems and vector optimization problems by many authors (see, for example, [2, 6, 7, 15, 38, 40, 42, 47, 48] and the references therein).

In recent years, the concept of well-posedness has been generalized to several related problems: variational inequality problems [4, 5, 9, 11, 12, 24, 25, 34, 44],

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saddle point problems [3], Nash equilibrium problems [24, 35-37, 39], inclusion problems [5, 11, 19, 20], and fixed point problems [5, 11, 19, 20, 41, 46].

Recently, Fang et al. [10] generalized the concept of well-posedness to equilibrium problems and optimization problems with equilibrium constraints and established some metric characterizations of well-posedness for equilibrium problems and optimization problems with equilibrium constraints. Kimura et al. [17] further generalized it to vector equilibrium problems. Long et al. [32] generalized the concept of Levitin-Polyak well-posedness to equilibrium problems with functional constraints and obtained some metric characterizations and sufficient conditions for Levitin-Polyak well-posedness of equilibrium problems with functional constraints. Also, Long and Huang [33] introduced and studied $\alpha-$well-posedness for sysmetric quasi-equilibrium problems. Li and Li [22] introduced and studied Levitin-Polyak well-posedness for vector equilibrium problems. Huang et al. [14] generalized it to vector quasi-equilibrium problems. Li et al. [23] further generalized it to generalized vector quasi-equilibrium problems. They obtained some criteria and metric characterizations of the Levitin-Polyak well-posedness and established the relations between Levitin-Polyak well-posedness of optimization problems and Levitin-Polyak well-posedness of generalized vector quasi-equilibrium problems. Very recently, Lin and Chuang [31] further extended the notion of well-posedness to variational inclusion and disclusion problems and optimization problems with variational inclusion and disclusion problems as constraints. They proved some results concerned with the well-posedness in the generalized sense for variational inclusion problems and variational disclusion problems, the well-posedness for optimization problems with variational inclusion problems, variational disclusion problems and scalar equilibrium problems as constraints.

On the other hand, it is well known that the quasi-variational inclusion problem is an important generalization of the variational inclusion problem, which contains lots of important problems as special cases and has many applications, like variational disclusion problems, minimax inequalities, equilibrium problems, saddle point problems, optimization theory, bilevel problems, mathematical program with equilibrium constraint, variational inequalities, fixed point problems, coincidence point problems, Ekeland's variational principle, etc. For details, we refer the reader to [13, 26-31, 43] and the references therein.

Motivated and inspired by the work mentioned above, in this paper, we shall investigate Levitin-Polyak well-posedness (for short, LP well-posedness) for generalized quasi-variational inclusion and disclusion problems and optimization problems with generalized quasi-variational inclusion problems, generalized quasi-variational disclusion problems and scalar generalized quasi-equilibrium problems as constraints. The results presented in this paper improve and generalize some known results in Huang et al. [14], Li and Li [22] and Li et al. [23].

2. Preliminaries

In this section, we shall recall some definitions and lemmas used in the sequel.

Definition 2.1. ([1]). Let X and Y be two topological spaces. A multivalued mapping $T : X \to 2^Y$ is said to be

(i) upper semi-continuous (for short, $u.s.c.$) at $x \in X$ if, for each open set V in Y with $T(x) \subseteq V$, there exists an open neighborhood $U(x)$ of x such that $T(x') \subseteq V$ for all $x' \in U(x)$;

(ii) lower semi-continuous (for short, $l.s.c.$) at $x \in X$ if, for each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $T(x') \cap V \neq \emptyset$ for all $x' \in U(x)$;

(iii) $u.s.c.$ (resp. $l.s.c.$) on X if it is $u.s.c.$ (resp. $l.s.c.$) at every point $x \in X$;

(iv) continuous on X if it is both $u.s.c.$ and $l.s.c.$ on X;

(v) closed if the graph of T is closed, i.e., the set $Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$;

(vi) open if the graph of T is open in $X \times Y$.

Lemma 2.1. ([1]). *Let X and Y be two topological spaces, $T : X \to 2^Y$ a multivalued mapping.*

(*i*) *If T is u.s.c. and closed-valued, then T is closed;*

(*ii*) *If T is closed and Y is compact, then T is $u.s.c.$;*

(*iii*) *If T is compact-valued, then T is u.s.c. at $x \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x$ and for any net $\{y_\alpha\} \subseteq Y$ with $y_\alpha \in T(x_\alpha)$, there exist a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ and $y \in T(x)$ such that $y_\beta \to y$;*

(*iv*) *T is $l.s.c.$ at $x \in X$ if and only if for any $y \in T(x)$ and for any net $\{x_\alpha\}$ with $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ with $y_\alpha \in T(x_\alpha)$ such that $y_\alpha \to y$.*

Definition 2.2. Let A and B be nonempty subsets of a metric space (E, d). The Hausdorff distance $\mathcal{H}(\cdot, \cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) := \max\{e(A, B), e(B, A)\},$$

where $e(A, B) := \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} d(a, b)$. Let $\{A_n\}$ be a sequence of nonempty subsets of E. We say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic, we refer the readers to [18].

3. LP Well-posedness for Generalized Quasi-variational Inclusion and Disclusion Problems

Throughout this paper, unless otherwise specified, we use the following notations and assumptions. Let (E, d) be a metric space, $X \subseteq E$ and $X_0 \subseteq X$ be nonempty closed subsets. Let F and Z be Hausdorff topological vector spaces and $Y \subseteq F$ be a nonempty closed subset. Let $K : X \multimap X$, $T : X \multimap Y$ and $G_1, G_2 : X \times Y \times X \multimap Z$ be multivalued mappings. Let $e : X \to Z$ be a continuous mapping.

We consider the following generalized quasi-variational inclusion and disclusion problems.

(GQVIP): Find $\bar{x} \in X_0$ such that $\bar{x} \in K(\bar{x})$ and there exists $\bar{y} \in T(\bar{x})$ satisfying

$$0 \in G_1(\bar{x}, \bar{y}, u),\ \forall\, u \in K(\bar{x}).$$

(GQVDP): Find $\bar{x} \in X_0$ such that $\bar{x} \in K(\bar{x})$ and there exists $\bar{y} \in T(\bar{x})$ satisfying

$$0 \notin G_2(\bar{x}, \bar{y}, u),\ \forall\, u \in K(\bar{x}).$$

Denote by S_1 and S_2 the solution sets of (GQVIP) and (GQVDP), respectively.

Definition 3.1. A sequence $\{x_n\} \subseteq X$ is called

(i) an LP approximating solution sequence for (GQVIP) if there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each n,

$$d(x_n, X_0) \le \varepsilon_n, \tag{3.1}$$

$$d(x_n, K(x_n)) \le \varepsilon_n, \tag{3.2}$$

$$0 \in G_1(x_n, y_n, u) + \varepsilon_n e(x_n),\ \forall\, u \in K(x_n); \tag{3.3}$$

(ii) an LP approximating solution sequence for (GQVDP) if there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each n,

$$0 \notin G_2(x_n, y_n, u) + \varepsilon_n e(x_n),\ \forall\, u \in K(x_n). \tag{3.4}$$

and (3.1) and (3.2) hold.

Definition 3.2. (GQVIP)(resp. (GQVDP)) is said to be LP well-posed if the solution set S_1 (resp. S_2) of (GQVIP)(resp. (GQVDP)) is nonempty and every LP approximating solution sequence for (GQVIP) (resp. (GQVDP)) has a subsequence which converges to some point of S_1 (resp. S_2).

Remark 3.1. (i) If $F = E$, $Y = X_0 = X$, $T = I$(identical mapping) and for any $x, u \in X, y \in Y$, $e(x) \in intC(x)$ and

$$G_1(x, y, u) = G_1(x, u) = G(x, u) + [intC(x)]^c,$$

where A^c denotes the coset of A and $G : X \times X \multimap Z$ and $C : X \multimap Z$ are multivalued mappings such that for each $x \in X$, $C(x)$ is a proper, closed and convex cone with nonempty interior, i.e., $intC(x) \neq \emptyset$, then the LP well-posedness for (GQVIP) reduces to the LP well-posedness for (GVQEP 1) introduced by Li et al. [23]. Moreover, if

$$G_2(x, y, u) = G_2(x, u) = G(x, u) + [C(x)]^c, \quad \forall (x, y, u) \in X \times Y \times X,$$

and the mapping $e(x)$ is replaced by $-e(x)$, then the LP well-posedness for (GQVDP) reduces to the LP well-posedness for (GVQEP 2) introduced by Li et al. [23].

(ii) If E is a real Banach space, $F = E$, $Y = X_0 = X$, $T = I$ and for any $x, u \in X, y \in Y$, $e(x) = e \in intC$ and

$$G_1(x, y, u) = G_1(x, u) = f(x, u) + [intC]^c,$$

where C is a pointed, closed and convex cone in Z with $intC \neq \emptyset$ and $f : X \times X \to Z$ is a vector-valued mapping, then the LP well-posedness for (GQVIP) reduces to the well-posedness for (VQE) introduced by Huang et al. [14].

(iii) If $Y = F = X = E$, $T = I$ and for any $x, u \in X, y \in Y$, $e(x) \in intC(x)$, $K(x) = X_0$ and

$$G_1(x, y, u) = G_1(x, u) = f(x, u) + [intC(x)]^c,$$

where $f : E \times E \to Z$ is a vector-valued mapping and $C : E \multimap Z$ is a multivalued mapping such that for each $x \in X$, $C(x)$ is a pointed, closed and convex cone with $intC(x) \neq \emptyset$, then the LP well-posedness for (GQVIP) reduces to the type I LP well-posedness for (VEP) introduced by Li and Li [22].

(iv) If E is a real Banach space, $X_0 = X$, $Z = R$, $F = Y = E^*$(the dual space of E) and for all $x, u \in X, y \in Y$, $e(x) = -1$ and

$$G_1(x, y, u) = \langle y, \eta(x, u) \rangle + f(x) - f(u) + R_+,$$

where $R_+ = [0, +\infty)$ and $\eta : X \times X \to E$ is a vector-valued mapping and $f : X \to R$ is a real-valued function, then the LP well-posedness for (GQVIP) reduces to the well-posedness for (MQVLI) introduced by Ceng et al. [4].

(v) If E is a normed space, $F = E$, $Y = X$, $Z = R$, $T = I$ and for any $x, u \in X, y \in Y$, $e(x) = 1$, $K(x) = X_0$ and

$$G_1(x, y, u) = G_1(x, u) = f(x, u) - R_+,$$

where $f : X \times X \to R$ is a real-valued function, then the LP well-posedness for (GQVIP) reduces to the type I LP well-posedness for (EP) introduced by Long et al. [32].

(vi) If E is a real reflexive Banach space, $F = E$, $Y = X_0 = X$, $Z = R$, $T = I$ and for any $x, u \in X, y \in Y$, $e(x) = 1$ and

$$G_1(x, y, u) = G_1(x, u) = \langle A(x), u - x\rangle - R_+,$$

where $A : X \to E^*$ is a vector-valued mapping and E^* is the dual spaces of E, then the LP well-posedness for (GQVIP) reduces to the well-posedness in the generalized sense for (QVI) introduced by Lignola [25].

For each $\varepsilon > 0$, let

$$\begin{aligned} M_1(\varepsilon) =&\{x \in X : d(x, X_0) \leq \varepsilon,\ d(x, K(x)) \leq \varepsilon \\ &\text{and } \exists\, y \in T(x) \text{ s.t. } 0 \in G_1(x, y, u) + \varepsilon e(x),\ \forall\, u \in K(x)\}; \\ M_2(\varepsilon) =&\{x \in X : d(x, X_0) \leq \varepsilon,\ d(x, K(x)) \leq \varepsilon \\ &\text{and } \exists\, y \in T(x) \text{ s.t. } 0 \notin G_2(x, y, u) + \varepsilon e(x),\ \forall\, u \in K(x)\}. \end{aligned}$$

Define the approximating solution sets for (GQVIP) and (GQVDP), respectively, by

$$\Omega_1(\varepsilon) = M_1(\varepsilon) \cup S_1 \quad \text{and} \quad \Omega_2(\varepsilon) = M_2(\varepsilon) \cup S_2.$$

Next, we consider the properties for $\Omega_1(\varepsilon)$ and $\Omega_2(\varepsilon)$.

Property 3.1. Assume that K is closed-valued and T is compact-valued. For each $(x, u) \in X \times X$,

(i) if $y \multimap G_1(x, y, u)$ is closed, then $S_1 = \cap_{\varepsilon>0}\Omega_1(\varepsilon)$;

(ii) if $y \multimap G_2(x, y, u)$ is open, then $S_2 = \cap_{\varepsilon>0}\Omega_2(\varepsilon)$.

Proof. (i) Clearly, $S_1 \subseteq \cap_{\varepsilon>0}\Omega_1(\varepsilon)$. Hence, we only need to show that $\cap_{\varepsilon>0}\Omega_1(\varepsilon) \subseteq S_1$. Suppose to the contrary that there exists some $x^* \in \cap_{\varepsilon>0}\Omega_1(\varepsilon)$ such that $x^* \notin S_1$. Then, for each $\varepsilon > 0$, $x^* \in \Omega_1(\varepsilon)\backslash S_1$. Hence, for each $n \in N$, $x^* \in \Omega_1(\frac{1}{n})\backslash S_1$, and so there exists $y_n \in T(x^*)$ such that

$$d(x^*, X_0) \leq \tfrac{1}{n}, \tag{3.5}$$

$$d(x^*, K(x^*)) \leq \tfrac{1}{n}, \tag{3.6}$$

$$0 \in G_1(x^*, y_n, u) + \tfrac{1}{n}e(x^*),\ \forall\, u \in K(x^*). \tag{3.7}$$

Note that X_0 and $K(x^*)$ are closed sets. Then, by (3.5) and (3.6), we have $x^* \in X_0$ and $x^* \in K(x^*)$. Since $\{y_n\} \subseteq T(x^*)$ and $T(x^*)$ is a compact set, there exist $y^* \in T(x^*)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y^*$ as $k \to \infty$, and so, for each $k \in N$,

$$0 \in G_1(x^*, y_{n_k}, u) + \frac{1}{n_k} e(x^*), \ \forall\, u \in K(x^*).$$

For each $u \in K(x^*)$, since $y \multimap G_1(x^*, y, u)$ is closed, we get

$$0 \in G_1(x^*, y^*, u).$$

Thus $x^* \in S_1$. This is a contradiction and so $\cap_{\varepsilon>0}\Omega_1(\varepsilon) \subseteq S_1$. Therefore $S_1 = \cap_{\varepsilon>0}\Omega_1(\varepsilon)$.

(ii) Let $G_1 : X \times Y \times X \multimap Z$ be defined by $G_1(x, y, u) = Z \backslash G_2(x, y, u)$ for each $(x, y, u) \in X \times Y \times X$. Then $S_1 = S_2$ and $M_1(\varepsilon) = M_2(\varepsilon)$, and so $\Omega_1(\varepsilon) = \Omega_2(\varepsilon)$. Since $y \multimap G_2(x, y, u)$ is open, $y \multimap G_1(x, y, u)$ is closed. By (i), the proof is completed.

Example 3.1. Let $E = F = Z = R$, $X = Y = [0, +\infty)$, and $X_0 = [0, 1]$. For each $x, u \in X, y \in Y$, let

$$e(x) = 1, \quad K(x) = [x, +\infty), \quad T(x) = [0, x],$$
$$G_1(x, y, u) = (-\infty, x - y + u], \quad G_2(x, y, u) = (x - y + u, +\infty).$$

Then, it is easy to see that all the conditions of Property 3.1 are satisfied. Moreover, by simple computation, we have, for each $i = 1, 2$, $S_i = [0, 1]$ and $\Omega_i(\varepsilon) = [0, 1+\varepsilon]$ for all $\varepsilon > 0$, and so $S_i = \cap_{\varepsilon>0}\Omega_i(\varepsilon),\ i = 1, 2$.

Property 3.2. Assume that K is continuous and closed-valued, T is $u.s.c.$ and compact-valued.

(i) If G_1 is closed, then S_1 is a closed subset of X_0; Furthermore, if K is also compact-valued, then, for each $\varepsilon > 0$, $M_1(\varepsilon)$ is a closed subset of X and so is $\Omega_1(\varepsilon)$;

(ii) if G_2 is open, then S_2 is a closed subset of X_0; Furthermore, if K is also compact-valued, then, for each $\varepsilon > 0$, $M_2(\varepsilon)$ is a closed subset of X and so is $\Omega_2(\varepsilon)$.

Proof. (i) Let $x \in clS_1$. Then, there exists a sequence $\{x_n\}$ in S_1 such that $x_n \to x$ as $n \to \infty$. It follows that, for each $n \in N$, $x_n \in X_0$, $x_n \in K(x_n)$ and there exists some $y_n \in T(x_n)$ such that

$$0 \in G_1(x_n, y_n, u), \ \forall\, u \in K(x_n).$$

Since X_0 is closed, we have $x \in X_0$. Moreover, since K is $u.s.c.$ and closed-valued, K is closed and so $x \in K(x)$. Since T is $u.s.c.$ and compact-valued, there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in T(x)$ such that $y_{n_k} \to y$ as $k \to \infty$. It follows that, for each $k \in N$,

$$0 \in G_1(x_{n_k}, y_{n_k}, u),\ \forall\, u \in K(x_{n_k}).$$

For each $u \in K(x)$, since K is $l.s.c.$, there exists a sequence $\{u_k\}$ with $u_k \in K(x_{n_k})$ such that $u_k \to u$ as $k \to \infty$, and so

$$0 \in G_1(x_{n_k}, y_{n_k}, u_k),\ \forall\, k \in N.$$

Since G_1 is closed, we get$0 \in G_1(x, y, u)$ and so $x \in S_1$. This implies that S_1 is a closed subset of X_0.

Next, suppose that K is also compact-valued. We show that, for each $\varepsilon > 0$, $M_1(\varepsilon)$ is a closed subset of X. Indeed, if $x \in cl(M_1(\varepsilon)) \subseteq X$, then there exists a sequence $\{x_n\}$ in $M_1(\varepsilon)$ such that $x_n \to x$ as $n \to \infty$. It follows that, for each $n \in N$, there exists $y_n \in T(x_n)$ such that

$$d(x_n, X_0) \leq \varepsilon, \tag{3.8}$$

$$d(x_n, K(x_n)) \leq \varepsilon, \tag{3.9}$$

$$0 \in G_1(x_n, y_n, u) + \varepsilon e(x_n),\ \forall\, u \in K(x_n). \tag{3.10}$$

By (3.8), we have $d(x, X_0) \leq \varepsilon$. By (3.9), for each $n \in N$, there exists $u_n \in K(x_n)$ such that

$$d(x_n, u_n) \leq \varepsilon + \frac{1}{n}. \tag{3.11}$$

Since K is u.s.c. and compact-valued, there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in K(x)$ such that $u_{n_k} \to u$ as $k \to \infty$. It follows that

$$d(x, u) = \lim_{k \to \infty} d(x_{n_k}, u_{n_k}) \leq \varepsilon.$$

Since $u \in K(x)$, we get

$$d(x, K(x)) \leq \varepsilon. \tag{3.12}$$

By (3.10) and similar arguments as in the first part of the proof, we can show that there exists $y \in T(x)$ such that

$$0 \in G_1(x, y, u) + \varepsilon e(x),\ \forall\, u \in K(x).$$

Thus $x \in M_1(\varepsilon)$ and so $M_1(\varepsilon)$ is a closed subset of X. Now it follows from the closedness of S_1 that $\Omega_1(\varepsilon)$ is a closed subset of X.

(ii) Let $G_1 : X \times Y \times X \multimap Z$ be defined by $G_1(x, y, u) = Z \backslash G_2(x, y, u)$ for each $(x, y, u) \in X \times Y \times X$. Then $S_1 = S_2$ and $M_1(\varepsilon) = M_2(\varepsilon)$, and so $\Omega_1(\varepsilon) = \Omega_2(\varepsilon)$. Since G_2 is open, we know that G_1 is closed. By (i), it is easy to see that conclusions of (ii) hold. This completes the proof.

Remark 3.2. In Property 3.2, if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed but not necessarily compact, and the condition "G_1 is closed" can be weakened by "$\forall\, u \in \tilde{X}, (x, y) \multimap G_1(x, y, u)$ is closed" and the condition "G_2 is open" can be weakened by "$\forall\, u \in \tilde{X}, (x, y) \multimap G_2(x, y, u)$ is open".

If E is finite-dimensional, then the assumption that "K is also compact-valued" in Property 3.2 can be removed.

Property 3.3. Let E be finite-dimensional. Assume that K is continuous and closed-valued, T is $u.s.c.$ and compact-valued. For each $\varepsilon > 0$,

(i) if G_1 is closed, then S_1 is a closed subset of X_0, $M_1(\varepsilon)$ and $\Omega_1(\varepsilon)$ are closed subsets of X;

(ii) if G_2 is open, then S_2 is a closed subset of X_0, $M_2(\varepsilon)$ and $\Omega_2(\varepsilon)$ are closed subsets of X.

Proof. We can proceed the proof exactly as that of Property 3.2 except for using the assumption that E is finite-dimensional to get (3.12). In fact, since $x_n \to x$, it follows that $\{x_n\}$ is bounded. By (3.11), we know that $\{u_n\}$ is also bounded. Thus, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges to some $u \in X$ as $k \to \infty$. Since K is $u.s.c.$ and closed-valued, K is closed and so $u \in K(x)$. It follows that $d(x, u) = \lim_{k\to\infty} d(x_{n_k}, u_{n_k}) \le \varepsilon$ and so $d(x, K(x)) \le \varepsilon$, i.e., (3.12) holds. This completes the proof.

Remark 3.3. In Property 3.3, if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed, and the condition "G_1 is closed" can be weakened by "$\forall\, u \in \tilde{X}, (x, y) \multimap G_1(x, y, u)$ is closed" and the condition "G_2 is open" can be weakened by "$\forall\, u \in \tilde{X}, (x, y) \multimap G_2(x, y, u)$ is open".

Theorem 3.1. *If (GQVIP) is LP well-posed, then we have*

(i) *for each* $\varepsilon > 0$, $\Omega_1(\varepsilon) \neq \emptyset$;

(ii) $\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0$ *as* $\varepsilon \to 0$.

Proof. Clearly, $S_1 \subseteq \Omega_1(\varepsilon)$ for all $\varepsilon > 0$. If (GQVIP) is LP well-posed, then $S_1 \neq \emptyset$ and so $\Omega_1(\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$.

Now we show that

$$\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0 \text{ as } \varepsilon \to 0. \tag{3.13}$$

For each $\varepsilon > 0$, since $S_1 \subseteq \Omega_1(\varepsilon)$, we have $e(S_1, \Omega_1(\varepsilon)) = 0$ and so

$$\mathcal{H}(\Omega_1(\varepsilon), S_1) = \max\{e(\Omega_1(\varepsilon), S_1), e(S_1, \Omega_1(\varepsilon)\} = e(\Omega_1(\varepsilon), S_1).$$

To prove (3.13), it is sufficient to show that $e(\Omega_1(\varepsilon), S_1) \to 0$ as $\varepsilon \to 0$. Suppose it is not true. Then there exist a real number $r > 0$, a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ as $n \to \infty$ and a sequence $\{x_n\}$ with $x_n \in \Omega_1(\varepsilon_n)$ such that

$$x_n \notin S_1 + B(0, r), \ \forall\, n \in N, \tag{3.14}$$

where $B(0, r)$ denotes the closed ball centered at 0 with radius r. For each $n \in N$, since $x_n \in \Omega_1(\varepsilon_n)$ and $x_n \notin S_1 + B(0, r)$, we have $x_n \in M_1(\varepsilon_n)$. Hence $\{x_n\}$ is an LP approximating solution sequence for (GQVIP). Since (GQVIP) is LP well-posed, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point of S_1. This contradicts to (3.14) and so $e(\Omega_1(\varepsilon), S_1) \to 0$ as $\varepsilon \to 0$. This completes the proof.

By Property 3.2 and Theorem 3.1, we can get the following result.

Corollary 3.1. *Let X_0 be a nonempty compact subset of X. Assume that K is continuous and closed-valued, T is $u.s.c.$ and compact-valued and G_1 is closed. If (GQVIP) is LP well-posed, then we have*

(i) *S_1 is a nonempty compact subset of X_0;*

(ii) *for each $\varepsilon > 0$, $\Omega_1(\varepsilon) \neq \emptyset$;*

(iii) *$\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0$ as $\varepsilon \to 0$.*

Proof. By the definition of LP well-posedness and Property 3.2, S_1 is a nonempty closed subset of X_0. Since X_0 is compact, S_1 is a nonempty compact subset of X_0. By Theorem 3.1, the proof is completed.

Remark 3.4. In Corollary 3.1,

(i) the assumption "X_0 is a nonempty compact subset of X" can be replaced by "X is a nonempty compact subset of E". Indeed, if X is a nonempty compact subset of E, then, by the closedness of X_0, X_0 is a nonempty compact subset of X;

(ii) if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed, and the condition "G_1 is closed" can be weakened by "$\forall\, u \in \tilde{X}, (x, y) \multimap G_1(x, y, u)$ is closed".

Theorem 3.2. *Assume that*

(i) S_1 *is a nonempty compact subset of* X_0*;*

(ii) $\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0$ *as* $\varepsilon \to 0$.

Then (GQVIP) is LP well-posed.

Proof. Let $\{x_n\} \subseteq X$ be an approximating solution sequence for (GQVIP). Then, there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each $n \in N$, $d(x_n, X_0) \leq \varepsilon_n$ and $d(x_n, K(x_n)) \leq \varepsilon_n$ with

$$0 \in G_1(x_n, y_n, u) + \varepsilon_n e(x_n),\ \forall\, u \in K(x_n).$$

Thus, $x_n \in M_1(\varepsilon_n) \subseteq \Omega_1(\varepsilon_n)$ and so, by (ii), we have $d(x_n, S_1) \to 0$ as $n \to \infty$. Since S_1 is compact, for each $n \in N$, there exists $\bar{x}_n \in S_1$ such that

$$d(x_n, \bar{x}_n) = d(x_n, S_1) \to 0 \text{ as } n \to \infty. \tag{3.15}$$

Again from the compactness of S_1, there exist a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ and $\bar{x} \in S_1$ such that $\bar{x}_{n_k} \to \bar{x}$ as $k \to \infty$. Hence, by (3.15), the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x}$. Therefore, (GQVIP) is LP well-posed. This completes the proof.

Example 3.2. Let $E = F = Z = R$, $X = Y = [0, +\infty)$, and $X_0 = [0, 1]$. For each $x, u \in X$ and $y \in Y$, let

$$e(x) = 1,\quad K(x) = [0, x],\quad T(x) = [2x, +\infty),\quad G_1(x, y, u) = [u - y, u + x].$$

Then, it is easy to compute that $S_1 = [0, 1]$ and $\Omega_1(\varepsilon) = [0, 1 + \varepsilon]$ for all $\varepsilon > 0$. It follows that S_1 is compact and $\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0$ as $\varepsilon \to 0$. By Theorem 3.2, (GQVIP) is LP well-posed.

The following example illustrates that the compactness condition in Theorem 3.2 is essential.

Example 3.3. Let E, F, Z, X, Y, e, K, T and G_1 be as in Example 3.2. Let $X_0 = [0, +\infty)$. Then, it is easy to compute that $S_1 = [0, +\infty)$ and $\Omega_1(\varepsilon) = [0, +\infty)$ for all $\varepsilon > 0$. Thus $\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0$ as $\varepsilon \to 0$. Let $x_n = n$ for $n = 1, 2, \cdots$. Then, $\{x_n\}$ is an approximating solution sequence for (GQVIP), which has no convergent subsequence. This implies that (GQVIP) is not LP well-posed.

From Theorem 3.2, we can get the following corollary.

Corollary 3.2. *Let* X_0 *be a nonempty compact subset of* X*. Let* K *be continuous and closed-valued,* T *be* $u.s.c.$ *and compact-valued and* G_1 *be closed. Assume that*

(i) $\cap_{\varepsilon>0}\Omega_1(\varepsilon) \neq \emptyset$*;*

(ii) $\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0$ *as* $\varepsilon \to 0$.

Then (GQVIP) is LP well-posed.

Proof. By (i) and Properties 3.1 and 3.2, S_1 is a nonempty closed subset of X_0. Since X_0 is compact, S_1 is a nonempty compact subset of X_0. Then, by Theorem 3.2, (GQVIP) is LP well-posed. This completes the proof.

Remark 3.5. In Corollary 3.2, if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed, and the condition "G_1 is closed" can be weakened by "$\forall\, u \in \tilde{X}, (x, y) \multimap G_1(x, y, u)$ is closed".

From Theorems 3.1 and 3.2, we have the following theorem which gives a metric characterization for LP well-posedness of (GQVIP).

Theorem 3.3. *Assume that* S_1 *be a nonempty compact subset of* X_0*. Then (GQVIP) is LP well-posed if and only if*

$$\Omega_1(\varepsilon) \neq \emptyset,\ \forall\, \varepsilon > 0, \quad \textit{and} \quad \mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0 \textit{ as } \varepsilon \to 0.$$

Theorem 3.4. *Let* K *be continuous and closed-valued,* T *be* $u.s.c.$ *and compact-valued and* G_1 *be closed. Assume that*

(i) E *is finite-dimensional;*

(ii) $S_1 \neq \emptyset$*;*

(iii) *there exists some* $\bar{\varepsilon} > 0$ *such that* $\Omega_1(\bar{\varepsilon})$ *is bounded and* $\Omega_1(\varepsilon') \subseteq \Omega_1(\bar{\varepsilon})$ *for every* $\varepsilon' \in (0, \bar{\varepsilon})$.

Then the following conclusions hold:

(a) *(GQVIP) is LP well-posed;*

(b) $\Omega_1(\varepsilon) \neq \emptyset,\ \forall\, \varepsilon > 0$*, and* $\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0$ *as* $\varepsilon \to 0$.

Proof. (a) Let $\{x_n\} \subseteq X$ be an LP approximating solution sequence for (GQVIP). Then, there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that (3.1), (3.2) and (3.3) hold. Thus, for each $n \in N$, $x_n \in M_1(\varepsilon_n) \subseteq \Omega_1(\varepsilon_n)$. Since $\varepsilon_n \to 0$, there exists $n_0 \in N$ such that $\varepsilon_n < \bar{\varepsilon}$ for all $n \geq n_0$. Hence, $x_n \in \Omega_1(\varepsilon_n) \subseteq \Omega_1(\bar{\varepsilon})$ for all $n \geq n_0$, and so $\{x_n\}$ is a bounded sequence in E. Since E is finite-dimensional, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x} \in X$ as $k \to \infty$. Then, by (3.1), (3.2) and (3.3), we have, for each $k \in N$,

$$(3.16) \qquad d(x_{n_k}, X_0) \le \varepsilon_{n_k},$$

$$(3.17) \qquad d(x_{n_k}, K(x_{n_k})) \le \varepsilon_{n_k},$$

$$(3.18) \qquad 0 \in G_1(x_{n_k}, y_{n_k}, u) + \varepsilon_{n_k} e(x_{n_k}),\ \forall\, u \in K(x_{n_k}).$$

Since X_0 is closed and (3.16) holds, we have $\bar{x} \in X_0$. By (3.17), for each $k \in N$, there exists some $u_k \in K(x_{n_k})$ such that

$$d(x_{n_k}, u_k) \le 2d(x_{n_k}, K(x_{n_k})) \le 2\varepsilon_{n_k}.$$

Since $x_{n_k} \to \bar{x}$ and $\varepsilon_{n_k} \to 0$, we get $u_k \to \bar{x}$ as $k \to \infty$. Since K is $u.s.c.$ and closed-valued, K is closed and so $\bar{x} \in K(\bar{x})$. Furthermore, by similar arguments as in the proof of Property 3.2, we can show that there exists some $\bar{y} \in T(\bar{x})$ such that

$$0 \in G_1(\bar{x}, \bar{y}, u),\ \forall\, u \in K(\bar{x}).$$

Therefore $\bar{x} \in S_1$, and this implies that (GQVIP) is LP well-posed.

(b) By (a), (GQVIP) is LP well-posed. Then, by Theorem 3.1, we have $\Omega_1(\varepsilon) \neq \emptyset,\ \forall\, \varepsilon > 0$ and $\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0$ as $\varepsilon \to 0$. This completes the proof.

Remark 3.6. In Theorem 3.4, if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed, and the condition "G_1 is closed" can be weakened by "$\forall\, u \in \tilde{X}, (x, y) \multimap G_1(x, y, u)$ is closed".

Remark 3.7. (a) By Lemma 3.2 of Lin et al. [29] and Remark 3.1, it is easy to see that Theorems 3.4 generalizes Theorems 3.4 of Li et al. [23]; (b) By Remarks 3.1 and 3.6, we know that Theorems 3.4 generalizes Theorems 3.3 of Li and Li [22]; (c) By Lemma 2.1(ii) and Remark 3.1, we can see that Theorems 3.4 generalizes Theorems 3.4 of Huang et al. [14].

Example 3.4. Let $E = F = Z = R$, $X = Y = [0, +\infty)$, and $X_0 = [0, 1]$. For each $x, u \in X, y \in Y$, let

$$e(x) = 1, \quad K(x) = [x, +\infty), \quad T(x) = [0, 2x],$$

$$G_1(x, y, u) = (-\infty, y - x + u].$$

By simple computation, we have $S_1 = [0, 1]$ and $\Omega_1(\varepsilon) = [0, 1 + \varepsilon]$ for all $\varepsilon > 0$. Then, it is easy to see that all the conditions of Theorem 3.4 are satisfied. By Theorem 3.4, (GQVIP) is LP well-posed and $\mathcal{H}(\Omega_1(\varepsilon), S_1) \to 0$ as $\varepsilon \to 0$.

The following example illustrates that the boundedness condition in Theorem 3.4 is essential.

Example 3.5. Let E, F, Z, X, Y, e, K, T and G_1 be as in Example 3.4. Let $X_0 = [0, +\infty)$. By simple computation, we have $S_1 = [0, +\infty)$ and $\Omega_1(\varepsilon) = [0, +\infty)$ for all $\varepsilon > 0$. Then, it is easy to see that all the conditions of Theorem 3.4 are satisfied except for the boundedness condition. Let $x_n = n$ for $n = 1, 2, \cdots$. Then, $\{x_n\}$ is an LP approximating solution sequence for (GQVIP), which has no convergent subsequence. This implies that (GQVIP) is not LP well-posed.

By Theorems 3.1-3.4, we can get the following results.

Theorem 3.5. *If* (*GQVDP*) *is LP well-posed, then we have*

(i) *for each* $\varepsilon > 0$, $\Omega_2(\varepsilon) \neq \emptyset$*;*

(ii) $\mathcal{H}(\Omega_2(\varepsilon), S_2) \to 0$ *as* $\varepsilon \to 0$.

Theorem 3.6. *Assume that*

(i) S_2 *is a nonempty compact subset of* X_0*;*

(ii) $\mathcal{H}(\Omega_2(\varepsilon), S_2) \to 0$ *as* $\varepsilon \to 0$.

Then (*GQVDP*) *is LP well-posed.*

Theorem 3.7. *Assume that* S_2 *be a nonempty compact subset of* X_0*. Then* (*GQVDP*) *is LP well-posed if and only if*

$$\Omega_2(\varepsilon) \neq \emptyset, \ \forall\, \varepsilon > 0, \quad \textit{and} \quad \mathcal{H}(\Omega_2(\varepsilon), S_2) \to 0 \ \textit{as} \ \varepsilon \to 0.$$

Theorem 3.8. *Let* K *be continuous and closed-valued,* T *be* $u.s.c.$ *and compact-valued and* G_2 *be open. Assume that*

(i) E *is finite-dimensional;*

(ii) $S_2 \neq \emptyset$*;*

(iii) *there exists some* $\bar{\varepsilon} > 0$ *such that* $\Omega_2(\bar{\varepsilon})$ *is bounded and* $\Omega_2(\varepsilon') \subseteq \Omega_2(\bar{\varepsilon})$ *for every* $\varepsilon' \in (0, \bar{\varepsilon})$.

Then the following conclusions hold:

(a) *(GQVDP) is LP well-posed;*

(b) $\Omega_2(\varepsilon) \neq \emptyset, \ \forall\, \varepsilon > 0$*, and* $\mathcal{H}(\Omega_2(\varepsilon), S_2) \to 0$ *as* $\varepsilon \to 0$.

Remark 3.8. In Theorem 3.8, if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed, and the condition "G_2 is open" can be weakened by "$\forall\, u \in \tilde{X}, (x, y) \multimap G_2(x, y, u)$ is open".

Remark 3.9. By Lemma 3.1 of Lin et al. [29] and Remark 3.1, it is easy to see that Theorems 3.8 is a generalization of Theorems 3.5 due to Li et al. [23].

4. LP Well-posedness for Optimization Problems with Constraints

In this section, we study LP well-posedness for optimization problems with generalized quasi-variational inclusion problems, generalized quasi-variational disclusion problems and scalar generalized quasi-equilibrium problems as constraints.

Let W be a normed space and $C \subseteq W$ be a pointed, closed and convex cone with $intC \neq \emptyset$. Let $H : X \multimap W$ be a multivalued mapping with nonempty values and $f : X \times Y \times X \to R$ be a real-valued function. We consider the following problems:

(OPVI) Min $H(x)$ subject to $x \in S_1$;

(OPVD) Min $H(x)$ subject to $x \in S_2$;

(OPEC) Min $H(x)$ subject to $x \in S_3$, where

$$S_3 = \{x \in X_0 : \; x \in K(x) \text{ and } \exists\, y \in T(x) \text{ s.t. } f(x, y, u) \geq 0, \; \forall\, u \in K(x)\}.$$

Suppose that S_1, S_2 and S_3 are nonempty closed subsets of X_0. A point $p \in H(S_1)$ (resp. $H(S_2), H(S_3)$) is called a minimal point of $H(S_1)$ (resp. $H(S_2), H(S_3)$) if $H(S_1) \cap (p - C\backslash\{0\}) = \emptyset$ (resp. $H(S_2) \cap (p - C\backslash\{0\}) = \emptyset, H(S_3) \cap (p - C\backslash\{0\}) = \emptyset$)). A point $x \in S_1$ (resp. S_2, S_3) is called an efficient solution of (OPVI) (resp. (OPVD), (OPEC)) if there exists $p \in H(x)$ such that p is a minimal point of $H(S_1)$ (resp. $H(S_2), H(S_3)$).

For each $p, q \in W$ and each $\delta > 0$ and $\varepsilon > 0$, let

$$\begin{aligned}
M_3(\varepsilon) &:= \{x \in X : d(x, X_0) \leq \varepsilon, \; d(x, K(x)) \leq \varepsilon \text{ and} \\
&\qquad \exists\, y \in T(x) \text{ s.t. } f(x, y, u) + \varepsilon \geq 0, \; \forall\, u \in K(x)\}; \\
A(p, q, \delta) &:= \{x \in X : H(x) \cap (p + \delta q - C) \neq \emptyset\}; \\
L_1(p, q, \delta, \varepsilon) &:= A(p, q, \delta) \cap \Omega_1(\varepsilon); \\
L_2(p, q, \delta, \varepsilon) &:= A(p, q, \delta) \cap \Omega_2(\varepsilon); \\
L_1^*(p, q, \delta, \varepsilon) &:= A(p, q, \delta) \cap M_1(\varepsilon); \\
L_2^*(p, q, \delta, \varepsilon) &:= A(p, q, \delta) \cap M_2(\varepsilon); \\
L_3(p, q, \delta, \varepsilon) &:= A(p, q, \delta) \cap M_3(\varepsilon).
\end{aligned}$$

Clearly, we have the following inclusions:

$$L_1^*(p, q, \delta, \varepsilon) \subseteq L_1(p, q, \delta, \varepsilon) \subseteq A(p, q, \delta)$$

and

$$L_2^*(p, q, \delta, \varepsilon) \subseteq L_2(p, q, \delta, \varepsilon) \subseteq A(p, q, \delta).$$

Definition 4.1. Let x be an efficient solution of (OPVI) (resp. (OPVD), (OPEC)). A sequence $\{x_n\} \subseteq X$ is said to be an LP approximating solution sequence for (OPVI) (resp. (OPVD), (OPEC)) at x if

(i) there exist a sequence $\{k_n\}$ in W with $k_n \to 0$ and $p \in H(x)$ with p is a minimal point of $H(S_1)$(resp. $H(S_2)$, $H(S_3)$) such that $H(x_n) \cap (p + k_n - C) \neq \emptyset$ for all $n \in N$;

(ii) there exists a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ such that $x_n \in M_1(\varepsilon_n)$ (resp. $M_2(\varepsilon_n)$, $M_3(\varepsilon_n)$) for all $n \in N$.

Definition 4.2. Let $q \in intC$ and let x be an efficient solution of (OPVI) (resp. (OPVD), (OPEC)). Then (OPVI) (resp. (OPVD), (OPEC)) is said to be LP well-posed at x if every LP approximating solution sequence for (OPVI) (resp. (OPVD), (OPEC)) at x converges to x.

Remark 4.1. Definitions of LP well-posedness for (OPVI), (OPVD) and (OPEC) are similar to ones defined by Lin and Chuang [31].

The following lemma is a useful tool in this paper.

Lemma 4.2. ([8]). *Let $\{k_n\} \subseteq W$ be any sequence with $k_n \to 0$. Then, for each $q \in intC$, there exists a sequence $\{\delta_n\}$ of real positive numbers with $\delta_n \to 0$ such that $\delta_n q - k_n \in intC$ for all $n \in N$.*

Theorem 4.1. *Let $q \in intC$ and let $x \in S_1$ be an efficient solution of (OPVI). Assume that, for each $p \in H(x)$ with p is a minimal point of $H(S_1)$, $\mathrm{diam}(L_1(p, q, \delta, \varepsilon)) \to 0$ as $(\delta, \varepsilon) \to (0, 0)$. Then (OPVI) is LP well-posed at x.*

Proof. Let $\{x_n\} \subseteq X$ be an LP approximating solution sequence for (OPVI) at x. Then, we have the following conclusions:

(a) there exist a sequence $\{k_n\}$ in W with $k_n \to 0$ and $p \in H(x)$ with p is a minimal point of $H(S_1)$ such that $H(x_n) \cap (p + k_n - C) \neq \emptyset$ for all $n \in N$;

(b) there exists a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ such that $x_n \in M_1(\varepsilon_n)$ for all $n \in N$.

By (a), for each $n \in N$, there exists $p_n \in H(x_n)$ such that $p_n \in p + k_n - C$. Note that $\{k_n\} \subseteq W$ and $k_n \to 0$. It follows from Lemma 4.2 that there exists a sequence $\{\delta_n\}$ of real positive numbers with $\delta_n \to 0$ such that $k_n \in \delta_n q - intC$. Hence,

$$p_n \in p + k_n - C \subseteq p + \delta_n q - intC - C \subseteq p + \delta_n q - intC,$$

and this implies that $p_n \in H(x_n) \cap (p + \delta_n q - intC)$. Thus we have $x_n \in L_1(p, q, \delta_n, \varepsilon_n)$. Moreover, for each $n \in N$, $x \in S_1 \subseteq \Omega_1(\varepsilon_n)$ and $p \in H(x) \cap (p + \delta_n q - C)$. Hence, $x \in L_1(p, q, \delta_n, \varepsilon_n)$. Then, we have

$$d(x_n, x) \leq \mathrm{diam}(L_1(p, q, \delta_n, \varepsilon_n)).$$

By the assumption, we get $x_n \to x$ as $n \to \infty$, and this implies that (OPVI) is LP well-posed at x. This completes the proof.

Theorem 4.2. *Let $q \in intC$ and let $x \in S_1$ be an efficient solution of (OPVI). If (OPVI) is LP well-posed at x, then, for each $p \in H(x)$ with p is a minimal point of $H(S_1)$, $\operatorname{diam}(L_1^*(p,q,\delta,\varepsilon)) \to 0$ as $(\delta,\varepsilon) \to (0,0)$.*

Proof. Suppose that there exists some $p \in H(x)$ with p is a minimal point of $H(S_1)$ such that diam $(L_1^*(p,q,\delta,\varepsilon)) \not\to 0$ as $(\delta,\varepsilon) \to (0,0)$. Then there exist a positive number r and two sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of real positive numbers and two sequences $\{x_n\}$ and $\{x_n'\}$ in X such that

(a) $(\delta_n,\varepsilon_n) \to (0,0)$ as $n \to \infty$;

(b) for each $n \in N$, $x_n, x_n' \in L_1^*(p,q,\delta_n,\varepsilon_n)$ and $d(x_n,x_n') \geq r$.

By (b), for each $n \in N$, we have the following conclusions:

(c) $H(x_n) \cap (p+\delta_n q - C) \neq \emptyset$ and $H(x_n') \cap (p+\delta_n q - C) \neq \emptyset$;

(d) $x_n, x_n' \in M_1(\varepsilon_n)$.

By (a), $\delta_n q \to 0$ as $n \to \infty$. Thus, by (c) and (d), $\{x_n\}$ and $\{x_n'\}$ are both LP approximating solution sequences for (OPVI) at x. Since (OPVI) is LP well-posed at x, $x_n \to x$ and $x_n' \to x$ as $n \to \infty$. This leads to a contradiction. Therefore, for each $p \in H(x)$ with p is a minimal point of $H(S_1)$, diam$(L_1^*(p,q,\delta,\varepsilon)) \to 0$ as $(\delta,\varepsilon) \to (0,0)$. This completes the proof.

The following corollary is a special case of Theorems 4.1 and 4.2.

Corollary 4.3. *Let $q \in intC$ and let $x \in S_3$ be an efficient solution of $(OPEC)$. Then $(OPEC)$ is LP well-posed at x if and only if for each $p \in H(x)$ with p is a minimal point of $H(S_3)$,*

$$\operatorname{diam}(L_3(p,q,\delta,\varepsilon)) \to 0 \text{ as } (\delta,\varepsilon) \to (0,0). \tag{4.19}$$

Proof. For each $x \in X$, let $e(x) = 1$. Define a multivalued mapping $G_1 : X \times Y \times X \multimap R$ as follows:

$$G_1(x,y,u) = f(x,y,u) - R_+,\ \forall\,(x,y,u) \in X \times Y \times X.$$

Then, $0 \in G_1(x,y,u)$ if and only if $f(x,y,u) \geq 0$ and $0 \in G_1(x,y,u) + \varepsilon e(x)$ if and only if $f(x,y,u) + \varepsilon \geq 0$. Hence, for each $\delta > 0$ and $\varepsilon > 0$, $S_1 = S_3$, $M_1(\varepsilon) = M_3(\varepsilon)$,

$$\begin{aligned} L_1(p,q,\delta,\varepsilon) &= A(p,q,\delta) \cap [M_1(\varepsilon) \cup S_1] \\ &= A(p,q,\delta) \cap [M_3(\varepsilon) \cup S_3] \\ &= A(p,q,\delta) \cap M_3(\varepsilon) \\ &= L_3(p,q,\delta,\varepsilon) \end{aligned}$$

and

$$\begin{aligned} L_1^*(p,q,\delta,\varepsilon) &= A(p,q,\delta)\cap M_1(\varepsilon) \\ &= A(p,q,\delta)\cap M_3(\varepsilon) \\ &= L_3(p,q,\delta,\varepsilon). \end{aligned}$$

If (4.19) holds, then all the conditions of Theorem 4.1 are satisfied and it follows from Theorem 4.1 that (OPVI) is LP well-posed at x. Thus, if $\{x_n\} \subseteq X$ is an LP approximating solution sequence for (OPVI) at x, then $x_n \to x$. That is, if $\{x_n\} \subseteq X$ is an LP approximating solution sequence for (OPEC) at x, then $x_n \to x$. Therefore, (OPEC) is LP well-posed at x.

Conversely, if (OPEC) is LP well-posed at x, then all the conditions of Theorem 4.2 are satisfied and it follows from Theorem 4.2 that diam$(L_1^*(p,q,\delta,\varepsilon)) \to 0$ as $(\delta,\varepsilon) \to (0,0)$. That is

$$\text{diam}(L_3(p,q,\delta,\varepsilon)) \to 0 \text{ as } (\delta,\varepsilon) \to (0,0).$$

This completes the proof.

Similar to proofs of Theorems 4.1 and 4.2, we have the following results.

Theorem 4.3. *Let $q \in intC$ and $x \in S_2$ be an efficient solution of (OPVD). Assume that, for each $p \in H(x)$ with p is a minimal point of $H(S_2)$, $\operatorname{diam}(L_2(p,q,\delta,\varepsilon))$ $\to 0$ as $(\delta,\varepsilon) \to (0,0)$. Then $($OPVD$)$ is LP well-posed at x.*

Theorem 4.4. *Let $q \in intC$ and let $x \in S_2$ be an efficient solution of $($OPVD$)$. If $($OPVD$)$ is LP well-posed at x, then, for each $p \in H(x)$ with p is a minimal point of $H(S_2)$, $\operatorname{diam}(L_2^*(p,q,\delta,\varepsilon)) \to 0$ as $(\delta,\varepsilon) \to (0,0)$.*

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