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ON PARTIAL SUMS OF GENERALIZED DEDEKIND SUMS

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Abstract. The main purpose of this paper is to use the mean value theorems of Dirichlet L-function to study the distribution properties of the generalized Dedekind sums, and gives two interesting asymptotic formulas.

1. INTRODUCTION

Let k be a positive integer, for arbitrary integers h and n, the generalized Dedekind sum $S(h,n,k)$ is defined by

$$S(h,n,k)=\sum_{a=1}^{k}\bar{B}_n\left(\frac{a}{k}\right)\bar{B}_n\left(\frac{ah}{k}\right),$$

where

$$\bar{B}_n(x)=\begin{cases}B_n(x-[x]), & \text{if } x \text{ is not an integer,}\\ 0, & \text{if } x \text{ is an integer,}\end{cases}$$

with $B_n(x)$ the Bernoulli polynomial. Some arithmetical properties of $S(h,n,k)$ have been studied [1-2]. For $n=1$, $S(h,n,k)=S(h,k)$ is the classical Dedekind sum, it plays such a great role in the study of the modular forms theory that it has attracted many experts in number theory [3-5]. In 1996, J. B.Conrey, E. Fransen, R. K lein, and C. Scott [6] deduced a mean value formula of the classical Dedekind sum:

$$\sum_{h=1}^{k}{}'\,|S(h,k)|^{2m}=f_m(k)\left(\frac{k}{12}\right)^{2m}+O\left(\left(k^{\frac{9}{5}}+k^{2m-1+\frac{1}{m+1}}\right)\log^3 k\right),$$

where $\sum{}'_h$ denotes the summation runs through integers coprime with k, and $f_m(k)$ is defined by

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$$\sum_{n=1}^{\infty}\frac{f_m(n)}{n^s}=2\frac{\zeta^2(2m)}{\zeta(4m)}\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s).$$

In 2000, Wenpeng Zhang and Yuan Yi [7] gave a sharper mean value formula for partial sums of $S(h,k)$. Min Xie and Wenpeng Zhang [8] obtained the square mean value formulas for generalized Dedekind sums.

In this paper, we will use the estimates of the character sums and the mean value theorems of Dirichlet L-function to study the distribution properties of the generalized Dedekind sum $S(h,n,k)$, and give two asymptotic formulas, namely

Theorem. *Let k be an integer with $k \geqslant 3$. Then for any positive real number N, we have*

(1) *If $n>1$ is an odd number, then*

$$\begin{aligned}\sum_{h\leqslant N}{}' S(h,n,k) &= \frac{(n!)^2}{2^{2n-1}\pi^{2n}}\zeta(2n)\zeta(n)k\prod_{p|k}(1-p^{-n})\\ &\quad +O\left(N^{-n}k^{2+\varepsilon}+N^{2n}k^{-2n+1+\varepsilon}+N^{n}k^{-n+1+\varepsilon}\right).\end{aligned}$$

(2) *If n is a positive even number, then*

$$\begin{aligned}\sum_{h\leqslant N}{}' S(h,n,k) &= \frac{(n!)^2}{2^{2n-1}\pi^{2n}}\zeta(2n)\zeta(n)k\prod_{p|k}(1-p^{-n})\\ &\quad +O\left(N+N^{-n}k^{2+\varepsilon}+N^{2n}k^{-2n+1+\varepsilon}+N^{n}k^{-n+1+\varepsilon}\right),\end{aligned}$$

where $\sum_{h\leqslant N}{}'$ denotes the summation over all $h\leqslant N$ such that $(h,k)=1$, ε is a sufficiently small positive real number which can be different at each occurrence. These results are obviously nontrivial for $k^{\varepsilon}<N<k^{1-\varepsilon}$.

2. Some Lemmas

To complete the proof of the theorem, we need the following lemmas.

Lemma 1. *Let $k\geqslant 3$ be an integer. Then for any integer h with $(h,k)=1$, we have the identities*

(1) *for n a positive odd number,*

$$S(h,n,k)=\frac{(n!)^2}{4^{n-1}k^{2n-1}\pi^{2n}}\sum_{d|k}\frac{d^{2n}}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)|L(n,\chi)|^2;$$

(2) *for n a positive even number,*

$$S(h,n,k)=\frac{(n!)^2}{4^{n-1}k^{2n-1}\pi^{2n}}\sum_{d|k}\frac{d^{2n}}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=1}}\chi(h)|L(n,\chi)|^2-\frac{(n!)^2}{4^{n-1}\pi^{2n}}\zeta^2(n),$$

where χ denotes a Dirichlet character modulo d, $L(n,\chi)$ denotes the Dirichlet L-function corresponding to χ, $\phi(d)$ and $\zeta(s)$ are the Euler function and Riemann zeta-function, respectively.

Proof. See reference [9].

Lemma 2. *Suppose that k, a and λ are positive integers, $q \geqslant 2$, $q|k$, then for any real number $1 < N < q$ and any integer $s \geqslant 2$, we have the asymptotic formula*

$$\sum_{\substack{a\leqslant N\\(a,k)=1}} \sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^\lambda}} \chi(a)|L(s,\chi)|^2 = \frac{\phi(q)}{2}\zeta(2s)\zeta(s)\prod_{p|q}(1-p^{-2s})\prod_{p|k}(1-p^{-s}) + O\left(\phi(q)N^{-s+1} + \frac{\phi(q)}{q^s}q^{\varepsilon}N^s\right)$$

Proof.

(i) If $\lambda \equiv 1 \bmod 2$, χ is an odd character mod q, Abel's identity implies that

$$L(s,\chi) = \sum_{n\leqslant \frac{q}{a}} \frac{\chi(n)}{n^s} + s\int_{\frac{q}{a}}^{+\infty} \frac{A(\chi,y)}{y^{s+1}}dy = \sum_{n\leqslant q} \frac{\chi(n)}{n^s} + s\int_{q}^{+\infty} \frac{B(\chi,y)}{y^{s+1}}dy$$

where $A(\chi,y) = \sum\limits_{\frac{q}{a}<n\leqslant y} \chi(n)$, $B(\chi,y) = \sum\limits_{q<n\leqslant y}\chi(n)$. Thus

$$\begin{aligned}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)|L(s,\chi)|^2 &= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{n\leqslant \frac{q}{a}}\frac{\chi(n)}{n^s} + s\int_{\frac{q}{a}}^{+\infty}\frac{A(\chi,y)}{y^{s+1}}dy\right)\\ &\quad\times\left(\sum_{m\leqslant q}\frac{\overline{\chi}(m)}{m^s} + s\int_q^{+\infty}\frac{B(\overline{\chi},y)}{y^{s+1}}dy\right)\\ &= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{n\leqslant \frac{q}{a}}\frac{\chi(n)}{n^s}\right)\left(\sum_{m\leqslant q}\frac{\overline{\chi}(m)}{m^s}\right)\\ &\quad+s\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{n\leqslant \frac{q}{a}}\frac{\chi(n)}{n^s}\right)\left(\int_q^{+\infty}\frac{B(\overline{\chi},y)}{y^{s+1}}dy\right)\end{aligned}$$

$$
\begin{aligned}
&+s\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)\left(\sum_{m\leqslant q}\frac{\overline{\chi}(m)}{m^s}\right)\left(\int_{\frac{q}{a}}^{+\infty}\frac{A(\chi,y)}{y^{s+1}}dy\right)\\
&+s^2\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)\left(\int_{\frac{q}{a}}^{+\infty}\frac{A(\chi,y)}{y^{s+1}}dy\right)\left(\int_{q}^{+\infty}\frac{B(\overline{\chi},y)}{y^{s+1}}dy\right)\\
=&\ M_1+M_2+M_3+M_4
\end{aligned}
$$

say. We then have

$$
(1)\qquad \sum_{\substack{a\leqslant N\\ (a,k)=1}}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)|L(s,\chi)|^2\equiv\sum_{\substack{a\leqslant N\\ (a,k)=1}}(M_1+M_2+M_3+M_4).
$$

We need to estimate M_1, M_2, M_3 and M_4 respectively.

(I) for $(q,mn)=1$,

$$
\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(n)\overline{\chi}(m)=\begin{cases}\frac{1}{2}\phi(q), & \text{if}\quad n\equiv m \bmod q,\\ -\frac{1}{2}\phi(q), & \text{if}\quad n\equiv -m \bmod q,\\ 0, & \text{otherwise.}\end{cases}
$$

We can deduce that when $a\geqslant 2$,

$$
\begin{aligned}
M_1 =&\ \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)\left(\sum_{n\leqslant\frac{q}{a}}\frac{\chi(n)}{n^s}\right)\left(\sum_{m\leqslant q}\frac{\overline{\chi}(m)}{m^s}\right)\\
=&\ \frac{1}{2}\phi(q)\sideset{}{'}\sum_{\substack{n\leqslant\frac{q}{a}\\ na\equiv m \bmod q}}\sideset{}{'}\sum_{m\leqslant q}\frac{1}{n^sm^s}-\frac{1}{2}\phi(q)\sideset{}{'}\sum_{\substack{n\leqslant\frac{q}{a}\\ na\equiv -m \bmod q}}\sideset{}{'}\sum_{m\leqslant q}\frac{1}{n^sm^s}\\
=&\ \frac{1}{2}\phi(q)\sideset{}{'}\sum_{n\leqslant\frac{q}{a}}\frac{1}{a^sn^{2s}}-\frac{1}{2}\phi(q)\sideset{}{'}\sum_{n\leqslant\frac{q}{a}}\frac{1}{n^s(q-na)^s}\\
=&\ \frac{\phi(q)}{2a^s}\sideset{}{'}\sum_{n=1}^{\infty}\frac{1}{n^{2s}}+O\left(\frac{\phi(q)}{a^s}\sum_{n>\frac{q}{a}}\frac{1}{n^{2s}}\right)+O\left(\phi(q)\sum_{n\leqslant\frac{q}{2a}}\frac{1}{n^sq^s}\right)\\
&+O\left(\phi(q)\sum_{\frac{q}{2a}<n\leqslant\frac{q}{a}-1}\frac{a^s}{q^s(q-na)^s}\right)+O\left(\frac{\phi(q)a^s}{q^s(q-a[\frac{q}{a}])^s}\right)\\
=&\ \frac{1}{2}\frac{\phi(q)}{a^s}\zeta(2s)\prod_{p|q}(1-p^{-2s})+O\left(\frac{\phi(q)}{q^s}\right)+O\left(\frac{\phi(q)a^s}{q^s(q-a[\frac{q}{a}])^s}\right),
\end{aligned}
$$

while in the case $a = 1$, the result holds with the last O-term not appearing. So that

$$\sum_{\substack{a\leqslant N\\(a,k)=1}} M_1 = \frac{\phi(q)}{2}\zeta(2s)\prod_{p|q}(1-p^{-2s}) \sum_{\substack{a\leqslant N\\(a,k)=1}} \frac{1}{a^s} + O\left(\frac{\phi(q)}{q^s}\sum_{\substack{a\leqslant N\\(a,k)=1}} 1\right)$$

$$+O\left(\frac{\phi(q)}{q^s}\sum_{\substack{2\leqslant a\leqslant N\\(a,k)=1}} \frac{a^s}{(q-a[\frac{q}{a}])^s}\right). \tag{2}$$

Note that

$$\sum_{\substack{a\leqslant N\\(a,k)=1}} \frac{1}{a^s} = \zeta(s)\prod_{p|k}(1-p^{-s}) + O\left(N^{-s+1}\right), \tag{3}$$

and

$$\sum_{\substack{2\leqslant a\leqslant N\\(a,k)=1}} \frac{a^s}{\left(q-a[\frac{q}{a}]\right)^s} \ll N^s \sum_{u\leqslant q-1} \sum_{\substack{a\leqslant N\\ q-a[\frac{q}{a}]=u}} \frac{1}{u^s} \ll N^s \sum_{u\leqslant q-1} \frac{d(q-u)}{u^s} \ll N^s q^{\varepsilon}, \tag{4}$$

where $d(q-u)$ is the divisor function. Inserting (3) and (4) into (2), we have

$$\sum_{\substack{a\leqslant N\\(a,k)=1}} M_1 = \frac{\phi(q)}{2}\zeta(2s)\zeta(s)\prod_{p|q}(1-p^{-2s})\prod_{p|k}(1-p^{-s})$$
$$+O\left(\frac{\phi(q)}{q^s}q^{\varepsilon}N^s\right) + O\left(\phi(q)N^{-s+1}\right). \tag{5}$$

(II) Changing the order of the summation and the integration implies that

$$M_2 = s\int_q^{+\infty}\left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{n\leqslant \frac{q}{a}} \frac{\chi(n)}{n^s}\right) B(\overline{\chi}, y)\right]\frac{1}{y^{s+1}}dy.$$

In order to estimate the integrand in M_2, we may replace $B(\overline{\chi}, y)$ by $\sum\limits_{m\leqslant y<q} \overline{\chi}(m)$ and get

$$
\begin{aligned}
&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{n\leqslant \frac{q}{a}} \frac{\chi(n)}{n^s}\right) \sum_{m\leqslant y<q} \overline{\chi}(m)\\
&\ll \phi(q) \sum_{\substack{n\leqslant \frac{q}{a}\\ na\equiv m \bmod q}}{}' \sum_{m<q}{}' \frac{1}{n^s} + \phi(q) \sum_{\substack{n\leqslant \frac{q}{a}\\ na\equiv -m \bmod q}}{}' \sum_{m<q}{}' \frac{1}{n^s}\\
&\ll \phi(q).
\end{aligned}
$$

so that

$$
\begin{aligned}
&\sum_{\substack{a\leqslant N\\ (a,k)=1}} M_2\\
(6) \qquad &= s\sum_{\substack{a\leqslant N\\ (a,k)=1}} \int_q^{+\infty} \left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{n\leqslant \frac{q}{a}} \frac{\chi(n)}{n^s}\right) \sum_{m\leqslant y<q} \overline{\chi}(m)\right] \frac{1}{y^{s+1}} dy\\
&\ll \sum_{\substack{a\leqslant N\\ (a,k)=1}} \int_q^{+\infty} \frac{\phi(q)}{y^{s+1}} dy \ll \frac{\phi(q)}{q^s} N.
\end{aligned}
$$

(III)

$$
\begin{aligned}
&\sum_{\substack{a\leqslant N\\ (a,k)=1}} M_3\\
&= \sum_{\substack{a\leqslant N\\ (a,k)=1}} s \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{m\leqslant q} \frac{\overline{\chi}(m)}{m^s}\right)\left(\int_{\frac{q}{a}}^{+\infty} \frac{A(\chi,y)}{y^{s+1}} dy\right)\\
&= \sum_{\substack{a\leqslant N\\ (a,k)=1}} s \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{m\leqslant q} \frac{\overline{\chi}(m)}{m^s}\right)\left(\int_{\frac{q}{a}}^{q} \frac{A(\chi,y)}{y^{s+1}} dy\right)\\
(7) \qquad &\quad + \sum_{\substack{a\leqslant N\\ (a,k)=1}} s \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{m\leqslant q} \frac{\overline{\chi}(m)}{m^s}\right)\left(\int_{q}^{+\infty} \frac{A(\chi,y)}{y^{s+1}} dy\right)\\
&= \sum_{\substack{a\leqslant N\\ (a,k)=1}} s \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{m\leqslant q} \frac{\overline{\chi}(m)}{m^s}\right)\left(\int_{\frac{q}{a}}^{q} \frac{A(\chi,y)}{y^{s+1}} dy\right)\\
&\quad + \sum_{\substack{a\leqslant N\\ (a,k)=1}} s \int_q^{+\infty} \left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{m\leqslant q} \frac{\overline{\chi}(m)}{m^s}\right) A(\chi,y)\right] \frac{1}{y^{s+1}} dy.
\end{aligned}
$$

Note that

(8)
$$\begin{aligned}
&\sum_{\substack{a\leqslant N\\(a,k)=1}} s \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{m\leqslant q}\frac{\overline{\chi}(m)}{m^s}\right)\left(\int_{\frac{q}{a}}^{q}\frac{A(\chi,y)}{y^{s+1}}dy\right)\\
&\ll \phi(q)\sum_{m\leqslant q}\frac{1}{m^s}\int_{\frac{q}{N}}^{q}\frac{1}{y^{s+1}}\left(\sum_{\substack{\frac{q}{y}<a\leqslant N\\(a,k)=1}}\ \sum_{\substack{\frac{q}{a}<n\leqslant y<q+\frac{q}{a}\\ na\equiv\pm m \bmod q}} 1\right)dy\\
&\ll \phi(q)\sum_{m\leqslant q}\frac{1}{m^s}\int_{\frac{q}{N}}^{q}\frac{Nyq^{\varepsilon}}{qy^{s+1}}dy \ll \frac{\phi(q)}{q^s}N^s q^{\varepsilon},
\end{aligned}$$

where we have used the fact that for any fixed positive integers l and m, the number of the solutions of equation $an = lq+m$ (for all positive integers a and n) is $\ll q^{\varepsilon}$.

On the other hand,

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{m\leqslant q}\frac{\overline{\chi}(m)}{m^s}\right)\sum_{\frac{q}{a}<n\leqslant y}\chi(n) \ll \phi(q){\sum_{m\leqslant q}}' \ {\sum_{\substack{\frac{q}{a}<n\leqslant y<q+\frac{q}{a}\\ na\equiv\pm m \bmod q}}}' \frac{1}{m^s} \ll \phi(q),$$

hence

(9)
$$\begin{aligned}
&\sum_{\substack{a\leqslant N\\(a,k)=1}} s\int_{q}^{+\infty}\left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(\sum_{m\leqslant q}\frac{\overline{\chi}(m)}{m^s}\right)A(\chi,y)\right]\frac{1}{y^{s+1}}dy\\
&\ll \sum_{\substack{a\leqslant N\\(a,k)=1}}\int_{q}^{+\infty}\frac{\phi(q)}{y^{s+1}}dy \ll \frac{\phi(q)}{q^s}N.
\end{aligned}$$

With (7) and (8), we obtain

(10)
$$\sum_{\substack{a\leqslant N\\(a,k)=1}} M_3 \ll \frac{\phi(q)}{q^s}N^s q^{\varepsilon}.$$

(IV)

$$\begin{aligned}
&\sum_{\substack{a\leqslant N\\(a,k)=1}} M_4\\
&= \sum_{\substack{a\leqslant N\\(a,k)=1}}\left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\left(s\int_{\frac{q}{a}}^{+\infty}\frac{A(\chi,y)}{y^{s+1}}dy\right)\left(s\int_{q}^{+\infty}\frac{B(\overline{\chi},y)}{y^{s+1}}dy\right)\right]\\
&\ll \int_{\frac{q}{N}}^{+\infty}\int_{q}^{+\infty}\left(\sum_{\substack{\frac{q}{y}<a\leqslant N\\(a,k)=1}}\sideset{}{'}\sum_{\frac{q}{a}<n<y}\sideset{}{'}\sum_{q<m<z}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(an)\overline{\chi}(m)\right)\frac{1}{y^{s+1}}\frac{1}{z^{s+1}}dzdy \qquad (11)\\
&\ll \phi(q)\int_{\frac{q}{N}}^{+\infty}\int_{q}^{+\infty}\left(\sum_{\substack{\frac{q}{y}<a\leqslant N\\(a,k)=1}}\sideset{}{'}\sum_{\frac{q}{a}<n\leqslant\frac{q}{a}+q}\sideset{}{'}\sum_{\substack{m\leqslant q\\ na\equiv\pm m \bmod q}}1\right)\frac{1}{y^{s+1}}\frac{1}{z^{s+1}}dzdy\\
&\ll \phi(q)\int_{\frac{q}{N}}^{+\infty}\int_{q}^{+\infty}qN\frac{1}{y^{s+1}}\frac{1}{z^{s+1}}dzdy \ll \frac{\phi(q)}{q^{2s-1}}N^{s+1}.
\end{aligned}$$

The lemma for $\lambda\equiv 1(\bmod 2)$ follows from (1), (5), (6), (10) and (11).

(ii) If $\lambda\equiv 0(\bmod 2)$, χ is an even character mod q. Note that when $(q,mn)=1$,

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\chi(n)\overline{\chi}(m)=\begin{cases}\frac{1}{2}\phi(q), & \text{if } \quad n\equiv\pm m \bmod q,\\ 0, & \text{otherwise.}\end{cases}$$

Using the same methods as previous, we can easily obtain the lemma for this case. This completes the proof of the lemma.

3. Proof of the Theorem

In this section, we shall complete the proof of the theorem.

(i) If n is an odd number with $n>1$, we will get from $1)$ of lemma 1 that

$$\begin{aligned}
\sideset{}{'}\sum_{h\leqslant N} S(h,n,k) &= \sideset{}{'}\sum_{h\leqslant N}\left[\frac{(n!)^2}{4^{n-1}k^{2n-1}\pi^{2n}}\sum_{d|k}\frac{d^{2n}}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)|L(n,\chi)|^2\right]\\
&= \frac{(n!)^2}{4^{n-1}k^{2n-1}\pi^{2n}}\sum_{d|k}\frac{d^{2n}}{\phi(d)}\sideset{}{'}\sum_{h\leqslant N}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)|L(n,\chi)|^2.
\end{aligned}$$

Without loss of generality, we may suppose that $d \nmid N$, since

$$\sideset{}{'}\sum_{h\leqslant N} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(h)|L(n,\chi)|^2 = 0$$

holds for $d \mid N$. From Lemma 2, we can deduce that

$$\begin{aligned}
&\sideset{}{'}\sum_{h\leqslant N} S(h,n,k)\\
&= \frac{(n!)^2}{4^{n-1}k^{2n-1}\pi^{2n}} \sum_{d|k} \frac{d^{2n}}{\phi(d)} \Bigg[\frac{\phi(d)}{2}\zeta(2n)\zeta(n) \prod_{p|d}(1-p^{-2n}) \prod_{p|k}(1-p^{-n})\\
&\quad + O\left(\phi(d) d^{-n+1} \left\{\frac{N}{d}\right\}^{-n+1} + \phi(d) d^{\varepsilon} \left\{\frac{N}{d}\right\}^{n} \right)\Bigg] \qquad (12)\\
&= \frac{(n!)^2}{2^{2n-1}k^{2n-1}\pi^{2n}} \zeta(2n)\zeta(n) \sum_{d|k} d^{2n} \prod_{p|d}(1-p^{-2n}) \prod_{p|k}(1-p^{-n})\\
&\quad + O\left(\frac{1}{k^{2n-1}} \left[\sum_{d|k} d^{n+1} \left\{\frac{N}{d}\right\}^{-n+1} + \sum_{d|k} d^{2n+\varepsilon} \left\{\frac{N}{d}\right\}^{n} \right] \right).
\end{aligned}$$

Note

$$\sum_{d|k} d^{2n} \prod_{p|d} (1-p^{-2n}) = k^{2n}, \tag{13}$$

$$\begin{aligned}
\sum_{d|k} d^{n+1}\left\{\frac{N}{d}\right\}^{-n+1} &= \sum_{\substack{d|k\\ d>N}} d^{n+1}\left\{\frac{N}{d}\right\}^{-n+1} + \sum_{\substack{d|k\\ d<N}} d^{n+1}\left\{\frac{N}{d}\right\}^{-n+1}\\
&\ll \sum_{\substack{d|k\\ d>N}} d^{n+1}\left(\frac{N}{d}\right)^{-n+1} + \sum_{\substack{d|k\\ d<N}} d^{n+1}\left(\frac{1}{d}\right)^{-n+1} \qquad (14)\\
&\ll N^{-n}k^{2n+1+\varepsilon} + N^{2n}k^{\varepsilon},
\end{aligned}$$

and

$$\sum_{d|k} d^{2n+\varepsilon}\left(\frac{N}{d}\right)^{n} \ll N^n k^{n+\varepsilon}, \tag{15}$$

the last O-term of (12) can be estimated as

$$\begin{aligned}
&\frac{1}{k^{2n-1}} \left[\sum_{d|k} d^{n+1} \left\{\frac{N}{d}\right\}^{-n+1} + \sum_{d|k} d^{2n+\varepsilon} \left\{\frac{N}{d}\right\}^{n} \right] \qquad (16)\\
&\ll N^{-n}k^{2+\varepsilon} + N^{2n}k^{-2n+1+\varepsilon} + N^{n}k^{-n+1+\varepsilon}.
\end{aligned}$$

Combining with (12) and (13), we get the first part of the theorem.

(ii) If n is a positive even number, we will get from $2)$ of Lemma 1 that

$$\begin{aligned}
&\sum_{h\leqslant N}{}' S(h,n,k)\\
&=\sum_{h\leqslant N}{}'\left[\frac{(n!)^2}{4^{n-1}k^{2n-1}\pi^{2n}}\sum_{d|k}\frac{d^{2n}}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=1}}\chi(h)|L(n,\chi)|^2-\frac{(n!)^2}{4^{n-1}\pi^{2n}}\zeta^2(n)\right]\\
&=\frac{(n!)^2}{4^{n-1}k^{2n-1}\pi^{2n}}\sum_{d|k}\frac{d^{2n}}{\phi(d)}\sum_{h\leqslant N}{}'\sum_{\substack{\chi \bmod d\\ \chi(-1)=1}}\chi(h)|L(n,\chi)|^2+O(N),
\end{aligned}$$

Lemma 2 indicates that

$$\begin{aligned}
&\sum_{h\leqslant N}{}' S(h,n,k)\\
&=\frac{(n!)^2}{2^{2n-1}k^{2n-1}\pi^{2n}}\zeta(2n)\zeta(n)\sum_{d|k}d^{2n}\prod_{p|d}(1-p^{-2n})\prod_{p|k}(1-p^{-n})\\
&\quad+O\left(\frac{1}{k^{2n-1}}\left[\sum_{d|k}d^{n+1}\left\{\frac{N}{d}\right\}^{-n+1}+\sum_{d|k}d^{2n+\varepsilon}\left\{\frac{N}{d}\right\}^{n}\right]\right)+O(N)
\end{aligned}$$

We apply the same methods of $1)$ in the theorem to obtain $2)$. This completes the proof of the theorem.

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References

1. T. M. Apostol, Theorems on generalized Dedekind sums, *Pacific Journal of Mathematics*, **2** (1952), 1-9.
2. L. Carlitz, Some theorems on generalized Dedekind sums, *Pacific Journal of Mathematics*, **3** (1953), 513-522.
3. L. J. Mordell, The reciprocity formula for Dedekind sums, *Amer. J. Math.*, **73** (1951), 593-598.
4. W. Zhang, On the mean values of Dedekind sums, *Journal de Theorie des Nombres*, **8** (1996), 429-442.

5. W. Zhang, A note on the mean square values of Dedekind sums, *Acta Mathematica Hungarica*, **86** (2000), 275-289.

6. J. B. Conrey, E. Fransen, R. Klein and C. Scott, Mean values of Dedekind sums, *J. Number Theory*, **56** (1996), 214-226.

7. W. Zhang and Yuan Yi, Partial Sums of Dedekind Sums, *Progress in Natural Science*, **10(7)** (2000), 551-557.

8. M. Xie and W. Zhang, On the Mean of the Square of a Generalized Dedekind Sum, *The Ramnujan Journal*, **5** (2001), 227-236

9. W. Zhang, Some identities about general Dedekind sums and Dirichlet L-function, *Acta Mathematica Sinica*, **44** (2001), 269-272.

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