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EXISTENCE OF SOLUTIONS OF STRONG VECTOR EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we consider strong form of a vector equilibrium problem and establish some existence results for solutions of such a problem in the setting of topological vector spaces. We provide several coercivity conditions under which strong vector equilibrium problem has a solution. Our results generalize and extend the results of Bianchi and Pini [10] to the topological vector space setting.

1. Introduction

Let X be a Hasudorff topological vector space, K a nonempty convex subset of X and $f : K \times K \to \mathbb{R}$ a real-valued bifunction. The *equilibrium problem* (in short, EP) is to find $\bar{x} \in K$ such that

$$(1) \qquad f(\bar{x}, y) \geq 0, \quad \forall y \in K.$$

It is well known that EP is a unified model of several problems, namely, variational inequality problem, complementarity problem, fixed point problem, saddle point problem, Nash equilibrium problem, etc. For further details, we refer to [2, 6, 12, 14-16, 19] and the references therein.

Motivated by the pioneer work of Giannessi [17], many authors extended EP to the vector case in different ways, see for example [1, 3-5, 7] and the eferences therein.

Let Y be a topological vector space with its zero element is denoted by $\mathbf{0}$, C a closed convex pointed cone in Y and $f : K \times K \to Y$ a vector-valued bifunction. The *weak vector equilibrium problem* (in short, WVEP) is to find $\bar{x} \in K$ such that

$$(2) \qquad f(\bar{x}, y) \notin -\mathrm{intC}, \quad \forall y \in K,$$

where $\mathrm{intC} \neq \emptyset$.

The *strong vector equilibrium problem* (in short, SVEP) is to find $\bar{x} \in K$ such that

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$$(3) \qquad f(\bar{x}, y) \in C, \quad \forall y \in K.$$

In most the papers appeared in the literature on the different aspects, WVEP is considered and studied. It is worth to mention that by using the results related to WVEP and SVEP, we can obtain the corresponding results for weak efficient solutions and strong efficient solutions of a vector optimization problem. Only a few papers have appeared on the existence of solutions of SVEP; See for example [5, 16] and the references therein.

We also consider the following problem, termed as *dual strong vector equilibrium problem* (in short, DSVEP) is closely related to SVEP: Find $\bar{x} \in K$ such that

$$(4) \qquad f(y, \bar{x}) \in -C, \quad \forall y \in K.$$

We denote by S_K and S_K^D the set of all solutions of SVEP and DSVEP, respectively. Aussel and Hadisavvas [8] introduced the concept of local solutions of variational inequalities. Bianchi and Pini [10] extended this concept of local solutions to EP in the setting of reflexive Banach spaces. It is further studied in [2]. We extend this concept of local solutions for strong vector equilibrium problems in the setting of topological vector spaces.

The sets of local solutions for SVEP and DSVEP are denoted by $S_{K,loc}$ and $S_{K,loc}^D$, respectively, and defined as follows:

$$S_{K,loc}^D = \{x \in K : \text{ there exists an open neighborhood } V \text{ of } x \text{ such that } f(y, x) \in -C, \forall y \in V \cap K\},$$

$$S_{K,loc} = \{x \in K : \text{ there exists an open neighborhood } V \text{ of } x \text{ such that } f(x, y) \in C, \forall y \in V \cap K\}.$$

Obviously, $S_K^D \subseteq S_{K,loc}^D$ and $S_K \subseteq S_{K,loc}$.

The main motivation of this paper is to introduce several coercivity conditions and to study the existence of solutions for SVEP under these coercivity conditions. Our results generalize and extend the results in [2, 10] to the topological vector space setting.

2. Preliminaries and Basic Results

The following concept of upper sign continuity is vector version of the the upper sign continuity introduced by Bianchi and Pini [10] which extends the earlier notion in the framework of variational inequalities in [18].

Definition 1. Let $y \in K$ be any fixed element. A function $x \to f(x, y)$ is said to be *upper sign continuous* if for every $x \in K$,

$$f(u, y) \in C, \quad \forall u \in]x, y[\quad \Rightarrow f(x, y) \in C,$$

where $]x, y[$ denotes the open line segment joining x and y.

If f is hemicontinuous, that is, the restriction of f to line segments in K is continuous, then f is upper sign continuous. Even this fact is true when f is upper hemicontinuous.

Definition 2. A bifunction $f : K \times K \to Y$ is said to be

(i) C-*pseudomonotone* if $\forall x, y \in K$,
$$f(x, y) \in Y \setminus (-C) \quad \Rightarrow \quad f(y, x) \in -C \setminus \{\mathbf{0}\};$$

(ii) C-*quasimonotone* if $\forall x, y \in K$,
$$f(x, y) \in Y \setminus (-C) \quad \Rightarrow \quad f(y, x) \in -C;$$

(iii) C-*properly quasimonotone* if for any finite set $\{x_1, \ldots, x_n\} \subseteq K$ and for all $x \in \text{co}\{x_1, \ldots, x_n\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that $f(x_i, x) \in -C$, where $\text{co}\{x_1, \ldots, x_n\}$ denotes the convex hull of $\{x_1, \ldots, x_n\}$.

It is clear from the definition that C-pseudomonotonicity of f implies C-quasimonotonicity. But in general there is no relationship between C-properly quasimonotonicity and quasimonotonicity or pseudomonotonicity.

The following proposition is a vector version of Proposition 1 in [9]. It provides a criteria for the C-proper quasimonotonicity of a bifunction.

Proposition 1. *Let $f : K \times K \to Y$ be a vector-valued bifunction such that $f(x, x) = \mathbf{0}$ $\forall x \in K$. If one of the following conditions holds:*

(i) *the set $\{x \in K : f(x, y) \in Y \backslash (-C)\}$ is convex, or*

(ii) *the set $\{y : f(x, y) \in -C \backslash \{\mathbf{0}\}\}$ is convex and f is C-pseudomonotone,*

then, f is C-properly quasimonotone.

Proof. Suppose that (i) holds and assume contrary that f is not C-pseudomonotone. Then there exist a finite set $\{x_1, \ldots, x_n\} \subseteq K$ and $\tilde{x} \in \text{co}\{x_1, x_2, \ldots, x_n\}$ such that $f(x_i, \tilde{x}) \in Y \backslash (-C)$, $\forall i = 1, 2, \ldots, n$. Therefore, by (i) we have $f(\tilde{x}, \tilde{x}) \in Y \backslash (-C)$ and so in particular $f(\tilde{x}, \tilde{x}) \neq \mathbf{0}$ which is a contradiction with the assumption $f(x, x) = \mathbf{0}$, $\forall x \in K$. Hence f is C-properly quasimonotone.

Let (ii) be valid and on the contrary there exist a finite set $\{x_1, \ldots, x_n\} \subseteq K$ and $\tilde{x} \in \text{co}\{x_1, x_2, \ldots, x_n\}$ such that $f(x_i, \tilde{x}) \in Y \backslash (-C)$, $\forall i = 1, 2, \ldots, n$. Because f is C-pseudomonotone, we have $f(\tilde{x}, x_i) \in -C \backslash \{\mathbf{0}\}$ and so from (ii) we deduce $f(\tilde{x}, \tilde{x}) \in -C \backslash \{\mathbf{0}\}$ which is a contradiction of our assumption that $f(x, x) = \mathbf{0}$, $\forall x \in K$. Then f is C-properly quasimonotone. ∎

Remark 1.

(a) For each fixed $x \in K$, if the mapping $y \mapsto f(x, y)$ is C-convex, that is,

$$(5) \quad tf(x, y) + (1 - t)f(x, z) - f(x, ty + (1 - t)z) \in C, \quad \forall y, z \in K, \ \forall t \in [0, 1],$$

then set $\{y : f(x, y) \in -C\backslash\{0\}\}$ is convex, $\forall x \in K$. Indeed, let $\lambda \in]0, 1[$ and

$$f(x, y_i) \in -C\backslash\{\mathbf{0}\}, \quad \forall i = 1, 2,$$

and so since C is a pointed convex cone, we have

$$\lambda f(x, y_1) + (1 - \lambda)f(x, y_2) \in -C\backslash\{\mathbf{0}\}. \tag{6}$$

From (5) and (6), we obtain

$$f(x, ty + (1 - t)z) \in C\backslash\{\mathbf{0}\} + C \subseteq C\backslash\{\mathbf{0}\}.$$

(b) If $Y\backslash(-C)$ is convex and f is concave in the first variable, then (i) of Proposition 1 holds. The proof is straight forward by using

$$Y\backslash(-C) + C \subseteq Y\backslash(-C).$$

The following lemma plays a key role in this section and it provides a relationship between the solution sets $S^D_{K,loc}$ and S_K. Furthermore it is a vector version of Lemma 2.1 in [10] and moreover the strong condition quasiconvexity in the second variable for f could omit. For extending the quasiconvexity for vector valued functions we need our space be a topological vector space with a structure as a lattice.

Lemma 1. *Let K be a nonempty convex subset of X and $f : K \times K \to Y$ be a vector-valued bifunction such that the following conditions hold:*

(i) $f(x, x) \in C$ $\forall x \in K$;

(ii) *For each fixed $y \in K$, the mapping $x \mapsto f(x, y)$ is upper sign continuous;*

(iii) *If $f(x, y) \in Y\backslash C$ and $f(x, z) \in -C$, then $f(x, u) \in Y\backslash C$, $\forall u \in]y, z[$.*

Then, $S^D_{K,loc} \subseteq S_K$.

Proof. Let $z \in S^D_{K,loc}$. In order to show that $z \in S_K$, we assume contrary that there exists $y \in K$ such that

$$f(z, y) \in Y\backslash C. \tag{7}$$

From the definition of $S^D_{K,loc}$, there exists an open neighborhood V of z such that $f(v, z) \in -C$ for all $v \in K \cap V$. Since $V - z$ is a neighborhood of $\mathbf{0}$, there exists $t_0 \in]0, 1[$ such that $t(y - z) \in V - z$ for all $0 < t \leq t_0$. Let $\bar{y} = z + t_0(y - z)$ and $y_t = (1 - t)z + t\bar{y} \in [z, \bar{y}]$ for $t \in [0, 1]$. Then $y_t \in K \cap V$, since $y_t = (1-t)z+tz+t\,t_0(y-z) = z+t\,t_0(y-z)$ and $t\,t_0(y-z) \in V-z$. Hence (7) implies that $f(y_t, z) \in -C$ and by condition (i), $f(z, z) = \mathbf{0}$. Now we will show that $f(u, \bar{y}) \in C$, $\forall u \in]z, \bar{y}[$. Indeed, if $f(u, \bar{y}) \in Y\backslash C$ for some $u \in]z, \bar{y}[$, then as $f(u, z) \in -C$, we deduce from (iii) that $f(u, v) \in Y\backslash C$, $\forall v \in]z, \bar{y}[$ and in particular $f(u, u) = \mathbf{0} \in Y\backslash C$. Hence $\mathbf{0} \notin C$ which contradicts the fact that $\mathbf{0} \in C$

since C is a pointed cone. Therefore, $f(u, \bar{y}) \in C$ for all $u \in]z, \bar{y}[$. Thus by (ii), we have

$$f(z, \bar{y}) \in C. \tag{8}$$

Since $f(z, z) = 0$ and $f(z, y) \in Y \backslash C$, it follows from (iii) that $f(z, \overline{y}) \in Y \backslash C$ which contradicts (8). ■

The next example shows that condition (iii) of Lemma 1 is essential.

Example 1. Let $X = Y = \mathbb{R}$, $K = [-1, 1]$, $C = [0, \infty)$ and $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ be defined as

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) \in \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right] \text{ or } x = y, \\ -1 & \text{otherwise.} \end{cases}$$

It is clear that $f(x, x) = 0$, $\forall x \in K$ and if $f(u, y) \geq 0$, $\forall u \in]x, y[$, then $u = 0$, $\forall u \in]x, y[$, which is impossible. This shows that the mapping $x \to f(x, y)$ is upper sign continuous for each fixed $y \in K$. Since $f\left(\frac{1}{4}, \frac{1}{3}\right) < 0$ and $f\left(\frac{3}{4}, \frac{1}{3}\right) < 0$, we can easily see that $f\left(u, \frac{1}{3}\right) < 0$ does not hold $\forall u \in]\frac{1}{4}, \frac{3}{4}[$, for example, take $u = \frac{1}{3} \in]\frac{1}{4}, \frac{3}{4}[$ and so the example does not fulfill the condition (iii) of Lemma 1. Moreover, the result in Lemma 1 is not true for this example, since $x_0 = 0 \in S_{[\frac{-1}{2}, \frac{1}{2}]}$ and $S_K = \emptyset$.

The following illustration say us it is possible that $S^D_{K,loc}$ be singleton while S_K an uncountable set.

Example 2. Let $X = \mathbb{R}$, $K = [0, 1]$, $C = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $Y = \mathbb{R}^2$, and $f : K \to \mathbb{R}^2$ be defined by $f(x, y) = (x, y)$.

It is easy to verify that f satisfies all the assumptions of Lemma 1 and $S^D_{K,loc} = \{0\}$, $S_K = K$.

Remark 2.

(a) If f is C-convex in the second variable, then condition (iii) of Lemma 1 holds.

To see this, let $f(x, y) \in Y \backslash C$ and $f(x, z) \in -C$. Since $Y \backslash C$ and $-C$ are cone, we have $tf(x, u) \in Y \backslash C$ and $(1 - t)f(x, z) \in -C$ and also from $(Y \backslash C) - C \subseteq Y \backslash C$, we get

$$tf(x, u) + (1 - t)f(x, z) \in Y \backslash C. \tag{9}$$

Now by (8), (9) and $(Y \backslash C) - C \subseteq Y \backslash C$, we obtain

$$f(x, ty + (1 - t)z) \in Y \backslash C, \quad \forall t \in [0, 1].$$

This shows that condition (iii) of Lemma 1 holds.

(b) Lemma 1 improves and extends Lemma 2.4 in [19] and Lemma 2.1 in [10] to vector-valued bifunctions.

(c) If for all $x, y \in K$, $f(x,y) \in Y\backslash(-C\backslash\{\mathbf{0}\})$ implies $f(y,x) \in -C$, then $S_K \subseteq S_K^D$. Therefore, under this assumption, we have $S_K^D = S_{K,loc}^D = S_K$. Thus, if $Y = \mathbb{R}$ and $C = [0, \infty)$, we deduce Proposition 2.5 in [19]. Moreover, if f is $C-$ quasimonotone and $f(x,y) = \mathbf{0}$ implies $f(y,x) = \mathbf{0}$, then $S_K \subseteq S_K^D$. Hence we obtain the quasimonotone version of Proposition 2.5 in [19] for the vector case.

Throughout the paper, for a nonempty set A, we denote by 2^A (respectively, $\mathcal{F}(A)$) the family of all (respectively, nonempty finite) subsets of A. If A is a nonempty subset of a topological space, $\overline{A}$ and $int\ A$ denote the closure and interior of A, respectively.

Let K be a convex subset of a vector space X. Then a mapping $F : K \to 2^X$ is called a *KKM mapping* if for each nonempty finite subset A of K, co$A \subseteq F(A)$, where co$A$ denotes the convex hull of A and $F(A) = \bigcup\{F(x) : x \in A\}$.

The following lemma will be used in the sequel which is a special case of Fan-KKM principle [13].

Lemma 2. *Let X be a nonempty subset of a topological vector space E and $F : X \to 2^E$ be a KKM mapping with closed values. Assume that there exists a nonempty compact convex subset B of X such that $\bigcap_{x\in B} F(x)$ is compact. Then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

We establish the following existence result for a solution of DSVEP.

Theorem 1. *Let $f : K \times K \to Y$ be a bifunction with $f(x,x) = \mathbf{0}$, $\forall x \in K$. Assume that the following conditions hold:*

(i) *For each $A \in \mathcal{F}(K)$ and $\forall x \in coA$, there exists $y \in A$ such that $f(y,x) \in -C$;*

(ii) *For all $x \in K$, the set $\{y \in K : f(x,y) \in -C\}$ is closed in K;*

(iii) *There exist a nonempty compact subset D and a nonempty compact convex subset B of K such that $\forall y \in K\backslash D$, there exists $x \in B$ such that $f(x,y) \in Y\backslash -C$.*

Then, the solution set S_K^D of DSVEP is nonempty and compact.

Proof. For each $x \in K$, define $\Gamma : K \to 2^K$ as

$$\Gamma(x) = \{y \in K : f(x,y) \in -C\}.$$

Then, Γ is a KKM mapping. Indeed, let $A \in \mathcal{F}(K)$ and $x \in \mathrm{co}A$. Then by (i), there is $y \in A$ such that $f(y,x) \in -C$ and so $x \in \Gamma(y)$. This shows that $x \in coA\backslash A$. From $f(x,x) = 0$, we have $x \in \Gamma(x)$. Consequently, $\mathrm{co}A \subseteq \cup_{x\in A}\Gamma(x)$. So Γ is a KKM mapping. It is obvious that the solution set S_K^D equals to the set $\bigcap_{x\in K}\Gamma(x)$. Since Γ is a KKM mapping and B is compact and convex then by Lemma 2 we deduce that $\bigcap_{x\in B}\Gamma(x)$ is nonempty and by (ii) and (iii) it is a closed subset of D and so is compact. Now Lemma 2 entails

$$S_K = \bigcap_{x\in K}\Gamma(x) \neq \emptyset.$$

Furthermore the following inclusions

$$S_K = \bigcap_{x\in K}\Gamma(x) \subseteq \bigcap_{x\in K}\Gamma(x) \subseteq \bigcap_{x\in B}\Gamma(x) \subset D$$

and (ii) imply that S_K is a compact subset of K. ∎

The following lemma provides a relationship among C-quasimonotonicity, C-properly quasimonotone and $S_{K,loc}^D$. Also it is a vector version of Lemma 4.2 of [10] without assuming f is quasiconvex in the second variable.

Lemma 3. *Let $f : K \times K \to Y$ be C-quasimonotone bifunction. If for each $x \in K$, the set $\{y \in K : f(x,y) \in -C\}$ is closed in K and convex, then either f is C-properly quasimonotone or $S_{K,loc}^D \neq \emptyset$.*

Proof. If f is not C-properly quasimonotone then there exist a finite set $\{x_1,\ldots,x_n\} \subseteq K$ and $\bar{x} \in \mathrm{co}\{x_1,x_2,\ldots,x_n\}$ such that $f(x_i,\bar{x}) \in Y\backslash(-C)$, $\forall i = 1,2,\ldots,n$. Since the set $A_i = \{y \in K : f(x_i,y) \in -C\}$ is closed in K, $\forall i = 1,2,\ldots,n$ and $\bar{x} \notin \bigcup_{i=1}^n A_i$, there exists an open neighborhood V of $\bar{x}$ such that

$$f(x_i,y) \in Y\backslash(-C), \quad \forall y \in K \cap V,\ i = 1,2,\ldots,n,$$

and so by C-quasimonotonicity of f we have

$$f(y,x_i) \in -C, \quad \forall i = 1,2,\ldots,n \text{ and } y \in K \cap V,$$

and so by our assumption, that is, the set $\{y \in K : f(x,y) \in -C\}$ is convex, $\forall x \in K$, we get

$$f(y,\bar{x}) \in -C, \quad \forall y \in K \cap V.$$

Therefore, $\bar{x} \in S_{K,loc}^D$. ∎

Now we establish the following existence results for a solution of SVEP with or without C-quasimonotonicity assumption.

Proposition 2. *Let $f : K\times K \to Y$ be a C-quasimonotone bifunction satisfying the conditions of Lemmas 1 and 3 and condition (iii) of Theorem 1. Then $S_K \neq \emptyset$.*

Proof. If f is C-properly quasimonotone, then the result deduces from Theorem 1 through Lemma 1. Otherwise, we obtain the result from Lemmas 3 and 1. ∎

Proposition 3. *Let K be a nonempty convex subset of X. Assume that $f : K \times K \to Y$ satisfies condition (iii) of Lemma 1. Let $x_0 \in K$ be a local solution of SVEP in a neighborhood V of x_0. If there exists $\bar{y} \in K \cap$ int V such that $f(x_0, \bar{y}) \in -C$, then $x_0 \in S_K$.*

Proof. Assume on contrary that there exists $z \in K$ such that $f(x_0, z) \in Y \backslash C$. Then, by condition (iii) of Lemma 1, note that $f(x_0, \bar{y}) \in -C$, we have

$$f(x_0, u) \in Y \backslash C, \quad \forall u \in]\bar{y}, z[,$$

and so this is a contradiction by using $\bar{y} \in K \cap intV$ and x_0 is a local solution of SVEP in the neighborhood V of x_0. This completes the proof. ∎

3. Existence Results for SVEP in Locally Bounded Topological Vector Spaces

Recall the following definition of a locally bounded topological vector space.

A topological vector space X is said to be *locally bounded* if there is a bounded neighborhood of **0**, (see [20, pp. 156]).

Trivially, every normed space is locally bounded. A known example of locally bounded topological vector space is L^p for $0 < p < 1$ which is not normable (see [20]).

Throughout this section, we assume that X is a real Hausdorff locally bounded topological vector space, K is a nonempty unbounded convex subset of X and $f : K \times K \to Y$ is a vector-valued bifunction.

Condition C. There exists an open bounded neighborhood V of **0** such that

$$\forall x \in K \backslash \overline{V},\ \exists y \in K \cap \overline{V} \text{ satisfying } f(x, y) \in Y \backslash C.$$

This coercivity condition was considered in [2] for the scalar-valued bifunctions which extends the coercivity condition considered in [10] in the setting of reflexive Banach spaces.

The next result extends Proposition 4.1 in [2] and Proposition 2.2 in [10] and it provides a necessary condition for boundedness of the solution set of SVEP.

Proposition 4. *Let $f : K \times K \to Y$ be a vector-valued bifunction such the following conditions hold:*

(i) $f(x, x) = 0$, $\forall x \in K$;

(ii) *For each fixed $y \in K$, the set $\{x \in K : f(x, y) \in C\}$ is convex;*

(iii) *If $f(x, y) \in Y \backslash C$ and $f(x, z) \in -C$, then $f(x, u) \in Y \backslash C$, $\forall u \in]y, z[$.*

If the solution set S_K of SVEP is nonempty and bounded, then Condition C holds.

Proof. Suppose that the Condition C does not hold. Let V be an arbitrary open bounded, balanced, neighborhood of **0** and $x_0 \in S_K$. Consider positive integer n_0 such that $x_0 \in n_0 V$. Let $n > n_0$ and $W_n = \underbrace{V + V + \cdots + V}_{n}$. Since Condition C is not true and W_n is bounded, there exists $x_n \in K \setminus W_n$ such that

$$f(x_n, y) \in C, \quad \forall y \in K \cap W_n. \tag{10}$$

Since $(n-1)V \subseteq W_n$ and $x_n \notin W_n$, we have

$$t_0 = \sup\{t \in [0,1] : x_0 + t(x_n - x_0) \in (n-1)V\} < 1. \tag{11}$$

Therefore, for all positive number t' with $t_0 + t' < 1$, we deduce that

$$z_n = x_0 + (t_0 + t')(x_n - x_0) \notin (n-1)V. \tag{12}$$

We claim that $z_n \in W_n \cap K$. Indeed, we can choose small positive number t such that $t(x_n - x_0) \in V$ and $t < 2(1 - t_0)$. By (11), there exists t_1 such that $t_0 - \frac{t}{2} < t_1$ and

$$z_n = x_0 + (t_1 + t)(x_n - x_0) \in (n-1)V + V \subseteq W_n.$$

Since $x_0 \in S_K$, the convexity of the set $\{x \in K : f(x,y) \in C\}$ and (10) imply that

$$f(z_n, y) \in C, \quad \forall y \in K \cap W_n. \tag{13}$$

By (13) and Proposition 3, we obtain $z_n \in S_K$. Hence, the sequence $\{z_n\}$ is unbounded, which contradicts the boundedness of S_K. ∎

Now we establish necessary and sufficient conditions for non-emptiness of the solution set of SVEP.

Theorem 2. *Let $f : K \times K \to Y$ be a C-pseudomonotone bifunction such that the following conditions hold:*

- (i) *For all $x \in K$, $f(x,x) = 0$;*
- (ii) *For all $y \in K$, the mapping $x \mapsto f(x,y)$ is upper sign continuous;*
- (iii) *For each $x \in K$, the set $\{y \in K : f(x,y) \in -C\}$ is convex and closed in K;*
- (iv) *If $f(x,y) \in Y \setminus C$ and $f(x,z) \in -C$, then $f(x,u) \in Y \setminus C$, $\forall u \in]y,z[$.*

If S_K is nonempty and bounded, then Condition C holds. Moreover, if Condition C holds with bounded open neighborhood V and $f|co(K \cap W)$, the restriction of f to $co(K \cap W)$, satisfies conditions (i) and (iii) of Theorem 1 for every bounded neighborhood W with int $W \supseteq \overline{V}$. Then, S_K is nonempty compact and convex.

Proof. Suppose that the Condition C does not hold. Let V be a bounded open balanced neighborhood of 0, $x_0 \in S_K$ and n_0 positive integer such that $x_0 \in n_0 V$. Let $n > n_0$ and $W_n = \underbrace{V + V + \cdots + V}_{n}$. Since Condition C does not hold and W_n is a bounded neighborhood of 0, there exists $x_n \in K \setminus W_n$ such that

$$f(x_n, y) \in C, \quad \forall y \in K \cap W_n. \tag{14}$$

Since C is pointed, we have $C \cap (-C \backslash \{0\}) = \emptyset$ and therefore

$$f(x_n, y) \notin -C \backslash \{0\}, \quad \forall y \in K \cap W_n. \tag{15}$$

The C-pseudomonotonicity of f implies that

$$f(y, x_n) \in -C, \quad \forall y \in K \cap W_n. \tag{16}$$

Since f is C-psudomonotone and $x_0 \in S_K$, we have

$$f(y, x_0) \in -C, \quad \forall y \in K. \tag{17}$$

Since $(n-1)V \subseteq W_n$ and $x_n \notin W_n$, we have

$$t_0 = \sup\{t \in [0,1] : x_0 + t(x_n - x_0) \in (n-1)V\} < 1.$$

Therefore, for all positive number t' such that $t_0 + t' < 1$, we deduce

$$z_n = x_0 + (t_0 + t')(x_n - x_0) \notin (n-1)V.$$

By using (16) and (17), we obtain

$$f(y, z_n) \in -C, \quad \forall y \in W_n \cap K,$$

as the set $\{y \in K : f(x,y) \in -C\}$ is convex. Consequently, $z_n \in S^D_{K,loc}$. Therefore, by Lemma 1, we have $z_n \in S_K$. Hence, the sequence $\{z_n\}$ is unbounded, which contradicts the boundedness of S_K.

Conversely, let Condition C hold with an open neighborhood V and W be an open bounded balanced neighborhood of 0 containing V. By our assumptions, the mapping $f|co(K \cap W)$ satisfies all the conditions of Theorem 1. Then, by Theorem 1 there exists $\bar{x} \in S_{\mathrm{co}(K \cap W)}$. If $\bar{x}$ is an element of W, then by Proposition 3 $\bar{x} \in S_K$. Otherwise, by Condition C, there exists $y \in \overline{V}$ such that $f(\bar{x}, y) \in Y \backslash C$. Since $f(\bar{x}, \bar{x}) = \mathbf{0} \in -C$, by condition (iv), we have, $f(\bar{x}, u) \in Y \backslash C$ for all $u \in]\bar{x}, y[$ which contradicts with $\bar{x} \in S_{\mathrm{co}(K \cap W)}$. Therefore, $\bar{x} \in S_K$.

Now we show that S_K is a compact subset of K. To see this, let $x_\alpha \in S_K$ and $x_\alpha \to x$. Then, $f(x_\alpha, y) \in C$ for all $y \in K$ and all α. Since C is pointed, we have by $f(x_\alpha, y) \in Y \backslash (C \backslash \{\mathbf{0}\})$, $\forall y \in K$ and $\forall \alpha$. The C-pseudomonotonicity of f implies that $f(y, x_\alpha) \in -C$, $\forall y \in K$ and $\forall \alpha$. Since $x_\alpha \to x$ and the set

$\{y \in K : f(x,y) \in -C\}$ is closed for all $x \in K$ in K, we get $f(y,x) \in C$ for all $y \in K$. This means that $x \in S_K^D$ and by Lemma 1, we have $x \in S_K$ as $S_K^D \subseteq S_{K,loc}^D$. Consequently, S_K is closed in K. It follows from condition (iii) of Theorem 1 that S_K is a subset of the compact subset D of K. Moreover, C-pseudomonotonicity of f and convexity of the set $\{y \in K : f(x,y) \in -C\}$, $\forall x \in K$ imply that S_K is convex. ∎

Remark 3. Theorem 2 is the vector version of Theorem 4.1 in [2] which extends Theorem 3.7 in [14].

Condition C entails the boundedness of the solution set but the following coercivity condition allows that the solution set to be unbounded.

Condition 1. There exists an open bounded neighborhood V of **0** such that $\forall x \in K\backslash\overline{V}$ and for all $W \in \mathcal{B}$ with $W \supseteq \overline{V}$ containing x, there exists $y \in int\ W \cap K$ satisfying $f(x,y) \in -C$, where $\mathcal{B}$ is a base at **0** consists of neighborhoods of **0** for topological vector space X.

Theorem 3. *Let $f : K \times K \to Y$ be a C-pseudomonotone bifunction such that $f(x,x) = \mathbf{0}$, $\forall x \in K$. If S_K is nonempty, then Condition C1 holds. Moreover, if f satisfies conditions (ii) and (iii) of Lemma 1, Condition C1 with bounded open neighborhood V of* **0***, and conditions (i)-(iii) of Theorem 1 hold for $f|co(K \cap W)$ and for every $W \in \mathcal{B}$ with int $W \supseteq \overline{V}$, then S_K is nonempty.*

Proof. Suppose that $x_0 \in S_K$. Then $f(x_0,y) \in C$ for all $y \in K$. Since C is pointed, we have $f(x_0,y) \in Y\backslash(-C\backslash\{\mathbf{0}\})$, $\forall y \in K$. The C-pseudomonotonicity of f implies that $f(y,x_0) \in -C$, $\forall y \in K$. Then, Condition C1 trivially holds for $y = x_0$ and for every $x \in K\backslash V$, where V is an arbitrary bounded open neighborhood of 0 such that $x_0 \in V$.

To see the converse, let $W \in \mathcal{B}$. By Theorem 1 there exists $\bar{x} \in S_{co(K\cap W)}$. In the case that $\bar{x} \in \overline{V}$, by our assumption $\overline{V} \subset intW$ and Proposition 3, we obtain $\bar{x} \in S_K$. If $\overline{x} \notin \overline{V}$, by Condition C1, there exist $y \in intW \cap K$ such that $f(\bar{x},y) \in -C$. Proposition 3 implies that $\bar{x} \in S_K$. ∎

We deal with the C-quasimonotone bifunctions and establish the following existence results for a solution of SVEP in the presence of Conditions C and C1, respectively. The first theorem is a vector version of Theorem 4.1 in [10].

Theorem 4. *Let $f : K \times K \to Y$ be a C-quasimonotone bifunction satisfying the conditions of Lemma 1 and for each $x \in K$, the set $\{y \in K : f(x,y) \in -C\}$ is closed and convex. If the set S_K is bounded and S_K^D nonempty, then Condition C holds. Moreover, if f satisfies in Condition C with an open bounded balanced neighborhood V of 0, and the condition (iii) of Theorem 1 for $f|co(K \cap W)$ holds for every $W \in \mathcal{B}$ with int $W \supseteq \overline{V}$. Then, S_K is nonempty.*

Proof. There exists an integer $n_0 > 1$ such that the set S_K is a subset of $K \cap (n_0 - 1)V$, where V is a bounded open balanced neighborhood of 0. Assume that the Condition C does not hold. Then, for every $n > n_0$, there exists $x_n \in K \backslash K \cap W_n$ such that

$$f(x_n, y) \in C, \quad \forall y \in K \cap W_n. \tag{18}$$

where $W_n = \underbrace{V + V + \cdots + V}_{n}$.

We show that $f(x_n, y) \in C \backslash \{0\}$ whenever $y \in K \cap W_n$. Indeed, assume that $f(x_n, y) = 0$ for all y. By (18), $f(x_n, y) \in C$. Let $z \in K \backslash W_n$ such that $f(x_n, z) \in Y \backslash C$. From assumption (iii) of Lemma 1, we obtain

$$f(x_n, (1-t)y + tz)) \in Y \backslash C, \quad \forall t \in]0, 1[.$$

Using $(1-t)y + tz \to y$ if $t \to 0^+$ and $y \in K \cap W_n$, there exists t (small enough) such that $(1-t)y + tz \in W_n$, which is a contradiction of (18). Therefore, $f(x_n, y) \in C \backslash \{0\}$, $\forall y \in K \cap W_n$. Since P is pointed, $f(x_n, y) \in Y \backslash -C$, $\forall y \in K \cap W_n$. Thus by C-quasimonotonicity of f we have,

$$f(y, x_n) \in -C, \quad \forall y \in K \cap W_n. \tag{19}$$

Let x_0 be a point in S_K^D. It follows from $x_0 \in (n-1)V$ and $x_n \in K \backslash W_n$ (for n sufficiently large) that there exists a positive number $t \in]0, 1[$ such that

$$z_n = (1-t)x_n + tx_0 \notin (n-1)V, \quad \forall z_n \in W_n \cap K.$$

From (18), $x_0 \in S_K^D$ and the convexity of the set $\{y \in K : f(x, y) \in -C\}$, we have

$$f(y, z_n) \in -C, \quad \forall y \in K \cap W_n. \tag{20}$$

Hence, $z_n \in S_{K,loc}^D$ and so by Lemma 1, $z_n \in S_K$. Therefore, the sequence $\{z_n\}$ is unbounded which contradicts the boundedness of S_K.

For the second part, if f is $C-$properly quasimonotone, we get the result arguing as in proof of the converse part of Theorem 2. If f is not properly quasimonotone, from Lemma 3, $S_{K,loc}^D \neq \emptyset$ and so by Lemma 1, $S_K \neq \emptyset$. ■

The following theorem is vector version of Theorem 4.4 in [2] and Theorem 4.2 of [10].

Theorem 5. *Let $f : K \times K \to Y$ be a C-quasimonotone bifunction. If S_K^D is nonempty, then the Condition C1 holds. Conversely, if f satisfies the conditions of Lemmas 1, 3 and Condition C1 for a bounded neighborhood V of* **0** *and moreover condition (iii) of Theorem 1 for $f|co(K \cap W)$ holds for every bounded neighborhood W with int $W \supseteq \overline{V}$, then S_K is nonempty.*

Proof. We only prove the converse part. For this, if f is C-properly quasimonotone, we get $S_K \neq \emptyset$, by arguing as in Theorem 3. If f is not C-properly quasimonotone, then form Lemma 3, we have $S^D_{K,loc} \neq \emptyset$ and so by Lemma 1, $S_K \neq \emptyset$. ■

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