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GENERAL DECAY OF ENERGY FOR A VISCOELASTIC EQUATION WITH DAMPING AND SOURCE TERMS

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Abstract. The initial boundary value problem for a viscoelastic equation with linear damping and nonlinear source term in a bounded domain is considered. The decay rate of solution energy is discussed under some conditions on relaxation function g and initial data by adopting the perturbed energy method of [4] and modifying the methods of $[11, 17]$. Decay estimates of the energy function are also given.

1. Introduction

In this paper we consider the initial boundary value problem for the following nonlinear viscoelastic equation:

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + u_t = |u|^{p-2} u, \text{ in } \Omega \times (0,\ \infty), \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.2}$$

and boundary condition

$$u(x,t) = 0, x \in \partial\Omega, t \geq 0, \tag{1.3}$$

where $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ and $\Omega \subset R^N$, $N \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$ so that Divergence theorem can be applied. Here, $\rho > 0$, $p > 2$, and g represents the kernel of the memory term which will be stated later (see assumption (A1)).

Problem related to the equation:

$$f(u_t)u_{tt} - \Delta u - \Delta u_{tt} = 0$$

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are interesting not only from the point of view of PDE theory, but also due to its applications in mechanics. It describes a thin rod which possesses a rigid surface and whose interior is somehow permissive to slight deformations such that the material density varies according to the velocity. In this direction, Cavalcanti et al.[4] considered the following problem:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds - \gamma \Delta u_t = 0, \tag{1.4}$$

with the same initial and boundary conditions (1.2)-(1.3), where a global existence result for $\gamma \geq 0$ and an exponential decay result for $\gamma > 0$ were established under the assumptions $0 < \rho \leq \frac{2}{N-2}$ if $N \geq 3$ or $\rho > 0$ if $N = 1,\ 2$ and $g(t)$ decays exponentially. Lately, these decay results were extended by Messaoudi and Tatar [10] to a situation where a source term is present. Recently, Messaoudi and Tatar [11] studied problem (1.4) for case of $\gamma = 0$, they showed that the solution goes to zero with an exponential or polynomial rate under some restrictions on the relaxation function.

As $\rho = 0$ and there is no dispersion term, related problems have been extensively studied and several results concerning existence, decay and blow-up have been obtained $[5-8, 12, 13, 15, 16, 18]$. In this regard, Cavalcanti et al. [5] considered the following equation:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^\gamma u = 0, \text{ in } \Omega \times (0,\ \infty),$$

with the same initial and boundary conditions (1.2)-(1.3), where $a : \Omega \to R^+$ is a function which may vanish outside a subset $\omega \subset \Omega$ of positive measure and $g(t)$ decays exponentially, they proved an exponential decay result for the energy function. This result was later extended by Berrimi and Messaoudi [3] to the nonlinear damping case

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t|u_t|^m + |u_t|^\gamma u = 0,$$

by introducing a new a functional, they weakened the conditions in $a(x)$ and $g(t)$ and obtained the decay result.

Motivated by previous works, in this paper, we investigated the problem (1.1)-(1.3) with imposing nonlinear source and linear damping terms. We will use the perturbed energy method to show that the exponential or polynomial decay of the solution energy, depending on the decay rate of relaxation functions. In this way, we can extend the results of [17] where the authors considered (1.1) without source term and the results of [11] in the absence of the linear damping term. The content of this paper is organized as follows. In section 2, we give some lemmas and assumptions which will be used later, and we mention the local existence result Theorem 2.3.

In section 3, we first define an energy function $E(t)$ in (3.3) and show that it is a nonincreasing function of t. We obtain global existence and decay properties of the solutions of $(1.1)-(1.3)$ given in Theorem 3.6.

2. Preliminaries Results

In this section, we shall give some lemmas and assumptions which will be used throughout this work. We use the standard Lebesgue space $L^p(\Omega)$ and sobolev space $H_0^1(\Omega)$ with their usual products and norms.

Lemma 2.1. (Sobolev-Poincaré inequality [9]). *Let* $2 \le p \le \frac{2N}{N-2}$, *the inequality*

$$\|u\|_p \le c_s \|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega),$$

holds with some positive constant c_s.

Assume that ρ satisfies

$$0 < \rho \le \frac{2}{N-2} \text{ if } N \ge 3 \text{ or } \rho > 0 \text{ if } N = 1, 2. \tag{2.1}$$

In regard to the relaxation function $g(t)$, we assume that it verifies:
(A1) $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$1 - \int_0^\infty g(s)ds = l > 0, \tag{2.2}$$

and there exist positive constants ξ such that

$$g'(t) \le -\xi g^r(t), \ 1 \le r < \frac{3}{2}. \tag{2.3}$$

Remark 2.2. $r < \frac{3}{2}$ is imposed so that $\int_0^\infty g^{2-r}(s)ds < \infty$.

Now, we state the local existence result of the problem (1.1)-(1.3) which can be established by combining arguments of $[2, 4, 17]$.

Theorem 2.3. *Suppose that (2.1) and (A1) hold, and that* $u_0, u_1 \in H_0^1(\Omega)$. *Assume further* $2 < p \le \frac{2(N-1)}{N-2}$, *if* $N \ge 3$, $p \ge 2$, *if* $N = 1, 2$. *Then there exists a unique solution* u *of* $(1.1)-(1.3)$ *satisfying* $u, u_t \in C\left([0, T); H_0^1(\Omega)\right), T > 0.$

3. Global Existence and Energy Decay

In this section, we shall prove the exponential or polynomial decay of the solutions energy depending on the decay rate of the relaxation function. We use the perturbed energy method introduced by Cavalcanti et al. $[1,4,5]$ and some technical lemmas $[3,11]$. For the initial boundary problem (1.1)-(1.3), we define

$$I(t) \equiv I(u(t)) = \left(1 - \int_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 + (g \circ \nabla u)(t) - \|u(t)\|_p^p, \tag{3.1}$$

$$J(t) \equiv J(u(t)) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2}\|\nabla u_t(t)\|_2^2 - \frac{1}{p}\|u(t)\|_p^p, \tag{3.2}$$

and the energy function

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + J(t), \text{ for } t \geq 0, \tag{3.3}$$

where

$$(g \circ \nabla u)(t) = \int_0^t \int_\Omega g(t-s)\,|\nabla u(t) - \nabla u(s)|^2\,dxds. \tag{3.4}$$

Lemma 3.1. *$E(t)$ is a nonincreasing function on $[0,T]$ and*

$$E'(t) = -\|u_t\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \tag{3.5}$$

Proof. By multiplying the equation in (1.1) by u_t and integrating it over Ω, then using integration by parts and the assumption (A1), we obtain (3.5) for any regular solution. Then, by density arguments, we have the proof.

Lemma 3.2. *Let $u_0, u_1 \in H_0^1(\Omega)$, if $I(0) > 0$ and*

$$\alpha = \frac{c_s^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{3.6}$$

then $I(t) > 0$, for $t \in [0,T]$.

Proof. Since $I(0) > 0$, then there exists (by continuity of $u(t)$) $T^* < T$ such that

$$I(t) \geq 0, \tag{3.7}$$

for all $t \in [0, T^*]$. From (3.1) and (3.7), (3.2) gives that

$$J(t) = \frac{p-2}{2p}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + (g\circ\nabla u)(t)\right] + \frac{1}{p}I(t)$$
$$\geq \frac{p-2}{2p}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + (g\circ\nabla u)(t)\right]. \tag{3.8}$$

Thus, by (2.2), (3.3) and Lemma 3.1, we deduce

$$l\,\|\nabla u\|_2^2 \leq \left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 \leq \frac{2p}{p-2}J(t)$$
$$\leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0),\ \forall t\in[0,\ T^*]. \tag{3.9}$$

Applying Lemma 2.1, (3.9) and (3.6), we obtain

$$\|u\|_p^p \leq c_s^p\|\nabla u\|_2^p \leq \frac{c_s^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} l\,\|\nabla u\|_2^2$$
$$= \alpha l\,\|\nabla u\|_2^2 < \left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2,\ \forall t\in[0,\ T^*]. \tag{3.10}$$

Hence

$$I(t)=\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2+\|\nabla u_t\|_2^2+(g\circ\nabla u)(t)-\|u\|_p^p > 0,\ \forall t\in[0,\ T^*].$$

By repeating this procedure and using the fact that

$$\lim_{t\to T^*}\frac{c_s^p}{l}\left(\frac{2p}{l(p-2)}E(u(t),\ u_t(t))\right)^{\frac{p-2}{2}} \leq \alpha < 1.$$

This implies that we can take $T^* = T$.

Remark 3.3. It follows from Lemma 3.1 and Lemma 3.2 that the energy function is uniformly bounded and decreasing in t, which implies that

$$l\,\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 \leq \frac{2p}{p-2}E(0),\ \forall t\geq 0.$$

This infers that the solution of (1.1)-(1.3) is bounded and global in time.

Now, we define

$$G(t) = ME(t) + \varepsilon\Phi(t) + \Psi(t), \tag{3.11}$$

where M and ε are positive constants which will be specified later and

$$\Phi(t)=\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx+\int_\Omega \nabla u_t(t)\nabla u(t)dx \tag{3.12}$$

$$\Psi(t)=\int_\Omega\left(\Delta u_t-\frac{1}{\rho+1}|u_t|^\rho u_t\right)\int_0^t g(t-s)\left(u(t)-u(s)\right)dsdx. \tag{3.13}$$

Lemma 3.4. *Let* $u\in H_0^1(\Omega)$, *then, for* $\rho\geq 0$, *we have*

$$\begin{aligned}&\int_\Omega\left(\int_0^t g(t-s)\left(u(t)-u(s)\right)ds\right)^{\rho+2}dx\\ &\leq (1-l)^{\rho+1}c_s^{\rho+2}\left(\frac{4pE(0)}{l(p-2)}\right)^{\frac{\rho}{2}}\left(g\circ\nabla u\right)(t).\end{aligned} \tag{3.14}$$

Proof. By Hölder inequality, Lemma 2.1 and Remark 3.3, we get

$$\begin{aligned}&\int_\Omega\left(\int_0^t g(t-s)\left(u(t)-u(s)\right)ds\right)^{\rho+2}dx\\ &\leq\int_\Omega\left(\int_0^t g(t-s)ds\right)^{\rho+1}\left(\int_0^t g(t-s)\left|u(t)-u(s)\right|^{\rho+2}ds\right)dx\\ &\leq (1-l)^{\rho+1}c_s^{\rho+2}\int_0^t g(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|_2^{\rho+2}ds\\ &\leq (1-l)^{\rho+1}c_s^{\rho+2}\left(\frac{4pE(0)}{l(p-2)}\right)^{\frac{\rho}{2}}\left(g\circ\nabla u\right)(t).\end{aligned}$$

Lemma 3.5. *Let* u *be a solution of* (1.1)-(1.3), *then there exists two positive constants* β_1 and β_2 *such that*

$$\beta_1 E(t)\leq G(t)\leq \beta_2 E(t),$$

for ε *small enough and* M *sufficiently large*.

Proof. By Young's inequality, Lemma 2.1 and (3.9), we have

$$\begin{aligned}&\left|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx\right|\\ &\leq\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{(\rho+2)(\rho+1)}\|u\|_{\rho+2}^{\rho+2}\\ &\leq\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{c_s^{\rho+2}}{(\rho+2)(\rho+1)}\|\nabla u\|_2^{\rho+2}\\ &\leq\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{c_s^{\rho+2}}{(\rho+2)(\rho+1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{\frac{\rho}{2}}\|\nabla u\|_2^2\end{aligned} \tag{3.15}$$

and

$$\left|\int_\Omega \nabla u_t(t)\nabla u(t)dx\right| \le \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2. \tag{3.16}$$

It follows from (3.13) that

$$\begin{aligned}\Psi(t) &= -\int_\Omega \nabla u_t \int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx \\ &\quad -\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g(t-s)\left(u(t)-u(s)\right)dsdx.\end{aligned} \tag{3.17}$$

By Young's inequality and Hölder inequality, the first term in the right hand of (3.17) can be estimated as

$$\begin{aligned}&\left|-\int_\Omega \nabla u_t \int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx\right| \\ &\le \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)^2 dx \\ &\le \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1-l}{2}\left(g\circ\nabla u\right)(t).\end{aligned} \tag{3.18}$$

Like for (3.15) and using (3.14), we have

$$\begin{aligned}&\left|-\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g(t-s)\left(u(t)-u(s)\right)dsdx\right| \\ &\le \frac{1}{\rho+2}\left(\|u_t\|_{\rho+2}^{\rho+2} + \frac{(1-l)^{\rho+1}c_s^{\rho+2}}{\rho+1}\left(\frac{4pE(0)}{l(p-2)}\right)^{\frac{\rho}{2}}\left(g\circ\nabla u\right)(t)\right).\end{aligned} \tag{3.19}$$

Hence, using $(3.15)-(3.19)$, we have the following inequalities from (3.11)

$$\begin{aligned}G(t) &= ME(t) + \varepsilon\Phi(t) + \Psi(t) \\ &\le ME(t) + c_1\|u_t\|_{\rho+2}^{\rho+2} + c_2\|\nabla u\|_2^2 + c_3\|\nabla u_t\|_2^2 + c_4\left(g\circ\nabla u\right)(t)\end{aligned}$$

and

$$G(t) \ge ME(t) - c_5\left(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + \left(g\circ\nabla u\right)(t)\right)$$

where $c_1 = \frac{1+\varepsilon}{\rho+2}$, $c_2 = \varepsilon\left(\frac{c_s^{\rho+2}}{(\rho+2)(\rho+1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right)$, $c_3 = \frac{\varepsilon+1}{2}$, $c_4 = \frac{1-l}{2} + \frac{(1-l)^{\rho+1}c_s^{\rho+2}}{(\rho+2)(\rho+1)}\left(\frac{4pE(0)}{l(p-2)}\right)^{\frac{\rho}{2}}$, and $c_5 = \max(c_1, c_2, c_3, c_4)$. Thus, from the definition of

$E(t)$ by (3.3) and selecting M sufficiently large and ε small enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq G(t) \leq \beta_2 E(t).$$

Theorem 3.6. *Let $u_0, u_1 \in H_0^1(\Omega)$ be given. Suppose that (A1), (2.1), (3.6) and the hypotheses on p holds. Then for each $t_0 > 0$ the solution energy of (1.1)-(1.3) satisfies*

$$E(t) \leq L_1 e^{-kt},\ r = 1,$$
$$E(t) \leq L_2 (1+t)^{-\frac{1}{r-1}},\ r > 1,$$

where k, L_1 and L_2 are some positive constants given in the proof.

Proof. In order to obtain the decay result of $E(t)$, it is sufficient to prove that of $G(t)$. To this end, we need to estimate the derivative of $G(t)$. It follows from (3.12) that

$$\begin{aligned}\Phi'(t) =& -\|\nabla u\|_2^2 + \int_\Omega \nabla u(t) \int_0^t g(t-s)\nabla u(s)dsdx \\ & - \int_\Omega u_t u dx + \|u\|_p^p + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 .\end{aligned} \tag{3.20}$$

We estimate the second term in the right hand side of (3.20) as follows [11].

$$\begin{aligned}&\left|\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx\right| \\ \leq& \left|\int_\Omega \nabla u\int_0^t g(t-s)\left(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|\right)dsdx\right| \\ \leq& \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|\right)ds\right)^2 dx.\end{aligned} \tag{3.21}$$

Applying Hölder inequality, Young's inequality and since $\int_0^t g(s)ds \leq \int_0^\infty g(s)ds = 1-l$ by (2.2), for $\eta > 0$, we note that

$$\begin{aligned}&\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|\right)ds\right)^2 dx \\ \leq& \int_\Omega\left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\right)^2 dx \\ &+ \int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)|ds\right)^2 dx \\ &+ 2\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\right)\left(\int_0^t g(t-s)|\nabla u(t)|ds\right)dx\end{aligned}$$

$$\leq \left(\int_0^t g(s)ds\right)^2 \|\nabla u\|_2^2 + \int_\Omega \left(\int_0^t g^{2-r}(t-s)ds\right) \left(\int_0^t g^r(t-s)\left|\nabla u(s) - \nabla u(t)\right|^2 ds\right) dx$$
$$+\eta \int_\Omega \left(\int_0^t g(t-s)\left|\nabla u(t)\right| ds\right)^2 dx$$
$$+\frac{1}{\eta}\int_\Omega \left(\int_0^t g(t-s)\left|\nabla u(s) - \nabla u(t)\right| ds\right)^2 dx$$
$$\leq (1+\eta)\left(\int_0^t g(s)ds\right)^2 \|\nabla u\|_2^2 + \left(1+\frac{1}{\eta}\right)\left(\int_0^t g^{2-r}(s)ds\right)(g^r \circ \nabla u)(t)$$
$$\leq (1+\eta)(1-l)^2 \|\nabla u\|_2^2 + \left(1+\frac{1}{\eta}\right)\left(\int_0^t g^{2-r}(s)ds\right)(g^r \circ \nabla u)(t).$$

Then, substituting the above inequality into (3.21) to get

$$\left|\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx\right| \leq \frac{1+(1+\eta)(1-l)^2}{2}\|\nabla u\|_2^2 + \frac{\left(1+\frac{1}{\eta}\right)}{2}\left(\int_0^t g^{2-r}(s)ds\right)(g^r \circ \nabla u)(t). \tag{3.22}$$

For the third term, by Young's inequality and Lemma 2.1, for $\eta_1 > 0$, we have

$$\left|\int_\Omega u_t u dx\right| \leq \eta_1 c_s^2 \|\nabla u\|_2^2 + \frac{1}{4\eta_1}\|u_t\|_2^2. \tag{3.23}$$

Letting $\eta = \frac{l}{1-l}$ in (3.22) and $\eta_1 = \frac{l}{4c_s^2}$ in (3.23), we derive from (3.20) that

$$\Phi'(t) \leq -\frac{l}{4}\|\nabla u\|_2^2 + \frac{1}{2l}\left(\int_0^t g^{2-r}(s)ds\right)(g^r \circ \nabla u)(t) + \frac{c_s^2}{l}\|u_t\|_2^2 + \|u\|_p^p + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2. \tag{3.24}$$

Taking the derivative of $\Psi(t)$ in (3.13) and using the equation in (1.1), we get

$$\Psi'(t) = \int_\Omega \nabla u(t)\int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)dsdx - \int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right)\left(\int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)ds\right)dx \tag{3.25}$$

$$
\begin{aligned}
&+\int_\Omega u_t(t)\int_0^t g(t-s)\,(u(t)-u(s))\,dsdx\\
&-\int_\Omega |u|^{p-2}\,u\int_0^t g(t-s)\,(u(t)-u(s))\,dsdx\\
&-\int_\Omega \nabla u_t(t)\int_0^t g^{'}(t-s)\,(\nabla u(t)-\nabla u(s))\,dsdx\\
&-\frac{1}{\rho+1}\int_\Omega |u_t|^\rho\,u_t\int_0^t g^{'}(t-s)\,(u(t)-u(s))\,dsdx\\
&-\left(\int_0^t g(s)ds\right)\|\nabla u_t\|_2^2-\frac{1}{\rho+1}\left(\int_0^t g(s)ds\right)\|u_t\|_{\rho+2}^{\rho+2}.
\end{aligned}
$$

Similarly to (3.24), in what follows we will estimate the right hand side of (3.25). Using Young's inequality, for $\delta>0$, we get

$$
\begin{aligned}
&\left|\int_\Omega \nabla u(t)\int_0^t g(t-s)\,(\nabla u(t)-\nabla u(s))\,dsdx\right|\\
&\le \delta\,\|\nabla u\|_2^2+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-s)\,(\nabla u(t)-\nabla u(s))\,ds\right)^2dx \qquad (3.26)\\
&\le \delta\,\|\nabla u\|_2^2+\frac{1}{4\delta}\left(\int_0^t g^{2-r}(s)ds\right)(g^r\circ\nabla u)\,(t).
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\left(\int_0^t g(t-s)\,(\nabla u(t)-\nabla u(s))\,ds\right)dx\right| \qquad (3.27)\\
&\le \delta I_1+\tfrac{1}{4\delta}I_2,
\end{aligned}
$$

where

$$I_1=\int_\Omega\left(\int_0^t g(t-s)\,|\nabla u(s)|\,ds\right)^2dx$$

and

$$I_2=\int_\Omega\left(\int_0^t g(t-s)\,|\nabla u(t)-\nabla u(s)|\,ds\right)^2dx.$$

As in deriving (3.22), for $\eta>0$, we have

$$|I_1|\le(1+\eta)\,(1-l)^2\,\|\nabla u\|_2^2+\left(1+\frac{1}{\eta}\right)\left(\int_0^t g^{2-r}(s)ds\right)(g^r\circ\nabla u)\,(t) \qquad (3.28)$$

and

$$|I_2|\le\left(\int_0^t g^{2-r}(s)ds\right)(g^r\circ\nabla u)\,(t). \qquad (3.29)$$

Taking $\eta = 1$ in (3.28) and using (3.29), we then get from (3.27) that

$$\begin{aligned}&\left|\int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right)\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)dx\right| \\ &\le 2\delta\left(1-l\right)^2\|\nabla u\|_2^2 + \left(2\delta+\frac{1}{4\delta}\right)\left(\int_0^t g^{2-r}(s)ds\right)\left(g^r\circ\nabla u\right)(t).\end{aligned} \tag{3.30}$$

By Young's inequality and Lemma 2.1, the third term can be estimated as

$$\begin{aligned}&\left|\int_\Omega u_t(t)\int_0^t g(t-s)\left(u(t)-u(s)\right)dsdx\right| \\ &\le \delta\|u_t\|_2^2 + \frac{c_s^2}{4\delta}\left(\int_0^t g^{2-r}(s)ds\right)\left(g^r\circ\nabla u\right)(t).\end{aligned} \tag{3.31}$$

For the fourth term, it follows from Young's inequality, Lemma 2.1 and (3.9) that

$$\begin{aligned}&\left|\int_\Omega |u|^{p-2}u\int_0^t g(t-s)\left(u(t)-u(s)\right)dsdx\right| \\ &\le \delta\int_\Omega |u|^{2(p-1)}dx + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-s)\left(u(t)-u(s)\right)ds\right)^2dx \\ &\le \delta c_s^{2(p-1)}\|\nabla u\|_2^{2(p-1)} + \frac{c_s^2}{4\delta}\left(\int_0^t g^{2-r}(s)ds\right)\left(g^r\circ\nabla u\right)(t) \\ &\le \delta c_s^{2(p-1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{p-2}\|\nabla u\|_2^2+\frac{c_s^2}{4\delta}\left(\int_0^t g^{2-r}(s)ds\right)\left(g^r\circ\nabla u\right)(t).\end{aligned} \tag{3.32}$$

Using Young's inequality and (A1) to deal with the fifth term

$$\begin{aligned}&\left|\int_\Omega \nabla u_t(t)\int_0^t g'(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx\right| \\ &\le \delta\|\nabla u_t\|_2^2 + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g'(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)^2dx \\ &\le \delta\|\nabla u_t\|_2^2 - \frac{g(0)}{4\delta}\left(g'\circ\nabla u\right)(t).\end{aligned} \tag{3.33}$$

By Young's inequality, (2.1), Lemma 2.1 and Remark 3.3, we have

$$
\begin{aligned}
&\left|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g^{'}(t-s)\left(u(t)-u(s)\right)dsdx\right| \\
&\le \frac{1}{\rho+1}\left(\delta\left\|u_t\right\|_{2(\rho+1)}^{2(\rho+1)}+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g^{'}(t-s)\left(u(t)-u(s)\right)ds\right)^2 dx\right) \\
&\le \frac{1}{\rho+1}\left(\delta\left\|u_t\right\|_{2(\rho+1)}^{2(\rho+1)}-\frac{g(0)c_s^2}{4\delta}\int_\Omega\int_0^t g^{'}(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx\right) \\
&\le \frac{\delta c_s^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\left\|\nabla u_t\right\|_2^2-\frac{g(0)c_s^2}{4\delta\left(\rho+1\right)}\left(g^{'}\circ\nabla u\right)(t).
\end{aligned}
\tag{3.34}
$$

A substitution of (3.26)-(3.34) into (3.25) yields

$$
\begin{aligned}
\Psi^{'}(t) \le{}& \delta c_6\left\|\nabla u\right\|_2^2+c_7\left(\int_0^t g^{2-r}(s)ds\right)\left(g^r\circ\nabla u\right)(t)-c_8\left(g^{'}\circ\nabla u\right)(t) \\
&+c_9\left\|\nabla u_t\right\|_2^2+\delta\left\|u_t\right\|_2^2-\frac{1}{\rho+1}\left(\int_0^t g(s)ds\right)\left\|u_t\right\|_{\rho+2}^{\rho+2},
\end{aligned}
\tag{3.35}
$$

where $c_6 = 1+2\left(1-l\right)^2+c_s^{2(p-1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{p-2}$, $c_7 = \frac{1+c_s^2}{2\delta}+2\delta$, $c_8 = \frac{g(0)\left(1+c_s^2\right)}{4\delta}$, and $c_9 = \delta\left(1+\frac{c_s^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right)-\int_0^t g(s)ds$. Since g is positive, continuous and $g(0)>0$, then for any $t_0>0$, we have

$$
\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds = g_0,\ \forall\ t\ge t_0.
\tag{3.36}
$$

Hence, we conclude from (3.5), (3.11), (3.24), (3.35) and (3.36) that for any $t\ge t_0>0$,

$$
\begin{aligned}
G^{'}(t) ={}& ME'(t)+\varepsilon\Phi'(t)+\Psi'(t) \\
\le{}& \left(\frac{M}{2}-c_8\right)\left(g^{'}\circ\nabla u\right)(t)-\frac{g_0-\varepsilon}{\rho+1}\left\|u_t\right\|_{\rho+2}^{\rho+2}-\left(\frac{l\varepsilon}{4}-\delta c_6\right)\left\|\nabla u\right\|_2^2+\varepsilon\left\|u\right\|_p^p \\
&-\left(M-\delta-\frac{c_s^2\varepsilon}{l}\right)\left\|u_t\right\|_2^2-\left(g_0-\delta\left(1+\frac{c_s^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right)-\varepsilon\right)\left\|\nabla u_t\right\|_2^2 \\
&+\left(\frac{\varepsilon}{2l}+c_7\right)\left(\int_0^t g^{2-r}(s)ds\right)\left(g^r\circ\nabla u\right)(t).
\end{aligned}
$$

However, $g'(t)\le-\xi g^r(t)$ by (2.3), thus, we see that

$$
\begin{aligned}
G^{'}(t) \leq & -\left[\xi\left(\frac{M}{2}-c_8\right)-\left(\frac{\varepsilon}{2l}+c_7\right)\int_0^\infty g^{2-r}(s)ds\right]\left(g^{^r}\circ\nabla u\right)(t) \\
& -\frac{g_0-\varepsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2}-\left(\frac{l\varepsilon}{4}-\delta c_6\right)\|\nabla u\|_2^2+\varepsilon\|u\|_p^p \\
& -\left(M-\delta-\frac{c_s^2\varepsilon}{l}\right)\|u_t\|_2^2 \\
& -\left(g_0-\delta\left(1+\frac{c_s^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right)-\varepsilon\right)\|\nabla u_t\|_2^2 .
\end{aligned} \tag{3.36}
$$

At this point, we choose $\varepsilon < g_0$ and

$$
\delta < \min\left\{\frac{l\varepsilon}{4c_6},\ \frac{g_0-\varepsilon}{1+\frac{c_s^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho}\right\}.
$$

Once ε and δ fixed, we pick M sufficiently large so that

$$
\xi\left(\frac{M}{2}-c_8\right)-\left(\frac{\varepsilon}{2l}+c_7\right)\int_0^\infty g^{2-r}(s)ds > 0
$$

and

$$
M-\delta-\frac{c_s^2\varepsilon}{l} > 0.
$$

Therefore, for all $t \geq t_0$, we have

$$
\begin{aligned}
G^{'}(t) \leq & -c_{10}\left(g^r\circ\nabla u\right)(t)-c_{11}\|u_t\|_{\rho+2}^{\rho+2}-c_{12}\|\nabla u\|_2^2 \\
& -c_{13}\|\nabla u_t\|_2^2+\varepsilon\|u\|_p^p ,
\end{aligned} \tag{3.37}
$$

where $c_{10} = \xi\left(\frac{M}{2}-c_8\right)-\left(\frac{\varepsilon}{2l}+c_7\right)\int_0^\infty g^{2-r}(s)ds,\ c_{11} = \frac{g_0-\varepsilon}{\rho+1},\ c_{12} = \frac{l\varepsilon}{4}-\delta c_6$ and $c_{13} = g_0-\delta\left(1+\frac{c_s^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right)-\varepsilon$.

Case 1. $r = 1$

By virtue of the choice of ε, δ and M, estimates (3.37) yields, for some constant $\alpha_1 > 0$,

$$
G^{'}(t) \leq -\alpha_1 E(t),\ \forall t \geq t_0. \tag{3.38}
$$

Hence, combining (3.38) and Lemma 3.5, we have

$$
G^{'}(t) \leq -\frac{\alpha_1}{\beta_2}G(t),\ \forall t \geq t_0. \tag{3.39}
$$

An integration of (3.39) over $(t_0,\, t)$ leads to

$$G(t) \le G(t_0) e^{-\frac{\alpha_1}{\beta_2}(t-t_0)}, \ \forall t \ge t_0. \tag{3.40}$$

Therefore, (3.40) and Lemma 3.5 yield

$$E(t) \le L_1 e^{-k(t-t_0)}, \ \forall t \ge t_0, \tag{3.41}$$

where $L_1 = \frac{G(t_0)}{\beta_1}$ and $k = \frac{\alpha_1}{\beta_2}$.

Case 2. $1 < r < \frac{3}{2}$
Similar to the discussion in [11], we note that

$$\left(g^r \circ \nabla u\right)(t) \ge c_{14} \left(g \circ \nabla u\right)^r (t), \tag{3.42}$$

for some constant $c_{14} > 0$. Combining (3.37) and (3.42), we get

$$\begin{aligned} &G^{'}(t) \\ &\le -c_{15} \left(\left(g \circ \nabla u\right)^r (t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 - \|u\|_p^p \right), \forall t \ge t_0, \end{aligned} \tag{3.43}$$

here c_{15} is some positive constant. On the other hand, from the definition of $E(t)$ by (3.3) and Lemma 3.1, we have

$$\begin{aligned} &E^r(t) \\ &\le c_{16} \left[E^{r-1}(0) \left(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 - \|u\|_p^p \right) + \left(g \circ \nabla u\right)^r (t) \right], \end{aligned} \tag{3.44}$$

for all $t \ge t_0$ and some constant $c_{16} > 0$. A combination of the last two inequalities and using Lemma 3.5, we derive

$$G^{'}(t) \le -c_{17} G^r(t), \ \forall t \ge t_0, \tag{3.45}$$

for some constant $c_{17} > 0$. An integration of (3.45) over (t_0, t) gives

$$G(t) \le L_2 \left(1 + t\right)^{-\frac{1}{r-1}}, \ \forall t \ge t_0, \tag{3.46}$$

where L_2 is some positive constant. Therefore, by using Lemma 3.5 once more, we complete the proof.

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