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QUANTUM APPROXIMATION ON ANISOTROPIC SOBOLEV AND HÖLDER-NIKOLSKII CLASSES

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Abstract. We estimate the quantum query error of approximation to functions from the anisotropic Sobolev class $\mathcal{B}(W_p^{\mathbf{r}}([0,1]^d))$ and the Hölder-Nikolskii class $\mathcal{B}(H_p^{\mathbf{r}}([0,1]^d))$ in the $L_q([0,1]^d)$ norm for all $1 \le p, q \le \infty$. It turns out that for the class $\mathcal{B}(W_p^{\mathbf{r}}([0,1]^d))$ $(\mathbf{r} \in \mathbb{N}^d)$, when $p < q$, the quantum algorithms can essentially improve the rate of convergence of classical deterministic and randomized algorithms; while for the class $\mathcal{B}(H_p^{\mathbf{r}}([0,1]^d))$ and $\mathcal{B}(W_p^{\mathbf{r}}([0,1]^d))$ $(\mathbf{r} \in \mathbb{R}_+^d)$, when $p \ge q$, the optimal convergence rate is the same for all three settings.

1. Introduction and Results

The problem of the approximation of functions by their values at n points has been studied extensively in the classical settings, see [2] and the references therein. In [12], this problem was considered in the quantum model of computation for the first time. The first result of such type of problem with the exact bounds was Heinrich's analyzing the query complexity of the approximation for the embeddings between finite-dimensional L_p spaces, cf. [4]. Based on this work Heinrich determined the query complexity of approximation of embedding from classical Sobolev space $W_p^r([0,1]^d)$ into $L_q([0,1]^d)$, cf. [5]. Furthermore, in [7] Heinrich improved some lower bounds from [5]. His results show that when $p < q$ quantum algorithms can bring a squaring speedup over classical deterministic and randomized algorithms. These results are very remarkable since classical randomized methods are not better than deterministic ones. While when $p \ge q$ the optimal orders of the complexity of three settings are the same. Thus a natural question arises: assume that a $L_q([0,1]^d)$ space is given, then for which kind of function class quantum

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computation can bring speed-up on the approximation in this space and for which class quantum computation can not. In this paper we will give partial answer to this question. To this end we consider the approximation of the anisotropic Sobolev classes $\mathcal{B}(W_p^{\mathbf{r}}([0,1]^d))$ and the Hölder-Nikolskii classes $\mathcal{B}(H_p^{\mathbf{r}}([0,1]^d))$ $(\mathbf{r} \in \mathbb{R}_+^d)$ in the $L_q([0,1]^d)$ metric. By studying the corresponding n-th minimal query error, we show that for the class $\mathcal{B}(W_p^{\mathbf{r}}([0,1]^d))$ and $\mathcal{B}(H_p^{\mathbf{r}}([0,1]^d))$ $(\mathbf{r} \in \mathbb{R}_+^d)$, when $1 \le q \le p \le \infty$, the optimal convergence rate of quantum algorithm is the same as the classical algorithms, while for $1 \le p < q \le \infty$ there exists an essential speed-ups under quantum computation on the class $\mathcal{B}(W_p^{\mathbf{r}}([0,1]^d))$ $(\mathbf{r} \in \mathbb{N}^d)$.

Let Ω be a nonempty set and $\mathbb{R}$ be the field of real numbers. We denote the set of all functions from Ω to $\mathbb{R}$ by $\mathcal{F}(\Omega, \mathbb{R})$. Let G be a normed space over $\mathbb{R}$ and $S : F \to G$ be a mapping, where $F \subset \mathcal{F}(\Omega, \mathbb{R})$. We want to approximate $S(f)$ for $f \in F$ by quantum computations. We use the quantum computation model developed by Heinrich [5]. Given a quantum algorithm A for S, the output of A at input $f \in F$ is a probability measure $A(f)$ on G. The error of A for S on input f is defined as follows:

$$e(S, A, f) = \inf\{\epsilon \ge 0 : P\{\|S(f) - A(f)\| > \epsilon\} \le 1/4\}.$$

The error on F is defined as

$$e(S, A, F) = \sup_{f \in F} e(S, A, f).$$

Let $n \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. The n-th minimal query error is defined for $n \in \mathbb{N}_0$ as

$$e_n^q(S, F) := \inf\{e(S, A, F) : A \text{ is any quantum algorithm with } n_q(A) \le n\},$$

where $n_q(A)$ denotes the numbers of queries used by A.

Let $D = [0,1]^d$ be the d-dimensional unit cube and $C(D)$ be the space of continuous functions on D, equipped with the supremum norm. For $1 \le p \le \infty$, let $L_p(D)$ be the space of real-valued p-th power Lebesgue-integrable functions, endowed with the usual norm. For $F \subset C(D)$, let $I_{pq} : F \to L_q(D)$ be the identical imbedding operator $I_{pq} f = f$. For $r \in \mathbb{N}$, let $W_p^r(D)$ be the classical Sobolev space with the embedding condition $rp > d$ which consists of all functions $f \in L_p(D)$ such that for all multi-index vector $\mathbf{l} = (l_1, ..., l_d) \in \mathbb{N}^d$ with $|\mathbf{l}| = \sum_{j=1}^d l_j \le r$, the distributional partial derivative $\partial^{\mathbf{l}} f := \partial^{|\mathbf{l}|} f / \partial^{l_1} x_1 ... \partial^{l_d} x_d$ belongs to $L_p(D)$. It is well known that the space $W_p^r(D)$ is a Banach space with the norm

$$\|f\|_{W_p^r(D)} := \sum_{|\mathbf{l}| \le r} \|\partial^{\mathbf{l}} f\|_{L_p(D)}.$$

In what follows, for any Banach space X the unit ball centered at the origin is denoted by $\mathcal{B}(X)$, which is defined as $\{f \in X : \|f\|_X \le 1\}$. We use the asymptotic notation: $a_n \asymp_{\log_2} b_n$ which means that for sufficiently large n there exist constants $c_1,\ c_2 > 0$ and $\alpha_1,\ \alpha_2 \in \mathbb{R}$ such that

$$c_1(\log_2^{\alpha_1}(n+1))b_n \le a_n \le c_2(\log_2^{\alpha_2}(n+1))b_n.$$

In particular, if $\alpha_1 = \alpha_2 = 0$, then we write $a_n \asymp b_n$. Furthermore, we often use the same symbol c, c_1 for possibly different positive constants. These constants depend at most on $\mathbf{r}$ and p, q. Thus Heinrich's results, cf. [5], can be stated as follows:

Theorem A. *Let $r \in \mathbb{N}$, $1 \le p, q \le \infty$ and assume $rp > d$, then*

$$e_n^q(I_{pq}, \mathcal{B}(W_p^r(D)) \asymp \begin{cases} {}_{\log_2} n^{-r/d} & \text{if } r/d > 2/p - 2/q, \\ {}_{\log_2} n^{-2r/d+2/p-2/q} & \text{if } r/d \le 2/p - 2/q. \end{cases}$$

Moreover in [7] Heinrich proved that

Corollary A. *Let $r \in \mathbb{N}$, $1 \le p, q \le \infty$ and assume $r/d > \max(1/p, 2/p - 2/q)$. Then there is a constant $c > 0$ such that*

$$e_n^q(I_{pq}, \mathcal{B}(W_p^r(D)) \ge c \cdot n^{-r/d}.$$

Therefore when $1 \le p \le q \le \infty$, $e_n^q(I_{pq}, \mathcal{B}(W_p^r(D)) \asymp n^{-r/d}$.

Now we introduce the anisotropic function classes which we will study. Let $\delta_{i,j}$ be the Kronecker notation $\mathbf{e}_j = (\delta_{i,j})_{i=1}^d$. For a real number x, let $[x]$ denote the largest integer not exceeding x. For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}_+^d$ and $1 \le p \le \infty$, the anisotropic Sobolev space $W_p^{\mathbf{r}}(D)$ consists of all functions $f \in L_p(D)$ such that for $j \in \{1, ..., d\}$, $\partial^{[r_j]\mathbf{e}_j} f \in L_p(D)$ and

$$|f|_{W_p^{r_j}(D)} := \begin{cases} \|\partial^{r_j \mathbf{e}_j} f\|_{L_p(D)}, & r_j \in \mathbb{N}, \\ \sup\limits_{h_j > 0} \dfrac{\omega(\partial^{[r_j]\mathbf{e}_j} f, h_j, D)_p}{h_j^{r_j - [r_j]}} & r_j \in \mathbb{R}_+ \backslash \mathbb{N}, \end{cases}$$

is finite, where

$$\omega(f, h_j, D)_p = \sup_{0 \le \sigma_j \le h_j} \|f(\cdot + \sigma_j \mathbf{e}_j) - f(\cdot)\|_{L_p(D)}$$

is the p-th modulus of continuity of f at the j-th coordinate. The space $W_p^{\mathbf{r}}(D)$ is a Banach space with the norm

$$\|f\|_{W_p^{\mathbf{r}}(D)} := \|f\|_{L_p(D)} + \sum_{j=1}^d |f|_{W_p^{r_j}(D)}.$$

For $\mathbf{r} \in \mathbb{R}_+^d$, the Hölder-Nikolskii space $H_p^{\mathbf{r}}(D)$ consists of all functions that

$$|f|_{H_p^{r_j}(D)} = \sup_{h_j > 0} \frac{\omega_{a_j}(f, h_j, D)_p}{h_j^{r_j}}$$

is finite, where $a_j = [r_j] + 1, j = 1, \dots, d$

$$\omega_{a_j}(f, h_j, D)_p = \sup_{0 \le \sigma_j \le h_j} \|\Delta_{\sigma_j}^{a_j}(f, \cdot)\|_{L_p(D)}$$

is the modulus of smoothness of f at the j-th coordinate in $L_p(D)$, $\Delta_{\sigma_j}^{a_j}$ is the usual a_j-th forward partial difference of step length σ_j in the j-th coordinate direction. The space $H_p^{\mathbf{r}}(D)$ is a Banach space with the norm

$$\|f\|_{H_p^{\mathbf{r}}(D)} := \|f\|_{L_p(D)} + \sum_{j=1}^{d} |f|_{H_p^{r_j}(D)}.$$

We introduce the notation

$$g(\mathbf{r}) = \left(\sum_{j=1}^{d} \frac{1}{r_j} \right)^{-1}, \tag{1.1}$$

which will be used in our error estimates. We assume that $g(\mathbf{r}) > 1/p$, which implies that the space $W_p^{\mathbf{r}}(D)$ and $H_p^{\mathbf{r}}(D)$ can be continuously imbedded into $C(D)$, see [10].

Next we recall the results of the approximation problems on these classes in the deterministic setting. It is known from [14] that the n-th deterministic minimal error of the linear approximation of the embedding I_{pq} on the class F is defined as

$$e_n^{det}(I_{pq}, F) = \inf_{S_n} \sup_{f \in F} \left\| I_{pq} f - S_n f \right\|_{L_q(D)}$$

where $S_n f = \sum_{i=1}^{n} f(x_i)\phi_i$ and the infimum is taken over all $\{x_i\}_{i=1}^{n} \subset D$ and $\{\phi_i\}_{i=1}^{n} \subset L_q(D)$.

It is known from [14, 2] that

Theorem B. *Let* $\mathbf{r} \in \mathbb{R}_+^d$, $1 \le p, q \le \infty$ *and assume* $g(\mathbf{r}) > 1/p$. *Let* F *be one of the classes* $\mathcal{B}(W_p^{\mathbf{r}}(D))$ *or* $\mathcal{B}(H_p^{\mathbf{r}}(D))$. *Then*

$$e_n^{det}(I_{pq}, F) \asymp n^{-g(\mathbf{r}) + (1/p - 1/q)_+},$$

where $(1/p - 1/q)_+ = \max\{1/p - 1/q, 0\}$.

Note that by the method in [11] one can prove that the above asymptotic relation also holds for the randomized setting. That is, the randomized method could not bring improvement on these classes. In the quantum setting, we obtain the following results.

Theorem 1. *Let* $\mathbf{r} \in \mathbb{R}_+^d$, $1 \le p, q \le \infty$ *and assume* $g(\mathbf{r}) > 1/p$. *Then*

$$e_n^q(I_{pq}, \mathcal{B}(W_p^{\mathbf{r}}(D))) \ge \begin{cases} c \cdot n^{-g(\mathbf{r})} & \text{if } g(\mathbf{r}) > 2/p - 2/q, \\ c \cdot_{\log_2} \; n^{-2g(\mathbf{r}) + 2/p - 2/q} & \text{if } g(\mathbf{r}) \le 2/p - 2/q. \end{cases}$$

Theorem 2. *Let* $\mathbf{r} \in \mathbb{R}^d_+$, $1 \le q \le p \le \infty$ *and assume* $g(\mathbf{r}) > 1/p$. *Let* F *be one of the classes* $\mathcal{B}(W^{\mathbf{r}}_p(D))$ *or* $\mathcal{B}(H^{\mathbf{r}}_p(D))$. *Then*

$$e^q_n(I_{p,q}, F) \asymp n^{-g(\mathbf{r})}.$$

Theorem 3. *Let* $\mathbf{r} \in \mathbb{N}^d$, $1 \le p, q \le \infty$ *and assume* $g(\mathbf{r}) > 1/p$. *Then*

$$e^q_n(I_{pq}, \mathcal{B}(W^{\mathbf{r}}_p(D))) \asymp \begin{cases} {}_{\log_2}\ n^{-g(\mathbf{r})} & \text{if } g(\mathbf{r}) > 2/p - 2/q, \\ {}_{\log_2}\ n^{-2g(\mathbf{r})+2/p-2/q} & \text{if } g(\mathbf{r}) \le 2/p - 2/q. \end{cases}$$

2. Some Auxiliary Results

As in the study of the classical Sobolev class, the basic idea is reducing the estimate of the complexity of the anisotropic Sobolev embedding to that of the embedding of finite-dimensional L^N_p into L^N_q spaces. However, we will use a more elegant technique to define the reduction mapping directly without using the mappings γ and β to discretize the reduction mapping, cf. [5]. To this end, we reformulate our problem as a tuple $P = (F, G, S, \Omega)$. Note that here we also view Ω as a set of linear functionals on F, i.e. $\Omega = \{\mathbf{x}(f) : \mathbf{x} \in \Omega\}$, where $\mathbf{x}(f) = f(\mathbf{x})$ for $f \in F$. For a given problem $P = (F, G, S, \Omega)$ we will reduce the estimate of its n-th minimal quantum query error to that of another problem $\tilde{P} = (\tilde{F}, \tilde{G}, \tilde{S}, \tilde{\Omega})$. Let us specify the assumptions. Let $R : F \to \tilde{F}$ be a mapping such that there exist a $\kappa \in \mathbb{N}$, mappings $\eta_j : \tilde{\Omega} \to \Omega, j \in \mathbb{Z}[0, \kappa)$ and $\varrho : \tilde{\Omega} \times \mathbb{R}^\kappa \to \mathbb{R}$ with

$$R(f)(\tilde{\mathbf{x}}) = \varrho(\tilde{\mathbf{x}}, f(\eta_0(\tilde{\mathbf{x}})), ..., f(\eta_{\kappa-1}(\tilde{\mathbf{x}}))) \tag{2.1}$$

for all $f \in F$ and $\tilde{\mathbf{x}} \in \tilde{\Omega}$. Furthermore, let $\Phi : \tilde{G} \to G$ be a Lipschitz mapping and assume that for all $f \in F$

$$S(f) = \Phi \circ \tilde{S} \circ R(f). \tag{2.2}$$

The following Proposition which is used for our reductions is a corollary of Proposition 5 in [6].

Proposition. *Assume that* S, $\tilde{S}$, *and* R *satisfy the above* (2.1)-(2.2) *and* F *is the unit ball of a Banach space* X. *Suppose sup* $_{f \in F}|f(\mathbf{x})| < \infty$ *for each* $\mathbf{x} \in \Omega$ *and* S *is uniformly continuous on* F. *Then for all* $n \in \mathbb{N}_0$,

$$e^q_{2\kappa n}(S, F) \le \|\Phi\|_{Lip} e^q_n(\tilde{S}, \tilde{F}).$$

In the process of reductions we need the following lemmas.

Lemma 1. *Let G, $\tilde{G}$ be two normed spaces and S be a mapping from F to G, T be a bounded linear operator from G to $\tilde{G}$. Then*

$$e_n^q(T \circ S, F) \leq \|T\| e_n^q(S, F).$$

Lemma 2. *Let Ω and $F \subseteq \mathcal{F}(\Omega, \mathbb{R})$ be nonempty sets. Let $k \in \mathbb{N}_0$ and $S_l : F \to G$ $(l = 0, \ldots, k)$ be mappings. Define $S : F \to G$ by $S(f) = \sum_{l=0}^k S_l(f)$. Let $n_0, \ldots, n_k \in \mathbb{N}_0$. Assume that $v_0, \ldots, v_k \in \mathbb{N}$ satisfy $\sum_{l=0}^k e^{-v_l/8} \leq 1/4$. Put $n = \sum_{l=0}^k v_l n_l$. Then*

$$e_n^q(S, F) \leq 2 \sum_{l=0}^k e_{n_l}^q(S_l, F).$$

Lemma 3. *Let S, $T : F \to G$ be any mappings, $n \in \mathbb{N}_0$ and assume that $e_n^q(S, F)$ is finite. Then the following hold:*

(i) $e_n^q(T, F) \leq e_n^q(S, F) + \sup_{f \in F} |T(f) - S(f)|$.

(ii) *If S is a linear operator from $\mathcal{F}(\Omega, \mathbb{R})$ to G, then for all $\lambda \in \mathbb{R}$*

$$e_n^q(S, \lambda F) = |\lambda| \cdot e_n^q(S, F) .$$

Next we will exploit the results of the approximation of finite imbeddings, cf. [4,7]. Denote $\mathbb{Z}[0, N) := \{0, \ldots, N-1\}$ for $N \in \mathbb{N}$. Let L_p^N be the Banach space of all functions $f : \mathbb{Z}[0, N) \to \mathbb{R}$, equipped with the norm

$$\|f\|_{L_p^N} = \left(\frac{1}{N} \sum_{i=0}^{N-1} |f(i)|^p \right)^{1/p}$$

if $1 \leq p < \infty$, and

$$\|f\|_{L_\infty^N} = \max \Big\{ |f(i)| : i \in \mathbb{Z}[0, N) \Big\}.$$

Let $I_{pq}^N : L_p^N \to L_q^N$ be the identical imbedding operator $I_{pq}^N f = f$.

Theorem C. *Let $1 \leq p, q \leq \infty$. Then*
(*i*)

$$e_n^q(I_{pq}^N, \mathcal{B}(L_p^N)) \leq \begin{cases} c \cdot \min \left(\left(\frac{N}{n} \left(\log_2 \left(n/\sqrt{N} + 2 \right) \right) \right)^{2/p-2/q}, N^{1/p-1/q} \right) & \text{if } p < q, \\ 1 & \text{if } p \geq q. \end{cases}$$

(*ii*) *For $n \leq c_0 N$ then*

$$e_n^q(I_{pq}^N, \mathcal{B}(L_p^N)) \geq \begin{cases} c \cdot \min \left((\frac{N}{n})^{2/p-2/q} \left(\log_2 \left(n/\sqrt{N} + 2 \right) \right)^{-2/q}, N^{1/p-1/q} \right) & \text{if } p < q, \\ 1/8 & \text{if } p \geq q. \end{cases}$$

3. The Proof of Results

We first establish some lemmas which will be used in the proof of Theorem 1. For a subset $E \subset \mathbb{R}^d$, we denote its characteristic function by χ_E, that is, if $\mathbf{x} \in E$, then $\chi_E(\mathbf{x}) = 1$, otherwise $\chi_E(\mathbf{x}) = 0$. For $1 \leq p \leq \infty$, let q be the exponent conjugate to p, i.e., $1/p + 1/q = 1$. Define the bilinear functional $< \cdot, \cdot >$: $L_p(D) \times L_q(D) \to \mathbb{R}$ as

$$< f, g > := \int_D f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}.$$

For $f \in \mathcal{F}(\mathbb{R}^d, \mathbb{R})$ let $\operatorname{supp} f$ denote the closure of the set $\{\mathbf{x} \in \mathbb{R}^d : f(\mathbf{x}) \neq 0\}$.

Lemma 4. *Let $(\psi_i)_{i=1}^l$ be a collection of functions in $\mathcal{B}(L_p(D))$. Let $E_i = \operatorname{supp} \psi_i$. If $\sum_i^l \chi_{E_i}(\mathbf{x}) \leq M$ holds a.e., then for any $f \in L_q(D)$*

$$\left(\sum_{i=1}^{l} | < f, \psi_i > |^q \right)^{1/q} \leq M^{1/q} \|f\|_{L_q(D)} .$$

Proof. We consider the case $1 \leq q < \infty$ first. Since for each i, we have

$$\begin{aligned} | < f, \psi_i > | &\leq \int_D |f(\mathbf{x})||\psi_i(\mathbf{x})| d\mathbf{x} \\ &= \int_D |f(\mathbf{x})| \chi_{E_i}(\mathbf{x}) |\psi_i(\mathbf{x})| d\mathbf{x} \\ &\leq \left(\int_D |f(\mathbf{x})|^q \chi_{E_i}(\mathbf{x}) d\mathbf{x} \right)^{1/q} \|\psi_i\|_{L_p(D)} \\ &\leq \left(\int_D |f(\mathbf{x})|^q \chi_{E_i}(\mathbf{x}) d\mathbf{x} \right)^{1/q}, \end{aligned} \tag{3.1}$$

where the second inequality is derived from Hölder's inequality and the third inequality is yielded from $\psi_i \in \mathcal{B}(L_p(D))$. It follows that

$$\begin{aligned} \sum_{i=1}^{l} | < f, \psi_i > |^q &\leq \sum_{i=1}^{l} \int_D |f(\mathbf{x})|^q \chi_{E_i}(\mathbf{x}) d\mathbf{x} \\ &= \int_D |f(\mathbf{x})|^q \sum_{i=1}^{l} \chi_{E_i}(\mathbf{x}) d\mathbf{x} \\ &\leq M \int_D |f(\mathbf{x})|^q d\mathbf{x}. \end{aligned} \tag{3.2}$$

Next we consider the case $q = \infty$. For each i, we have $| < f, \psi_i > | \leq \|f\|_{L_\infty(D)}$ $\|\psi_i\|_{L_1(D)} \leq \|f\|_{L_\infty(D)}$. Thus the lemma is proved.

Lemma 5. *Let* $(\psi_i)_{i=1}^l$ *be a collection of functions in* $\mathcal{B}(L_p(D))$. *Let* $E_i = \operatorname{supp}\psi_i$. *If* $\sum_i^l \chi_{E_i}(\mathbf{x}) \leq M$ *holds a.e., then*

$$\left\|\sum_{i=1}^l a_i\psi_i\right\|_{L_p(D)} \leq M^{1/q}\left(\sum_{i=1}^l |a_i|^p\right)^{1/p}.$$

Proof. Let $g \in L_q(D)$ we have

$$< \sum_{i=1}^l a_i\psi_i, g >= \sum_{i=1}^l a_i < \psi_i, g > .$$

By Hölder's inequality we get

$$\sum_{i=1}^l |a_i < \psi_i, g > | \leq \left(\sum_{i=1}^l |a_i|^p\right)^{1/p}\left(\sum_{i=1}^l | < \psi_i, g > |^q\right)^{1/q}.$$

By lemma 4, we have

$$\left(\sum_{i=1}^l | < \psi_i, g > |^q\right)^{1/q} \leq M^{1/q}\|g\|_{L_q(D)}.$$

Hence

$$| < \sum_{i=1}^l a_i\psi_i, g > | \leq M^{1/q}\left(\sum_{i=1}^l |a_i|^p\right)^{1/p}\|g\|_{L_q(D)}.$$

Since this inequality holds for all $g \in L_q(D)$, the desired inequality is known from [12], see lemma 12 in Chapter 6.

Lemma 6. *Let* $(\psi_i)_{i=1}^l$ *be a collection of functions in* $\mathcal{B}(H_p^{\mathbf{r}}(D))$. *Let* $E_i = \operatorname{supp}\psi_i$. *If* $i \neq j$, $E_i \cap E_j = \emptyset$, *then for* $j = 1, 2...d$

$$\left|\sum_{i=1}^l b_i\psi_i\right|_{H_p^{r_j}(D)} \leq (a_j+1)^{1/q}\left(\sum_{i=1}^l |b_i|^p\right)^{1/p}, \tag{3.3}$$

where $a_j = [r_j] + 1$.

Proof. Let $f = \sum_{i=1}^l b_i\psi_i$. We have

$$\Delta_{\sigma_j}^{a_j} f = \sum_{i=1}^l b_i\Delta_{\sigma_j}^{a_j}\psi_i.$$

According to the definition of $\Delta_{\sigma_j}^{a_j}$, we have

$$F_i := \operatorname{supp}(\Delta_{\sigma_j}^{a_j}\psi_i) \subset \bigcup_{m=0}^{a_j}(E_i - m\sigma_j\mathbf{e}_j),$$

where $E_i - m\sigma_j\mathbf{e}_j = \{\mathbf{x} - m\sigma_j\mathbf{e}_j : \mathbf{x} \in E_i\}$. Since E_i are disjoint, we have $\sum_{i=1}^{l}\chi_{E_i}(\mathbf{x}) \le 1$. And hence

$$\sum_{i=1}^{l}\chi_{F_i}(\mathbf{x}) \le a_j + 1.$$

Since $\psi_i \in \mathcal{B}(H_p^{\mathbf{r}}(D))$, we have for any $0 < \sigma_j \le h_j$,

$$\|\Delta_{\sigma_j}^{a_j}\psi_i\|_{L_p(D)} \le \omega_{a_j}(\psi_i, h_j, D)_p \le h_j^{r_j}.$$

By lemma 5 we obtain

$$\|\Delta_{\sigma_j}^{a_j}f\|_{L_p(D)} \le (a_j+1)^{1/q}(\sum_{i=1}^{l}|b_i|^p)^{1/p}h_j^{r_j}.$$

Hence for $h_j > 0$

$$\omega_{a_j}(f, h_j, D)_p \le (a_j+1)^{1/q}(\sum_{i=1}^{l}|b_i|^p)^{1/p}h_j^{r_j}.$$

The lemma is proved.

Now we are ready to prove Theorem 1. For $\mathbf{m} \in \mathbb{R}^d, k \in \mathbb{N}$ set

$$\mathbf{m}^k := (m_1^k, ..., m_d^k).$$

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, we define

$$\mathbf{a} \circ \mathbf{b} := (a_1b_1, \ldots, a_db_d).$$

Proof of Theorem 1. We begin with the decomposition of the cube D as in [8, 9]. Let n_0 be sufficiently large integer such that $n_0^{\frac{g(\mathbf{r})}{r_j}} > 2$. let

$$m_j(\mathbf{r}) = [n_0^{\frac{g(\mathbf{r})}{r_j}}], \ \ j = 1, \ldots, d.$$

Define P_0 as

$$P_0 = \sum_{j=1}^{d}\log_2 m_j(\mathbf{r}).$$

Then we have

$$m_j^{r_j} \asymp 2^{P_0 g(\mathbf{r})}, \ j = 1, \ldots, d. \tag{3.4}$$

We split the cube D into $2^{P_0 k}$ congruent rectangles of disjoint interior, i.e.

$$D = \bigcup_{i=0}^{2^{P_0 k}-1} D_{li}$$

with side length vector $(\frac{1}{m_1(\mathbf{r})}, \ldots, \frac{1}{m_d(\mathbf{r})})$. Let $\mathbf{s}_{li}$ denote the point in D_{li} with the smallest Euclidean norm. We first consider the case that the smoothness index $\mathbf{r} \notin \mathbb{N}^d$. Let $C^\infty(\mathbb{R}^d)$ denote the set of infinitely differentiable functions on $\mathbb{R}^d$ and define its subset $C_0^\infty(\mathbb{R}^d)$ as

$$C_0^\infty(\mathbb{R}^d) := \{f : f \in C^\infty(\mathbb{R}^d), \, \mathrm{supp} f \subset (0,1)^d\}.$$

Let $\psi \in C_0^\infty(\mathbb{R}^d) \bigcap \mathcal{B}(W_p^{\mathbf{r}}(D))$ and assume $\sigma = \int_D \psi(x)dx > 0$. Let $k \in \mathbb{N}_0$, $N = 2^{P_0 k}$. For $i \in \mathbb{Z}[0, N)$, define the restriction operator $R_{ki} : \mathcal{F}(D, \mathbb{R}) \to \mathcal{F}(D, \mathbb{R})$ by

$$(R_{ki} f)(\mathbf{s}) := \begin{cases} f(\mathbf{m}^k \circ (\mathbf{s} - \mathbf{s}_{ki})) & \text{if } \mathbf{s} \in D_{ki}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

For $i \in \mathbb{Z}[0, N)$, set

$$\psi_i = R_{ki}\psi.$$

We have

$$\|\psi_i\|_{L_p(D)} \asymp 2^{-P_0 k/p} \|\psi\|_{L_p(D)}. \tag{3.6}$$

Let r_j be some coordinate of $\mathbf{r}$ such that $r_j \in \mathbb{R}^+ \backslash \mathbb{N}$. By the definition of modulus of continuity, we have

$$\omega(\partial^{[r_j]\mathbf{e}_j}\psi_i, h_j, D)_p \le c \cdot m_j^{[r_j]k} 2^{-P_0 k/p} \omega(\partial^{[r_j]\mathbf{e}_j}\psi, m_j^k h_j, D)_p$$

and together with (3.4) we have

$$|\psi_i|_{W_p^{\mathbf{r}}(D)} \le c \cdot 2^{(g(\mathbf{r}) - 1/p)P_0 k} |\psi|_{W_p^{\mathbf{r}}(D)}.$$

Combining this with (3.6) we have

$$\|\psi_i\|_{W_p^{\mathbf{r}}(D)} \le c \cdot 2^{(g(\mathbf{r}) - 1/p)P_0 k} \|\psi\|_{W_p^{\mathbf{r}}(D)} \le c \cdot 2^{(g(\mathbf{r}) - 1/p)P_0 k}.$$

Note that the supports of the ψ_i are disjoint. Therefore it follows from lemma 5 and lemma 6

$$\|\sum_{i=0}^{N-1} a_i \psi_i\|_{W_p^{\mathbf{r}}(D)} \le c \cdot 2^{P_0 g(\mathbf{r}) k} \|(a_i)_{i=0}^{N-1}\|_{L_p^N}. \tag{3.7}$$

When $\mathbf{r} \in \mathbb{N}^d$, it is not difficult to prove that (3.7) still holds.

We will reduce the problem $(\mathcal{B}(L_p^N), \, L_q^N, \, I_{pq}^N, \, \mathbb{Z}[0, N))$ to $(\mathcal{B}(W_p^{\mathbf{r}}(D)), \, L_q(D), \, I_{pq}, \, D)$. To this end, we define the reduction mapping $\Gamma : \mathcal{B}(L_p^N) \to W_p^{\mathbf{r}}(D)$ by

$$\Gamma(f) = \sum_{i=0}^{N-1} f(i)\psi_i.$$

It is known from (3.7)

$$\Gamma(\mathcal{B}(L_p^N)) \subset c \cdot 2^{P_0 g(\mathbf{r})k} \mathcal{B}(W_p^{\mathbf{r}}(D)). \tag{3.8}$$

Define $\Phi : L_q(D) \to L_q^N$ by

$$\Phi(f)(i) = N \int_{D_{ki}} f(\mathbf{t}) d\mathbf{t}.$$

Note that

$$\Phi\psi_i = \sigma \mathbf{e}_i^N$$

where $\mathbf{e}_i^N$ denotes the i-th unit vector in L_p^N. As in [5] we can prove $\|\Phi\|_{Lip} \leq 1$ and

$$\Phi \circ I_{pq} \circ \Gamma = \sigma I_{pq}^N. \tag{3.9}$$

Define $\eta : D \to \mathbb{Z}[0, N)$ by

$$\eta(\mathbf{s}) = \min\{i : \mathbf{s} \in D_{ki}\},$$

then $\Gamma(f)(\mathbf{s}) = f(\eta(\mathbf{s}))\psi_{\eta(s)}(\mathbf{s})$. For $f \in \mathcal{B}(L_p^N)$, we have

$$|f(i)| \leq N^{1/p}.$$

Hence by using (3.8)-(3.9) and the proposition, we have

$$e_{2n}^q(I_{pq}^N, \mathcal{B}(L_p^N)) \leq c \cdot 2^{P_0 g(\mathbf{r})k} e_n^q(I_{pq}, \mathcal{B}(W_p^{\mathbf{r}}(D))). \tag{3.10}$$

For the case $g(\mathbf{r}) > 2/p - 2/q$, let $k = [P_0^{-1}(\log_2(n/c_1) + 1]$, where c_0 is the constant from Theorem C. We have

$$n \asymp 2^{P_0 k}. \tag{3.11}$$

In the case $g(\mathbf{r}) \leq 2/p - 2/q$, which implies $p < q$, we set $k = [P_0^{-1}(\log_2(n^2/c_1) + 1]$. Then we have

$$n^2 \asymp 2^{P_0 k}. \tag{3.12}$$

The desired lower bounds follows from (3.10)-(3.12) and part ii) of Theorem C.

Proof of Theorem 2. By the inclusion that $\mathcal{B}(W_p^{\mathbf{r}}(D)) \subset \mathcal{B}(H_p^{\mathbf{r}}(D))$, see [10], it suffices to prove the upper bounds for $\mathcal{B}(H_p^{\mathbf{r}}(D)$ and the lower bounds for $\mathcal{B}(W_p^{\mathbf{r}}(D))$. The upper bound follows from Theorem B and the lower bound follows from Theorem 1.

The proof of Theorem 3 relies on the following results of the approximation of functions from anisotropic Sobolev space by polynomials of coordinate degree, cf. [1].

Lemma 7. Let $\mathbf{r} \in \mathbb{N}^d$, $P_{\mathbf{r}} := \mathrm{span}\{\prod_{j=1}^d t_j^{l_j} : \mathbf{l} \in \mathbb{N}^d, \mathbf{t} \in \mathbb{R}^d, l_j < r_j, \ j = 1, \ldots, d\}$. Then for each $f \in W_p^{\mathbf{r}}(D)$, $1 \leq p \leq \infty$, there exists a polynomial $g \in P_{\mathbf{r}}$ such that

$$\|f - g\|_{L_\infty(D)} \leq c \cdot \sum_{j=1}^{d} |f|_{W_p^{r_j}(D)}.$$

Proof of Theorem 3. According to Theorem 1 and Theorem 2 it suffices to prove the upper bounds for the case $p < q$. Let P be a projection of $C(D)$ onto $P_{\mathbf{r}}$ with the form

$$Pf = \sum_{i=0}^{\kappa-1} f(\mathbf{t}_i)\phi_i,$$

where $\phi_i \in P_{\mathbf{r}}(D)$, $\kappa = \mathrm{dim} P_{\mathbf{r}}$. Therefore for any $g \in P_{\mathbf{r}}$, $Pg = g$. By Lemma 7

$$\begin{aligned} \|f - Pf\|_{L_q(D)} &= \|f - g - P(f-g)\|_{L_q(D)} \\ &\leq (1 + \sum_{i=0}^{\kappa-1} \|\phi_i\|_{L_q(D)})\|f - g\|_{C(D)} \\ &\leq c \cdot |f|_{W_p^{\mathbf{r}}(D)} . \end{aligned} \tag{3.13}$$

Now we introduce the operator $E_{li} : C(D) \to C(D)$ by setting

$$(E_{li}f)(\mathbf{s}) = f(\mathbf{s}_{li} + \mathbf{m}^{-l} \circ \mathbf{s}). \tag{3.14}$$

For $l \in \mathbb{N}_0$, let

$$P_l f = \sum_{i=0}^{2^{P_0 l}-1} R_{li} P E_{li} f. \tag{3.15}$$

Then by using similar arguments in [5] we have

$$\|f - P_l f\|_{L_q(D)} \leq c \cdot 2^{-(g(\mathbf{r})+1/p-1/q)P_0 l} \|f\|_{W_p^{\mathbf{r}}(D)}. \tag{3.16}$$

Similarly to [5], we choose l^* so that f is approximated by $P_{l^*} f$ which provides the desired precision. Then we split P_{l^*} into the sum of a single operator P_{l_0} with number of function values of the order n, and a hierarchy of operators $P_l'(l = l_0, \ldots, l^* - 1)$. We compute P_{l_0} deterministically and reduce the computation of $P_l' f$ to that of the approximation of operators $I_{pq}^{N_l}$ for proper N_l. Then we can continue our error estimate by applying Theorem C. Define $P'f := (P_1 - P)f$. Then

$$P'f = \sum_{i=0}^{2^{P_0}-1} \sum_{j=0}^{\kappa-1} f(\mathbf{s}_{1i} + (\mathbf{m}^{-1} \circ \mathbf{t}_j)) R_{1,i}\phi_j - \sum_{j=0}^{\kappa-1} f(\mathbf{t}_j)\phi_j \tag{3.17}$$
$$= \sum_{j=0}^{\kappa'-1} \Big(\sum_{k=0}^{\kappa''-1} a_{jk} f(\mathbf{t'}_{jk}) \Big) \psi_j$$

where $\kappa', \kappa'' \le \kappa(2^{P_0}+1)$. The linear independence of $\{\psi_j\}$ implies for $1 \le u \le \infty$

$$\Big\| \sum_{j=0}^{\kappa'-1} \alpha_j \psi_j \|_{L_u(D)} \asymp \|(\alpha_j)\Big\|_{L_u^{\kappa'}}. \tag{3.18}$$

For $l \ge 1$ define $P_l' f := (P_{l+1} - P_l)f$. It is readily proved that:

$$P_{l+1} = \sum_{i=0}^{2^{P_0 l}-1} R_{li} P_1 E_{li}$$

and hence

$$P_l' = \sum_{i=0}^{2^{P_0 l}-1} R_{li} P' E_{li}\,. \tag{3.19}$$

Therefore by (3.16)

$$\|P_l' f\|_{L_p(D)} \tag{3.20}$$
$$\le \|P_{l+1} f - f\|_{L_p(D)} + \|P_l f - f\|_{L_p(D)} \le c\, 2^{-g(\mathbf{r})P_0 l} \|f\|_{W_p^{\mathbf{r}}(D)}\,.$$

Put $\psi_{lij} = R_{li}\psi_j$, $N_l = \kappa' 2^{P_0 l}$. Then by the disjointness of the D_{li} and (3.18) we have for $1 \le u \le \infty$

$$\| \sum_{i=0}^{2^{P_0 l}-1} \sum_{j=0}^{\kappa'-1} \alpha_{ij} \psi_{lij} \|_{L_u(D)} \asymp \|(\alpha_{ij})\|_{L_u^{N_l}}. \tag{3.21}$$

Let $\pi_l = \operatorname{span}\{\psi_{lij} : i \in \mathbb{Z}[0, 2^{P_0 l}),\ j \in \mathbb{Z}[0, \kappa')\}$. Define the operator $T_l : \pi_l \to \mathbb{R}^{N_l}$ by

$$T_l \sum_{i=0}^{2^{P_0 l}-1} \sum_{j=0}^{\kappa-1} \alpha_{ij} \psi_{lij} = (\alpha_{ij}).$$

It follows from (3.21) that for $f \in \pi_l$

$$\|T_l f\|_{L_p^{N_l}} \le c \|f\|_{L_p(D)}. \tag{3.22}$$

We define operator $U_l : W_p^{\mathbf{r}}(D) \to L_p^{N_l}$ by

$$U_l = T_l P_l' \tag{3.23}$$

By (3.20), (3.22) and (3.23)

$$\|U_l f\|_{L_p^{N_l}} \le c \cdot 2^{-g(\mathbf{r})P_0 l} \|f\|_{W_p^{\mathbf{r}}(D)}. \tag{3.24}$$

For $n \ge \max(\kappa, 5)$, let $l_0 = \lfloor \log_2(n/\kappa)/P_0 \rfloor$, and $l^* = 2l_0$. By the definition of P_l'

$$P_{l^*} = P_{l_0} + \sum_{l=l_0}^{l^*-1} P_l'. \tag{3.25}$$

By the definition of l_0, we have $\kappa 2^{P_0 l_0} \le n$. Thus $e_n^q(P_{l_0}, \mathcal{B}(W_p^{\mathbf{r}}(D)), 0) = 0$. Let v_l be natural number satisfying

$$\sum_{l=l_0}^{l^*-1} e^{-v_l/8} < 1/4\,. \tag{3.26}$$

Set

$$\tilde{n} = n + 2\kappa'' \sum_{l=l_0}^{l^*-1} v_l n_l. \tag{3.27}$$

As in [5] we can prove

$$e_{\tilde{n}}^q(P_{l^*}, \mathcal{B}(W_p^{\mathbf{r}}(D))) \;\; \le \;\; \sum_{l=l_0}^{l^*-1} e_{2\kappa'' n_l}^q(I_{pq}^{N_l} U_l, \mathcal{B}(W_p^{\mathbf{r}}(D))). \tag{3.28}$$

Now we reduce the problem $(\mathcal{B}(W_p^{\mathbf{r}}(D))$, $L_q^{N_l}$, $I_{pq}^{N_l} U_l$, $D)$ to $(\mathcal{B}(L_p^{N_l})$, $L_q^{N_l}$, $I_{pq}^{N_l}$, $\mathbb{Z}[0, N_l))$. Note that

$$U_l(i,j) = \sum_{k=0}^{\kappa''-1} a_{jk} f(\mathbf{s}_{l_i} + \mathbf{m}^{-l} \circ \mathbf{t}_{jk}'). \tag{3.29}$$

By the proposition

$$\begin{aligned} & e_{2\kappa'' n_l}^q(I_{pq}^{N_l} U_l, \mathcal{B}(W_p^{\mathbf{r}}(D))) \\ \le\; & e_{n_l}^q(I_{pq}^{N_l}, c 2^{-g(\mathbf{r})P_0 l} \mathcal{B}(L_p^{N_l})) \\ =\; & c 2^{-g(\mathbf{r})P_0 l} e_{n_l}^q(I_{pq}^{N_l}, \mathcal{B}(L_p^{N_l})) \\ \le\; & c \cdot 2^{-g(\mathbf{r})P_0 l} n_l^{-(\frac{2}{p}-\frac{2}{q})} N_l^{\frac{2}{p}-\frac{2}{q}} (\log_2(n_l/\sqrt{N_l}+2))^{\frac{2}{p}-\frac{2}{q}}. \end{aligned} \tag{3.30}$$

By part (i) of Lemma 4 and part (i) of Theorem C

$$
\begin{aligned}
& e_{\tilde{n}}^q(I_{pq}, \mathcal{B}(W_p^{\mathbf{r}}(D))) \\
\le & \sup_{f\in\mathcal{B}(W_p^{\mathbf{r}}(D))} \|I_{pq}f - P_{l^*}f\|_{L_q(D)} + e_{\tilde{n}}^q(P_{l^*}, \mathcal{B}(W_p^{\mathbf{r}}(D))) \\
\le & \; c2^{-(g(\mathbf{r})-(\frac{1}{p}-\frac{1}{q}))P_0 l^*} \\
& + c\sum_{l=l_0}^{l^*-1} 2^{-g(\mathbf{r})P_0 l} n_l^{-(\frac{2}{p}-\frac{2}{q})} N_l^{\frac{2}{p}-\frac{2}{q}} (\log_2(\frac{n_l}{\sqrt{N_l}}+2))^{\frac{2}{p}-\frac{2}{q}}.
\end{aligned} \tag{3.31}
$$

First assume that $g(\mathbf{r}) > 2/p - 2/q$. Take any $\delta > 0$ with

$$P_0 g(\mathbf{r}) > (2/p-2/q)(P_0+\delta) \tag{3.32}$$

and put for $l = l_0, \ldots, l^*-1$

$$n_l = \lceil 2^{P_0 l_0 - \mu(l-l_0)} \rceil, \tag{3.33}$$

$$v_l = \lceil 8(2\ln(l-l_0+1)+\ln 8) \rceil. \tag{3.34}$$

It is easy to check that (3.26) holds. By (3.27), (3.33) and (3.34),

$$\tilde{n} \le n + 2\kappa''(2^{P_0}+1)2^{P_0 l_0} \sum_{l=0}^{l^*-l_0-1} \lceil 8(2\ln(l+1)+\ln 8)\rceil \lceil 2^{-\mu l}\rceil \le c2^{P_0 l_0} \le cn. \tag{3.35}$$

Below for simplicity let F denote $\mathcal{B}(W_p^{\mathbf{r}}(D))$. According to (3.31)-(3.35), we have

$$
\begin{aligned}
& e_n^q(I_{pq}, F) \\
\le & \; c2^{-g(\mathbf{r})P_0 l^*/2} \\
& + \sum_{l=l_0}^{l^*-1} c\cdot 2^{-g(\mathbf{r})P_0 l - (\frac{2}{p}-\frac{2}{q})P_0 l_0 + (\frac{2}{p}-\frac{2}{q})\mu(l-l_0) + (\frac{2}{p}-\frac{2}{q})P_0 l} (l_0+1)^{\frac{2}{p}-\frac{2}{q}} \\
\le & \; c2^{-g(\mathbf{r})P_0 l_0} + c2^{-g(\mathbf{r})P_0 l_0}(l_0+1)^{\frac{2}{p}-\frac{2}{q}} \sum_{l=l_0}^{l^*-1} 2^{\left(-g(\mathbf{r})P_0 + (P_0+\mu)(\frac{2}{p}-\frac{2}{q})\right)(l-l_0)} \\
\le & \; c\cdot 2^{-P_0 g(\mathbf{r}) l_0}(l_0+1)^{\frac{2}{p}-\frac{2}{q}} \\
\le & \; c\cdot n^{-g(\mathbf{r})}(\log_2 n)^{\frac{2}{p}-\frac{2}{q}}.
\end{aligned} \tag{3.36}
$$

Next assume that $g(\mathbf{r}) < 2/p - 2/q$. Take any $\delta > 0$ with

$$P_0 g(\mathbf{r}) < (2/p-2/q)(P_0-\delta) \tag{3.37}$$

and put for $l = l_0, \dots, l^* - 1$

$$n_l = \lceil 2^{P_0 l_0 - \mu(l^* - l)} \rceil, \tag{3.38}$$

$$v_l = \lceil 8(2\ln(l^* - l) + \ln 8) \rceil. \tag{3.39}$$

It is easy to check that (3.26) holds and $\tilde{n} \le c 2^{P_0 l_0} \le cn$. Therefore it follows from (3.31) that

$$\begin{aligned}
& e_n^q(I_{pq}, F) \\
\le\ & c 2^{-(2g(\mathbf{r}) + \frac{2}{p} - \frac{2}{q}) P_0 l_0} \\
& + c \sum_{l=l_0}^{l^*-1} 2^{-g(\mathbf{r}) P_0 l + (\frac{2}{p} - \frac{2}{q}) P_0 l_0 + (\frac{2}{p} - \frac{2}{q}) \mu (l - l_0) + (\frac{2}{p} - \frac{2}{q}) P_0 l} (l^* - l + 1)^{\frac{2}{p} - \frac{2}{q}} \\
\le\ & c 2^{-(2g(\mathbf{r}) + \frac{2}{p} - \frac{2}{q}) P_0 l_0} + 2^{-(g(\mathbf{r}) l^* + (\frac{2}{p} - \frac{2}{q}) l_0) P_0} \\
& \sum_{l=l_0}^{l^*-1} (l^* - l + 1)^{\frac{2}{p} - \frac{2}{q}} 2^{(-g(\mathbf{r}) P_0 + (P_0 - \mu)(\frac{2}{p} - \frac{2}{q}))(l^* - l)} \\
\le\ & c \cdot 2^{-(2g(\mathbf{r}) + \frac{2}{p} - \frac{2}{q}) P_0 l_0} \\
\le\ & c \cdot n^{-(2g(\mathbf{r}) + \frac{2}{p} - \frac{2}{q})}.
\end{aligned} \tag{3.40}$$

Finally assume that

$$g(\mathbf{r}) = 2/p - 2/q \tag{3.41}$$

and put for $l = l_0, \dots, l^* - 1$

$$n_l = \lceil 2^{P_0 l_0} (l_0 + 1)^{-1} (\ln(l_0 + 2))^{-1} \rceil, \tag{3.42}$$

$$v_l = \lceil 8(\ln(l_0 + 2) + \ln 4) \rceil. \tag{3.43}$$

Again we can check that (3.26) holds and $\tilde{n} \le cn$. We get

$$\begin{aligned}
& e_n^q(I_{pq}, F) \\
\le\ & c 2^{-g(\mathbf{r}) P_0 l^*/2} \\
& + c \sum_{l=l_0}^{l^*-1} 2^{-g(\mathbf{r}) P_0 l - (\frac{2}{p} - \frac{2}{q}) P_0 l_0 + (\frac{2}{p} - \frac{2}{q}) P_0 l} (l_0 + 1)^{(\frac{4}{p} - \frac{4}{q})} \log_2 (l_0 + 2)^{\frac{2}{p} - \frac{2}{q}} \\
\le\ & c 2^{-g(\mathbf{r}) P_0 l_0} + (l_0 + 1)^{(\frac{4}{p} - \frac{4}{q})} \log_2 (l_0 + 2)^{\frac{2}{p} - \frac{2}{q}} \sum_{l=l_0}^{l^*-1} 2^{-g(\mathbf{r}) P_0 l_0} \\
\le\ & c \cdot 2^{-P_0 g(\mathbf{r}) l_0} (l_0 + 1)^{(\frac{4}{p} - \frac{4}{q})} (\log_2 (l_0 + 1))^{\frac{2}{p} - \frac{2}{q}} \\
\le\ & c \cdot n^{-g(\mathbf{r})} (\log_2 n)^{\frac{4}{p} - \frac{4}{q}} (\log_2 \log_2 n)^{\frac{2}{p} - \frac{2}{q}}.
\end{aligned} \tag{3.44}$$

Thus the proof of Theorem 3 is complete.

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