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INTEGRAL REPRESENTATIONS FOR SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTIONS

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Abstract. While investigating the Lauricella's list of 14 complete second-order hypergeometric series in three variables, Srivastava noticed the existence of three additional complete triple hypergeometric series of the second order, which were denoted by H_A, H_B and H_C. Each of these three triple hypergeometric functions H_A, H_B and H_C has been investigated extensively in many different ways including, for example, in the problem of finding their integral representations of one kind or the other. Here, in this paper, we aim at presenting further integral representations for each of Srivastava's triple hypergeometric functions H_A, H_B and H_C.

1. Introduction and Preliminaries

In the theory of hypergeometric functions of several variables, a remarkably large number of triple hypergeometric functions have been introduced and investigated. A comprehensive table of 205 distinct triple hypergeometric functions is provided in the work of Srivastava and Karlsson [13, Chapter 3]. Out of these 205 distinct triple hypergeometric functions, Lauricella [7, p. 114] introduced fourteen complete triple hypergeometric functions of the second order. He denoted his triple hypergeometric functions by the symbols $F_1, \cdots, F_{14}$ of which F_1, F_2, F_5 and F_9 correspond, respectively, to the three-variable Lauricella functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$ that are the three-variable cases of the n-variable Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ (*cf.* [7, p. 113]; see also [1, p. 114, Equations (1) to (4)], [13, p. 33 *et seq.*] and [4, 5]). Saran [9] initiated a systematic study of ten of the triple

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hypergeometric functions from Lauricella's set. Exton [3] introduced 20 distinct triple hypergeometric functions, which he denoted by $X_1 \cdots X_{20}$, and investigated their twenty Laplace integral representations whose kernels include the confluent hypergeometric functions ${}_0F_1$ and ${}_1F_1$, and the Humbert hypergeometric functions Φ_2 and Ψ_2 of two variables. The four Appell hypergeometric functions $F_1, \cdots F_4$ of two variables are simply the special case of Lauricella's n-variable functions when $n = 2$, that is,

$$F_1 = F_D^{(2)}, \quad F_2 = F_A^{(2)}, \quad F_3 = F_B^{(2)} \qquad \text{and} \qquad F_4 = F_C^{(2)}.$$

While transforming Pochhammer's double-loop contour integrals associated with the functions F_8 and F_{14} (that is, F_G and F_F, respectively) belonging to Lauricella's set of hypergeometric functions of three variables, Srivastava [10, 11] discovered the existence of three additional complete triple hypergeometric functions H_A, H_B and H_C of the second order, which are defined as follows (see also [13, p. 43, Equations 1.5(11) to 1.5(13)]):

$$\begin{aligned} &H_A\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right) \\ &\quad = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}\,(a_2)_{m+n}\,(a_3)_{n+p}}{(c_1)_m\,(c_2)_{n+p}}\;\frac{x^m}{m!}\;\frac{y^n}{n!}\;\frac{z^p}{p} \\ &\big(|x| =: \mathfrak{r} < 1;\; |y| =: \mathfrak{s} < 1;\; |z| =: \mathfrak{t} < (1-\mathfrak{r})(1-\mathfrak{s})\big), \end{aligned} \tag{1.1}$$

$$\begin{aligned} &H_B(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) \\ &\quad = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}(a_2)_{m+n}(a_3)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p}\;\frac{x^m}{m!}\;\frac{y^n}{n!}\;\frac{z^p}{p!} \\ &\big(\mathfrak{r} := |x|;\; \mathfrak{s} := |y|;\; \mathfrak{t} := |z|;\; \mathfrak{r}+\mathfrak{s}+\mathfrak{t}+2\sqrt{\mathfrak{rst}} < 1\big) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &H_C\left(a_1, a_2, a_3; c; x, y, z\right) \\ &\quad = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}\,(a_2)_{m+n}\,(a_3)_{n+p}}{(c)_{m+n+p}}\;\frac{x^m}{m!}\;\frac{y^n}{n!}\;\frac{z^p}{p} \\ &\big(\mathfrak{r} := |x|;\; \mathfrak{s} := |y|;\; \mathfrak{t} := |z|;\; \mathfrak{r}+\mathfrak{s}+\mathfrak{t}-2\sqrt{(1-\mathfrak{r})(1-\mathfrak{s})(1-\mathfrak{t})}<2\big), \end{aligned} \tag{1.3}$$

where, with $\mathbb{C}$ and $\mathbb{Z}_0^-$ denoting the set of complex numbers and the set of nonpositive integers, respectively, $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N} := \{1,\, 2,\, 3,\, \cdots\}) \end{cases} \tag{1.4}$$

$\Gamma(z)$ being the well-known Gamma function. Of course, all 20 of Exton's triple hypergeometric functions $X_1, \cdots, X_{20}$ as well as Srivastava's triple hypergeometric functions H_A, H_B and H_C are included in the set of the aforementioned 205 distinct triple hypergeometric functions which were presented systematically by Srivastava and Karlsson [13, Chapter 3]. The above-stated three-dimensional regions of convergence of the triple hypergeometric series in (1.1), (1.2) and (1.3) for H_A, H_B and H_C, respectively, were given by Srivastava [10, 11] (see also Srivastava and Karlsson [13, Section 3.4]).

Various multivariable generalizations and cases of reducibility of Srivastava's functions H_A, H_B and H_C have been investigated (see, for details, [13, pp. 43–44]). Turaev [15] studied the Srivastava function H_A. Hasanov *et al.* [6] reproduced Srivastava's integral representations [10, 11] for the functions H_A, H_B and H_C in the following (potentially useful) forms:

$$\begin{aligned} H_A(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) &= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_1-\beta_1)\Gamma(\gamma_2-\beta_2)} \\ &\cdot \int_0^1\int_0^1 \xi^{\beta_1-1}\,\eta^{\beta_2-1}\,(1-\xi)^{\gamma_1-\beta_1-1}(1-\eta)^{\gamma_2-\beta_2-1}(1-y\eta)^{-\beta_1}(1-x\xi-z\eta)^{-\alpha} \\ &\cdot\left(1-\frac{xy\xi\eta}{(1-y\eta)(1-x\xi-z\eta)}\right)^{-\alpha} d\xi\, d\eta \\ &\big(\Re(\gamma_1)>\Re(\beta_1)>0;\ \Re(\gamma_2)>\Re(\beta_2)>0\big); \end{aligned} \tag{1.5}$$

$$\begin{aligned} &H_B(\alpha,\,\beta_1,\,\beta_2;\,\gamma_1,\,\gamma_2,\,\gamma_3;\,x,y,z) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-s-t-u}\,t^{\alpha-1}\,s^{\beta_1-1}\,u^{\beta_2-1} \\ &\cdot {}_0F_1(\text{—};\,\gamma_1;\,xst)\;{}_0F_1(\text{—};\,\gamma_2;\,yus)\;{}_0F_1(\text{—};\,\gamma_3;\,zut)\,ds\,dt\,du \\ &\big(\min\{\Re(\alpha),\,\Re(\beta_1),\,\Re(\beta_2)\}>0\big), \end{aligned} \tag{1.6}$$

where each of the confluent hypergeometric functions ${}_0F_1$ can be rewitten in terms of the Bessel function $J_\nu(z)$ and $I_\nu(z)$ given by

$$J_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\text{—};\,\nu+1;\,-\frac{z^2}{4}\right) \tag{1.7}$$

and

$$I_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\text{—};\,\nu+1;\,\frac{z^2}{4}\right), \tag{1.8}$$

respectively;

$$H_C(\alpha,\,\beta_1,\,\beta_2;\,\gamma;\,x,\,y,\,z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\gamma-\alpha-\beta_1)}$$
$$\cdot\int_0^1\int_0^1 \xi^{\alpha-1}\,\eta^{\beta_1-1}\,(1-\xi)^{\gamma-\alpha-1}(1-\eta)^{\gamma-\alpha-\beta_1-1}(1-x\xi)^{\beta_2-\beta_1}$$
$$\cdot\left(1-x\xi-y\eta-z\xi+y\xi\eta+zx\xi^2\right)^{-\beta_2}d\xi\,d\eta \tag{1.9}$$
$$\left(\min\{\Re(\alpha),\,\Re(\beta_1),\,\Re(\gamma-\alpha-\beta_1)\}>0\right).$$

Here, in this present sequel to some of the above-mentioned works, we aim at investigating some *further* integral representations for each of the three Srivatava functions H_A, H_B, and H_C.

2. Integral Representations of H_A

Theorem 1. *Each of the following integral representations for* H_A *holds true*:

$$H_A\,(a_1,a_2,a_3;c_1,c_2;x,y,z)$$
$$=\frac{\Gamma\,(c_2)}{\Gamma\,(a_3)\,\Gamma\,(c_2-a_3)}\int_0^1 \xi^{a_3-1}\,(1-\xi)^{c_2-a_3-1}$$
$$\cdot\,(1-y\xi)^{-a_2}\,(1-z\xi)^{-a_1}\;{}_2F_1\left(a_1,a_2;c_1;\frac{x}{(1-y\xi)\,(1-z\xi)}\right)d\xi \tag{2.1}$$
$$\left(\Re(c_2)>\Re(a_3)>0\right);$$

$$H_A\,(a_1,a_2,a_3;c_1,c_2;x,y,z)$$
$$=\frac{\Gamma\,(c_2)\,(1+\lambda)^{a_3}}{\Gamma\,(a_3)\,\Gamma\,(c_2-a_3)}\int_0^1 \xi^{a_3-1}\,(1-\xi)^{c_2-a_3-1}$$
$$\cdot\,(1+\lambda\xi)^{a_1+a_2-c_2}\,[1+\lambda\xi-(1+\lambda)\,\xi y]^{-a_2}\,[1+\lambda\xi-(1+\lambda)\,\xi z]^{-a_1} \tag{2.2}$$
$$\cdot\,{}_2F_1\left(a_1,a_2;c_1;\frac{x\,(1+\lambda\xi)^2}{[1+\lambda\xi-(1+\lambda)\,\xi y]\,[1+\lambda\xi-(1+\lambda)\,\xi z]}\right)d\xi$$
$$\left(\Re(c_2)>\Re(a_3)>0;\;\lambda>-1\right);$$

$$H_A\,(a_1,a_2,a_3;c_1,c_2;x,y,z)$$
$$=\frac{\Gamma\,(c_2)}{\Gamma\,(a_3)\,\Gamma\,(c_2-a_3)}\frac{(\beta-\gamma)^{a_3}\,(\alpha-\gamma)^{c_2-a_3}}{(\beta-\alpha)^{c_2-a_1-a_2-1}}$$
$$\cdot\int_\alpha^\beta (\beta-\xi)^{c_2-a_3-1}\,(\xi-\alpha)^{a_3-1}\,(\xi-\gamma)^{a_1+a_2-c_2} \tag{2.3}$$
$$\cdot\,[(\beta-\alpha)\,(\xi-\gamma)-(\beta-\gamma)\,(\xi-\alpha)\,y]^{-a_2}\,[(\beta-\alpha)\,(\xi-\gamma)-(\beta-\gamma)\,(\xi-\alpha)\,z]^{-a_1}$$
$$\cdot\,{}_2F_1\,(a_1,a_2;c_1;\sigma x)\;d\xi \qquad \left(\Re(c_2)>\Re(a_3)>0;\;\gamma<\alpha<\beta\right),$$

where

$$\sigma := \frac{(\beta-\alpha)^2(\xi-\gamma)^2}{[(\beta-\alpha)(\xi-\gamma)-(\beta-\gamma)(\xi-\alpha)y][(\beta-\alpha)(\xi-\gamma)-(\beta-\gamma)(\xi-\alpha)z]};$$

(2.4)
$$\begin{aligned} &H_A(a_1,a_2,a_3;c_1,c_2;x,y,z) \\ &\quad = \frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)} \int_0^\infty \xi^{a_3-1}(1+\xi)^{a_1+a_2-c_2} \\ &\quad \cdot (1+\xi-y\xi)^{-a_2}(1+\xi-z\xi)^{-a_1} \\ &\quad \cdot {}_2F_1\left(a_1,a_2;c_1;\frac{x(1+\xi)^2}{(1+\xi-y\xi)(1+\xi-z\xi)}\right)d\xi \qquad \big(\Re(c_2)>\Re(a_3)>0\big); \end{aligned}$$

(2.5)
$$\begin{aligned} &H_A(a_1,a_2,a_3;c_1,c_2;x,y,z) \\ &= \frac{2\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)} \int_0^{\frac{\pi}{2}} \left(\sin^2\xi\right)^{a_3-\frac{1}{2}} \left(\cos^2\xi\right)^{c_2-a_3-\frac{1}{2}} \\ &\cdot \left(1-y\sin^2\xi\right)^{-a_2}\left(1-z\sin^2\xi\right)^{-a_1} \\ &\cdot {}_2F_1\left(a_1,a_2;c_1;\frac{x}{\left(1-y\sin^2\xi\right)\left(1-z\sin^2\xi\right)}\right)d\xi \qquad \big(\Re(c_2)>\Re(a_3)>0\big). \end{aligned}$$

Here ${}_2F_1$ *denotes the well-known Gauss hypergeometric function defined by*

(2.6)
$$\begin{aligned} &{}_2F_1(a,\,b;\,c;\,z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \\ &\big(c\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ |z|<1;\ |z|=1\ \ (z\neq-1)\quad \textit{and}\quad \Re(c-a-b)>0; \\ &\qquad z=-1\quad \textit{and}\quad \Re(c-a-b)>-1\big). \end{aligned}$$

Proof. The integral representation (2.1) was derived by Srivastava himself [10, p. 100] as an intermediate result in his demonstration of the integral representation (1.5) [10, p. 100, Equation (3.3)]. In fact, Srivastava's derivation of (2.1) involved writing the triple hypergeometric series in (1.1) as a single series of the Appell function F_1 as follows:

(2.7)
$$\begin{aligned} &H_A(a_1,a_2,a_3;c_1,c_2;x,y,z) \\ &\qquad = \sum_{m=0}^\infty \frac{(a_1)_m(a_2)_m}{(c_1)_m} F_1[a_3,a_2+m,a_1+m;c_2;y,z]\ \frac{x^m}{m!} \end{aligned}$$

and then applying Picard's integral formula [1, p. 29, Equation (4)]:

(2.8)
$$\begin{aligned} &F_1\left[\alpha,\beta,\beta';\gamma;x,y\right] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \\ &\qquad \cdot \int_0^1 \tau^{\alpha-1}(1-\tau)^{\gamma-\alpha-1}(1-x\tau)^{-\beta}(1-y\tau)^{-\beta'}d\tau \end{aligned}$$

$$\big(\Re(\gamma) > \Re(\alpha) > 0;\ \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-\big)$$

to each term on the right-hand side of (2.7). The transition from (2.1) to Srivastava's final result (1.5) was made by appealing to the following *classical* result (see, for details, [10, pp. 99–100]):

$$(2.9)\quad {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 \tau^{\alpha-1}\,(1-\tau)^{\gamma-\alpha-1}\,(1-z\tau)^{-\beta}\,d\tau$$
$$\big(\Re(\gamma) > \Re(\alpha) > 0;\ \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-\big).$$

The assertions (2.4) and (2.5) of Theorem 1 would follow from Srivastava's result (2.1) upon setting

$$\xi \mapsto \frac{\xi}{1+\xi},\quad d\xi \mapsto \frac{d\xi}{(1+\xi)^2} \qquad \text{and} \qquad (0,1) \mapsto (0,\infty)$$

and

$$\xi \mapsto \sin^2\xi,\quad d\xi \mapsto 2\sin\xi\cos\xi\,d\xi \qquad \text{and} \qquad (0,1) \mapsto \left(0,\frac{\pi}{2}\right),$$

respectively.

Each of the integral representations (2.1) to (2.5) can also be proved *directly* by expressing the series definition of the involved hypergeometric function ${}_2F_1$ in each integrand and changing the order of the integral sign and the summation, and finally using one or the other of the following well-known relationships between the Beta function $B(\alpha,\beta)$, the Gamma function $\Gamma(z)$ and their various associated Eulerian integrals (see, for example, [2, pp. 9–11] and [14, p. 26 and p. 86, Problem 1]):

$$(2.10)\quad B(\alpha,\,\beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & \big(\min\{\Re(\alpha),\,\Re(\beta)\} > 0\big) \\ \displaystyle\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\,\beta \in \mathbb{C}\setminus\mathbb{Z}_0^-), \end{cases}$$

$$(2.11)\quad B(\alpha,\,\beta) = 2\int_0^{\frac{\pi}{2}} (\sin\theta^{2\alpha-1}(\cos\theta)^{2\beta-1}\,d\theta = \int_0^\infty \frac{\tau^{\alpha-1}}{(1+\tau)^{\alpha+\beta}}\,d\tau$$
$$\big(\min\{\Re(\alpha),\,\Re(\beta)\} > 0\big)$$

and

$$(2.12)\quad \begin{aligned} B(\alpha,\,\beta) &= \frac{(b-c)^\alpha\,(a-c)^\beta}{(b-a)^{\alpha+\beta-1}} \int_a^b \frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{(t-c)^{\alpha+\beta}}\,dt \quad (c<a<b) \\ &= (1+\lambda)^\alpha \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{(1+\lambda t)^{\alpha+\beta}}\,dt \qquad (\lambda > -1) \end{aligned}$$
$$\big(\min\{\Re(\alpha),\,\Re(\beta)\} > 0\big).$$

■

3. Integral Representations of H_B

Theorem 2. *Each of the following integral representations for* H_B *holds true*:

(3.1)
$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\int_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2-1}$$
$$\cdot X_4\left[a_1+a_2,a_3;c_1,c_2,c_3;x\xi\left(1-\xi\right),y\left(1-\xi\right),z\xi\right]d\xi$$
$$\bigl(\min\{\Re(a_1),\,\Re(a_2)\}>0\bigr);$$

(3.2)
$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\frac{\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\frac{\left(\beta-\gamma\right)^{a_1}\left(\alpha-\gamma\right)^{a_2}}{\left(\beta-\alpha\right)^{a_1+a_2-1}}$$
$$\cdot\int_\alpha^\beta\left(\beta-\xi\right)^{a_2-1}\left(\xi-\alpha\right)^{a_1-1}\left(\xi-\gamma\right)^{-a_1-a_2}$$
$$\cdot\,X_4\left(a_1+a_2,a_3;c_1,c_2,c_3;\sigma_1x,\sigma_2y,\sigma_3z\right)d\xi$$
$$\bigl(\min\{\Re(a_1),\,\Re(a_2)\}>0;\,\gamma<\alpha<\beta\bigr),$$

where
$$\sigma_1:=\frac{\left(\alpha-\gamma\right)\left(\beta-\gamma\right)\left(\xi-\alpha\right)\left(\beta-\xi\right)}{\left(\beta-\alpha\right)^2\left(\xi-\gamma\right)^2},\quad\sigma_2:=\frac{\left(\alpha-\gamma\right)\left(\beta-\xi\right)}{\left(\beta-\alpha\right)\left(\xi-\gamma\right)}$$

and
$$\sigma_3:=\frac{\left(\beta-\gamma\right)\left(\xi-\alpha\right)}{\left(\beta-\alpha\right)\left(\xi-\gamma\right)};$$

(3.3)
$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\frac{2\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}$$
$$\cdot\int_0^{\frac{\pi}{2}}\left(\sin^2\xi\right)^{a_1-\frac12}\left(\cos^2\xi\right)^{a_2-\frac12}X_4\left(a_1+a_2,a_3;c_1,c_2,c_3;\sigma_1x,\sigma_2y,\sigma_3z\right)d\xi$$
$$\bigl(\min\{\Re(a_1),\,\Re(a_2)\}>0\bigr),$$

where
$$\sigma_1:=\sin^2\xi\cos^2\xi,\quad\sigma_2:=\cos^2\xi\quad\textit{and}\quad\sigma_3:=\sin^2\xi;$$

(3.4)
$$H_B\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\frac{2\Gamma\left(a_1+a_2\right)\left(1+\lambda\right)^{a_1}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}$$
$$\cdot\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\xi\right)^{a_1-\frac12}\left(\cos^2\xi\right)^{a_2-\frac12}}{\left(1+\lambda\sin^2\xi\right)^{a_1+a_2}}X_4\left(a_1+a_2,a_3;c_1,c_2,c_3;\sigma_1x,\sigma_2y,\sigma_3z\right)d\xi$$
$$\bigl(\min\{\Re(a_1),\,\Re(a_2)\}>0;\,\lambda>-1\bigr),$$

where

$$\sigma_1 := \frac{(1+\lambda)\sin^2\xi\cos^2\xi}{\left(1+\lambda\sin^2\xi\right)^2},\quad \sigma_2 := \frac{\cos^2\xi}{1+\lambda\sin^2\xi} \quad \textit{and} \quad \sigma_3 := \frac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi};$$

(3.5)

$$H_B(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z) = \frac{2\Gamma(a_1+a_2)\,\lambda^{a_1}}{\Gamma(a_1)\,\Gamma(a_2)} \cdot \int_0^{\frac{\pi}{2}} \frac{\left(\sin^2\xi\right)^{a_1-\frac{1}{2}}\left(\cos^2\xi\right)^{a_2-\frac{1}{2}}}{\left(\cos^2\xi+\lambda\sin^2\xi\right)^{a_1+a_2}}\, X_4(a_1+a_2,a_3;c_1,c_2,c_3;\sigma_1 x,\sigma_2 y,\sigma_3 z)\,d\xi$$

$$\left(\min\{\Re(a_1),\ \Re(a_2)\} > 0;\ \lambda > 0\right),$$

where

$$\sigma_1 := \frac{\lambda\sin^2\xi\cos^2\xi}{\left(\cos^2\xi+\lambda\sin^2\xi\right)^2},\quad \sigma_2 := \frac{\cos^2\xi}{\cos^2\xi+\lambda\sin^2\xi} \quad \textit{and} \quad \sigma_3 := \frac{\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi}.$$

Here X_4 denotes one of Exton's twenty hypergeometric functions defined by (*see* [3] *and* [13, *p.* 84, *Entry* (45a)])

$$X_4(a_1,\,a_2;\,c_1,\,c_2,\,c_3;\,x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!} \tag{3.6}$$

$$\left(\mathfrak{r} := |x|;\ \mathfrak{s} := |y|;\ \mathfrak{t} := |z|;\ 2\sqrt{\mathfrak{r}} + \left(\sqrt{\mathfrak{s}}+\sqrt{\mathfrak{t}}\right)^2 < 1\right).$$

Proof. A similar argument as in the demonstration of Theorem 1 will establish the results asserted by Theorem 2. Indeed, instead of the Gauss hypergeometric series in (2.6), we make use of the double hypergeometric series in (3.6) for Exton's function X_4.

Alternatively, the assertions (3.1) to (3.5) of Theorem 2 can be proven *directly* and *much more systematically* by first writing the definition (1.2) in the following form:

$$\begin{aligned} &H_B(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z) \\ &= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}(a_2)_{m+n}(a_3)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!} \\ &= \sum_{m,n,p=0}^{\infty} \frac{(a_1+a_2)_{2m+n+p}(a_3)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p}\,\frac{(a_1)_{m+p}(a_2)_{m+n}}{(a_1+a_2)_{2m+n+p}}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!} \\ &= \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\,\Gamma(a_2)} \sum_{m,n,p=0}^{\infty} \frac{(a_1+a_2)_{2m+n+p}(a_3)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!} \\ &\qquad \cdot B(a_1+m+p,a_2+m+n), \end{aligned}$$

replacing the Beta function:

$$B(a_1+m+p, a_2+m+n) \qquad \bigl(\min\{\Re(a_1), \Re(a_2)\} > 0\bigr)$$

by one or the other of its numerous Eulerian integral representations in (for example) (2.10) to (2.12), and then interpreting the resulting triple hypergeometric series by means of the definition (2.6). In this manner, of course, we can derive a considerably large number of other integral representations for H_B involving the triple hypergeometric function X_4 defined by (2.6). ■

4. Integral representations of H_C

Theorem 3. *Each of the following integral representations for H_C holds true*:

$$\begin{aligned} H_C(a_1,a_2,a_3;c;x,y,z) &= \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(c-a_1)} \int_0^1 \xi^{a_1-1}(1-\xi)^{c-a_1-1} \\ &\quad \cdot (1-x\xi)^{-a_2}(1-z\xi)^{-a_3}\; {}_2F_1\left(a_2,a_3;c-a_1;\frac{y(1-\xi)}{(1-x\xi)(1-z\xi)}\right) d\xi \\ &\quad \bigl(\Re(c) > \Re(a_1) > 0\bigr); \end{aligned} \tag{4.1}$$

$$\begin{aligned} &H_C(a_1,a_2,a_3;c;x,y,z) \\ &= \frac{\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^1\int_0^1 \xi^{a_1-1}\eta^{a_1+a_2-1}(1-\xi)^{a_2-1}(1-\eta)^{a_3-1} \\ &\quad \cdot {}_2F_1\left(\frac{a_1+a_2+a_3}{2}, \frac{a_1+a_2+a_3+1}{2};\; c;\; \Xi(\xi,\eta;x,y,z)\right) d\xi\, d\eta \\ &\quad \bigl(\min\{\Re(a_1),\, \Re(a_2),\, \Re(a_3)\} > 0\bigr), \end{aligned} \tag{4.2}$$

where

$$\Xi(\xi,\eta;x,y,z) := 4\eta\left[x\xi(1-\xi)\eta + y(1-\xi)(1-\eta) + z\xi(1-\eta)\right];$$

$$\begin{aligned} H_C(a_1,a_2,a_3;c;x,y,z) &= \frac{\Gamma(c)(1+\lambda)^{a_1}}{\Gamma(a_1)\Gamma(c-a_1)} \int_0^1 \xi^{a_1-1}(1-\xi)^{c-a_1-1} \\ &\quad \cdot (1+\lambda\xi)^{a_2+a_3-c}\left[1+\lambda\xi-(1+\lambda)x\xi\right]^{-a_2}\left[1+\lambda\xi-(1+\lambda)z\xi\right]^{-a_3} \\ &\quad \cdot {}_2F_1\left(a_2,a_3;c-a_1;\frac{y(1+\lambda\xi)(1-\xi)}{[1+\lambda\xi-(1+\lambda)x\xi][1+\lambda\xi-(1+\lambda)z\xi]}\right) d\xi \\ &\quad \bigl(\Re(c) > \Re(a_1) > 0;\ \lambda > -1\bigr); \end{aligned} \tag{4.3}$$

$$\begin{aligned} &H_C\left(a_1,a_2,a_3;c;x,y,z\right) \\ &= \frac{\Gamma\left(c\right)\left(\beta-\gamma\right)^{a_1}\left(\alpha-\gamma\right)^{c-a_1}}{\Gamma\left(a_1\right)\Gamma\left(c-a_1\right)\left(\beta-\alpha\right)^{c-a_2-a_3-1}} \int_\alpha^\beta \left(\beta-\xi\right)^{c-a_1-1} \\ &\quad \cdot \left(\xi-\alpha\right)^{a_1-1}\left(\xi-\gamma\right)^{a_2+a_3-c}\left[\left(\beta-\alpha\right)\left(\xi-\gamma\right)-\left(\beta-\gamma\right)\left(\xi-\alpha\right)x\right]^{-a_2} \\ &\quad \cdot \left[\left(\beta-\alpha\right)\left(\xi-\gamma\right)-\left(\beta-\gamma\right)\left(\xi-\alpha\right)z\right]^{-a_3}\ {}_2F_1\left(a_2,a_3;c-a_1;\sigma y\right)\,d\xi \\ &\qquad \big(\Re(c)>\Re(a_1)>0;\ \gamma<\alpha<\beta\big), \end{aligned} \tag{4.4}$$

where

$$\sigma := \frac{\left(\beta-\alpha\right)\left(\alpha-\gamma\right)\left(\xi-\gamma\right)\left(\beta-\xi\right)}{\left[\left(\beta-\alpha\right)\left(\xi-\gamma\right)-\left(\beta-\gamma\right)\left(\xi-\alpha\right)x\right]\left[\left(\beta-\alpha\right)\left(\xi-\gamma\right)-\left(\beta-\gamma\right)\left(\xi-\alpha\right)z\right]};$$

$$\begin{aligned} &H_C\left(a_1,a_2,a_3;c;x,y,z\right) = \frac{\Gamma\left(c\right)}{\Gamma\left(a_1\right)\Gamma\left(c-a_1\right)} \int_0^\infty \xi^{a_1-1}\left(1+\xi\right)^{a_2+a_3-c} \\ &\quad \cdot \left(1+\xi-\xi x\right)^{-a_2}\left(1+\xi-\xi z\right)^{-a_3}\ {}_2F_1\left(a_2,a_3;c-a_1;\sigma y\right)\,d\xi \\ &\qquad \big(\Re(c)>\Re(a_1)>0\big), \end{aligned} \tag{4.5}$$

where

$$\sigma := \frac{\left(1+\xi\right)}{\left(1+\xi-\xi x\right)\left(1+\xi-\xi z\right)};$$

$$\begin{aligned} &H_C\left(a_1,a_2,a_3;c;x,y,z\right) \\ &= \frac{2\Gamma\left(c\right)}{\Gamma\left(a_1\right)\Gamma\left(c-a_1\right)} \int_0^{\frac{\pi}{2}} \left(\sin^2\xi\right)^{a_1-\frac{1}{2}}\left(\cos^2\xi\right)^{c-a_1-\frac{1}{2}} \\ &\quad \cdot \left(1-x\sin^2\xi\right)^{-a_2}\left(1-z\sin^2\xi\right)^{-a_3}\ {}_2F_1\left(a_2,a_3;c-a_1;\sigma y\right)\,d\xi \\ &\qquad \big(\Re(c)>\Re(a_1)>0\big), \end{aligned} \tag{4.6}$$

where

$$\sigma := \frac{\cos^2\xi}{\left(1-x\sin^2\xi\right)\left(1-z\sin^2\xi\right)}.$$

Here ${}_2F_1$ *denotes the Gauss hypergeometric function given by* (2.6).

Proof. Our proof of Theorem 3 is much akin to that of Theorem 1, which we have already presented in a reasonably detailed manner. ∎

5. Concluding Remarks and Observations

Integral representations for most of the special functions of mathematical physics and applied mathematics have been investigated in the existing literature. Here we have presented only some illustrative integral representations for each of Srivastava's functions H_A, H_B and H_C. A variety of integral representations of H_A, H_B and H_C, which may be different from those presented here, can also be provided. Furthermore, just as we mentioned in connection with the single- and double-integral representations (2.1) and (1.5) for H_A, Srivastava's double-integral representation (1.9) for H_C can easily be deduced from the assertion (4.1) of Theorem 3 by appealing to the classical result (2.9).

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References

1. P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.
2. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1953.
3. H. Exton, Hypergeometric functions of three variables, *J. Indian Acad. Math.*, **4** (1982), 113-119.
4. A. Hasanov and H. M. Srivastava, Some decomposition formulas associated with the Lauricella function $F_A^{(r)}$ and other multiple hypergeometric functions, *Appl. Math. Lett.*, **19** (2006), 113-121.
5. A. Hasanov and H. M. Srivastava, Decomposition formulas associated with the Lauricella multivariable hypergeometric functions, *Comput. Math. Appl.*, **53** (2007), 1119-1128.
6. A. Hasanov, H. M. Srivastava and M. Turaev, Decomposition formulas for some triple hypergeometric functions, *J. Math. Anal. Appl.*, **324** (2006), 955-969.
7. G. Lauricella, Sulle funzioni ipergeometriche a più variabili, *Rend. Circ. Mat. Palermo*, **7** (1893), 111-158.
8. P. A. Padmanabham, Two results on three variable hypergeometric function, *Indian J. Pure Appl. Math.*, **30** (1999), 1107-1109.

9. S. Saran, Hypergeometric functions of three variables, *Gaṇita*, **5** (1954), 71-91; see also Corrigendum, *Gaṇita*, **7** (1956), 65.

10. H. M. Srivastava, Hypergeometric functions of three variables, *Gaṇita*, **15** (1964), 97-108.

11. H. M. Srivastava, Some integrals representing triple hypergeometric functions, *Rend. Circ. Mat. Palermo* (*Ser.* 2), **16** (1967), 99-115.

12. H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

13. H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

14. H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

15. M. Turaev, Decomposition formulas for Srivastava's hypergeometric function H_A on Saran functions, *J. Comput. Appl. Math.*, **233** (2009), 842-846.

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