TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 6, pp. 2721-2731, December 2011
This paper is available online at http://tjm.math.ntu.edu.tw

SCHUR-CONVEXITY OF THE GENERALIZED HERONIAN MEANS INVOLVING TWO POSITIVE NUMBERS

Li-Li Fu, Bo-Yan Xi and H. M. Srivastava*

Abstract. In this paper, we give the sufficient as well as necessary condition of the Schur-convexity and Schur-harmonic-convexity of the generalized Heronian means with two positive numbers. Our main results provide the perfected versions of the results given in 2008 by Shi *et al.* [9].

1. INTRODUCTION

Throughout the this paper, we let

$$\mathbb{R} = (-\infty, +\infty),\ \mathbb{R}_0 = [0, +\infty) \quad \text{and} \quad \mathbb{R}_+ = (0, +\infty).$$

We also let

$$(a, b) \in \mathbb{R}_+^2,\ w \in \mathbb{R}_0 \quad \text{and} \quad p \in \mathbb{R}.$$

The well-known Heronian means of $(a, b) \in \mathbb{R}_+^2$ is defined by (see [1] and also [2, p. 399])

$$H_{1,1}(a, b) = \begin{cases} \dfrac{a + \sqrt{ab} + b}{3} & (a \neq b) \\ \sqrt{ab} & (a = b). \end{cases} \tag{1.1}$$

An analogue of the above-defined Heronian means is stated as follows (see [5]):

$$H_{1,4}(a, b) = \frac{a + 4(ab)^{\frac{1}{2}} + b}{6}. \tag{1.2}$$

Received May 5, 2011, accepted June 5, 2011.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: Primary 05E05; Secondary 26B25.
Key words and phrases: Heronian means, Generalized Heronian means, Schur-convexity, Schur-harmonic-convexity, Arithmetic-geometric-harmonic means inequalities, Schur-geometric-convexity.
*Corresponding author.

Recently, Janous [4] presented a family of the generalized Heronian means defined by

$$H_{1,w}(a,b) = \begin{cases} \dfrac{a + w(ab)^{\frac{1}{2}} + b}{w+2} & (w \in \mathbb{R}_0) \\ \sqrt{ab} & (w = \infty) \end{cases} \tag{1.3}$$

and compared it with the other means.

In 2006, Li *et al.* [6] gave the monotonicity and Schur-convexity of another generalized Heronian means as follows:

$$H_{p,1}(a,b) = \begin{cases} \left(\dfrac{a^p + (ab)^{\frac{p}{2}} + b^p}{3}\right)^{\frac{1}{p}} & (p \neq 0) \\ \sqrt{ab} & (p = 0). \end{cases} \tag{1.4}$$

Several variants as well as interesting applications of the Heronian means can be found in the recent papers [3], [9], [10] and [12] to [15]. We remark here that Shi *et al.* [9] discussed the Schur-convexity and Schur-geometric-convexity of a further generalization of the Heronian means given by

$$H_{p,w}(a,b) = \begin{cases} \left(\dfrac{a^p + w(ab)^{\frac{p}{2}} + b^p}{w+2}\right)^{\frac{1}{p}} & (p \neq 0) \\ \sqrt{ab} & (p = 0) \end{cases} \tag{1.5}$$

and proved Theorem 1 below.

Theorem 1. (see [9]). *Each of the following assertions holds true*:

(i) $H_{p,w}(a,b)$ *is increasing with respect to* w;

(ii) $H_{p,w}(a,b)$ *is Schur-convex if* $(p,w) \in E_{11}$;

(iii) $H_{p,w}(a,b)$ *is Schur-concave if* $(p,w) \in E_{21}$,

where

$$E_{11} := \{(p,w) : p \geqq 2 \quad \textit{and} \quad 0 \leqq w \leqq 2\} \tag{1.6}$$

and

$$\begin{aligned} E_{21} := &\left\{(p,w) : p \leqq 1 \ \textit{and} \ 0 \leqq w\right\} \cup \left\{(p,w) : 1 < p \leqq \frac{3}{2} \ \textit{and} \ w \geqq 1\right\} \\ &\cup \left\{(p,w) : \frac{3}{2} < p \leqq 2 \ \textit{and} \ w \geqq 2\right\}. \end{aligned} \tag{1.7}$$

Remark 1. Theorem 1 merely provides a sufficient condition of the Schur-convexity of the generalized Heronian means $H_{p,w}(a,b)$.

The main purpose of this paper is to give the sufficient as well as necessary condition of the Schur-convexity and Schur-harmonic-convexity of the generalized Heronian means $H_{p,w}(a,b)$ with $(a,b) \in \mathbb{R}_+^2$. As applications our results, a new refinement of the arithmetic-geometric-harmonic means inequalities is established.

2. Preliminaries Results

In order to prove our main results, we require a number of lemmas. Lemmas 1 and 2 involving the Schur-convexity and Schur-harmonic-convexity of a given function can be found in [8] and [11], respectively. Lemma 3 involving Bernoulli's inequality [2] is well-known.

Lemma 1. (see [8, pp. 54-57]). *Let $\Omega \subset \mathbb{R}^n$ be a convex set which is symmetric with respect to permutations and which has a nonempty interior set Ω°. If $\varphi : \Omega \to \mathbb{R}$ is continuous and symmetric on Ω and differentiable in Ω°, then φ is Schur-convex* (*Schur-concave*) *if and only if the following condition*:

$$S(x_1, x_2; \varphi) := (x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geqq 0 \, (\leqq 0)$$

holds true for any $x \in \Omega^\circ$.

Lemma 2. (see [11]). *Let $\Omega \subset \mathbb{R}_+^n$ be symmetric and have a nonempty interior set Ω°. Suppose also that*

$$\left\{\left(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}\right) : \boldsymbol{x} \in \Omega\right\}$$

is a convex set. If $\varphi : \Omega \to \mathbb{R}_+$ is continuous and symmetric on Ω, and differentiable in Ω°, then φ is Schur-harmonic-convex (*Schur-harmonic-concave*) *if and only if the following condition*:

$$H(x_1, x_2; \varphi) := (x_1 - x_2)\left(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2}\right) \geqq 0 \ (\leqq 0)$$

holds true for any $x \in \Omega^\circ$.

Lemma 3. [Bernoulli's Inequality (see [2, p. 4])]. *Let $x \geqq -1$. Then the following inequality*:

$$(1+x)^\alpha \geqq 1 + \alpha x \tag{2.1}$$

holds true if $\alpha \geqq 1$ or $\alpha \leqq 0$ $\ (x \neq -1)$. Furthermore, the inequality (2.1) *is reversed if* $0 < \alpha < 1$.

Lemma 4. *For* $u \in \mathbb{R}_0$, *let*

$$h_{p,w}(u) := (1+u)^{p-1} - 1 - \frac{w}{2}\, u(1+u)^{\frac{p}{2}-1}. \tag{2.2}$$

Then $h_{p,w}(u) \geqq 0$ *if and only if* $(p, w) \in E_1$. *Furthermore*, $h_{p,w}(u) \leqq 0$ *if and only if* $(p, w) \in E_2$, *where*

$$\begin{aligned} E_1 &:= \{(p,w) : p \geqq 2 \quad \textit{and} \quad 0 \leqq w \leqq 2(p-1)\} \\ &\quad \cup \{(p,w) : 1 < p \leqq 2 \quad \textit{and} \quad w = 0)\} \end{aligned} \tag{2.3}$$

and

$$E_2 := \{(p,w) : p \leqq 2 \quad \textit{and} \quad \max\{0, 2(p-1)\} \leqq w\}. \tag{2.4}$$

Proof. First of all, we prove the ***sufficiency***. Indeed, for $u \in \mathbb{R}_0$, one gets

$$h'_{p,w}(u) = (1+u)^{\frac{p}{2}-2}\left[(p-1)(1+u)^{\frac{p}{2}} - \frac{w}{2}\left(1 + \frac{p}{2}u\right)\right]. \tag{2.5}$$

From Lemma 3, it follows for any $u \in \mathbb{R}_0$ that

$$(1+u)^{\frac{p}{2}} \geqq 1 + \frac{p}{2}u \qquad (p \geqq 2) \tag{2.6}$$

and

$$(1+u)^{\frac{p}{2}} \leqq 1 + \frac{p}{2}u \qquad (0 \leqq p \leqq 2). \tag{2.7}$$

(i) We can easily see that $h_{p,w}(u) \geqq 0$ for $1 \leqq p \leqq 2$ and $w = 0$.

If $(p, w) \in E_1$ with $2(p-1) \geqq w \geqq 0$ and $p \geqq 2$, then, by using (2.5) and the inequality (2.6), we obtain

$$h'_{p,w}(u) \geqq (1+u)^{\frac{p}{2}-2}\frac{w}{2}\left((1+u)^{\frac{p}{2}} - 1 - \frac{p}{2}u\right) \geqq 0.$$

It is not difficult to find that $h_{p,w}(u)$ is increasing for $u \in \mathbb{R}_0$ and that

$$h_{p,w}(u) \geqq h_{p,w}(0) = 0.$$

(ii) If $(p, w) \in E_2$ with $p \leqq 1$ and $w \geqq 0$, then, obviously, $h_{p,w}(u) \leqq 0$ holds true.

If $(p, w) \in E_2$ with

$$1 \leqq p \leqq 2 \qquad \text{and} \qquad p \leqq 1 + \frac{w}{2},$$

then we find the from (2.5) and inequality (2.7) that

$$h'_{p,w}(u) \leqq (1+u)^{\frac{p}{2}-2}\frac{w}{2}\left((1+u)^{\frac{p}{2}} - 1 - \frac{p}{2}u\right) \leqq 0,$$

which implies that $h_{p,w}(u)$ is decreasing with respect to $u \in \mathbb{R}_0$ and

$$h_{p,w}(u) \leqq h_{p,w}(0) = 0.$$

We now give the proof of the ***necessity***.

(iii) For $w, u \in \mathbb{R}_0$, in view of $h_{p,w}(0) = 0$ and using the mean value theorem, we obtain

$$h_{p,w}(u) = u \cdot h'_{p,w}(u_0) \geqq 0 \qquad (u_0 \in [0, u]).$$

We thus find that

$$\lim_{u_0 \to 0^+} h'_{p,w}(u_0) = h'_{p,w}(0^+) = p - 1 - \frac{w}{2} \geqq 0,$$

that is, that

$$p - 1 \geqq \frac{w}{2}.$$

If we set $p - 1 = \frac{w}{2} > 0$, then we find from the mean value theorem that

$$h_p(u) := h_{p,2(p-1)}(u) = (1+u)^{p-1} - 1 - (p-1)u(1+u)^{\frac{p}{2}-1} = u \cdot h'_p(u_0) \geqq 0,$$

where $u_0 \in [0, u]$. It, therefore, follows that

$$h'_p(u_0) = (p-1)(1+u_0)^{\frac{p}{2}-2}\left[(1+u_0)^{\frac{p}{2}} - \left(1 + \frac{p}{2}u_0\right)\right] \geqq 0, \tag{2.8}$$

which implies that $p \geqq 2$ by means of Lemma 3.

If

$$p - 1 \geqq \frac{w}{2} = 0,$$

then we see that $p \geqq 1$ with

$$h_{p,0}(u) = (1+u)^{p-1} - 1 \geqq 0.$$

This also means that $h_{p,w}(u) \geqq 0$ must yield $(p, w) \in E_1$.

(iv) For $w, u \in \mathbb{R}_0$, according to $h_{p,w}(0) = 0$ and $h_{p,w}(u) \leqq 0$, one finds that $h'_{p,w}(0^+) \leqq 0$ and

$$p - 1 \leqq \frac{w}{2}.$$

If $0 \leqq p - 1 \leqq \frac{w}{2}$, by the same discussion as in the case of Part (iii) above, it is easy to obtain $1 \leqq p \leqq 2$ for $h_{p,w}(u) \leqq 0$.

If $p-1<0 \leqq \frac{w}{2}$, then, upon letting $w=0$, we have $p<1$ with

$$h_{p,0}(u)=(1+u)^{p-1}-1 \leqq 0. \tag{2.9}$$

Therefore, $h_{p,w}(u) \leqq 0$ must yield $(p,w) \in E_2$.

The proof of Lemma 4 is thus completed. ■

By using the same method as in the proof of Lemma 4 above, we can deduce the following analogous result.

Lemma 5. *Define*

$$k_{p,w}(u) := (1+u)^{p+1}-1+\frac{w}{2}\,u(1+u)^{\frac{p}{2}} \qquad (u \in \mathbb{R}_0). \tag{2.10}$$

Then $k_{p,w}(u) \geqq 0$ *if and only if* $(p,w) \in F_1$. *Furthermore*, $k_{p,w}(u) \leqq 0$ *if and only if* $(p,w) \in F_2$, *where*

$$F_1 := \{(p,w) : -2 \leqq p \quad \textit{and} \quad \max\{0,-2(p+1)\} \leqq w\} \tag{2.11}$$

and

$$\begin{aligned} F_2 &:= \{(p,w) : p \leqq -2 \quad \textit{and} \quad 0 \leqq w \leqq -2(p+1)\} \\ &\cup \{(p,w) : p \leqq -1 \quad \textit{and} \quad w=0\}. \end{aligned} \tag{2.12}$$

3. Main Results and Applications

Theorem 2. *The generalized Heronian means* $H_{p,w}(a,b)$ *is Schur-convex if and only if* $(p,w) \in E_1$, *and is also Schur-concave if and only if* $(p,w) \in E_2$, *where* E_1 *and* E_2 *are given by* (2.3) *and* (2.4), *respectively.*

Proof. It is easily observed that $H_{0,w}(a,b)=\sqrt{ab}$ is Schur-concave for $(a,b) \in \mathbb{R}_+^2$.

For $p \neq 0$, we readily arrive that

$$\frac{\partial H_{p,w}(a,b)}{\partial a}=\frac{1}{w+2}\left(a^{p-1}+\frac{wb}{2}(ab)^{\frac{p}{2}-1}\right)[H_{p,w}(a,b)]^{1-p}>0 \tag{3.1}$$

and

$$\frac{\partial H_{p,w}(a,b)}{\partial b}=\frac{1}{w+2}\left(b^{p-1}+\frac{wa}{2}(ab)^{\frac{p}{2}-1}\right)[H_{p,w}(a,b)]^{1-p}>0. \tag{3.2}$$

There is no loss of generality in supposing that

$$a \geqq b \quad \text{and} \quad 1+u=\frac{a}{b} \qquad (u \in \mathbb{R}_0),$$

which yields

$$\begin{aligned} S(a,b;H_{p,w}) &= \frac{a-b}{w+2}[H_{p,w}(a,b)]^{1-p}\left(a^{p-1}-b^{p-1}-\frac{w}{2}(a-b)(ab)^{\frac{p}{2}-1}\right) \\ &= \frac{(a-b)b^{p-1}}{w+2}[H_{p,w}(a,b)]^{1-p}h_{p,w}(u), \end{aligned} \tag{3.3}$$

where $h_{p,w}(u)$ is defined by (2.2).

This evidently completes the proof of Theorem 2 by means of Lemmas 1 and 4, and the expression given by (3.3). ■

Remark 2. In Figure 1 below, if we let

$$E_3 := \left\{(p,w) : 1+\frac{w}{2} < p < 2 \quad \text{and} \quad 0 < w < 2\right\} \tag{3.4}$$

and

$$E_4 := \left\{(p,w) : 2 < p < 1+\frac{w}{2} \quad \text{and} \quad 2 < w\right\}, \tag{3.5}$$

then we find that

$$\mathbb{R} \times \mathbb{R}_0 = E_1 \cup E_2 \cup E_3 \cup E_4$$

and

$$E_1 \cap E_3 = E_2 \cap E_3 = E_1 \cap E_4 = E_2 \cap E_4 = \phi.$$

Remark 3. In the case when $(p,w) \in E_3 \cup E_4$, we cannot determine the Schur-convexity of $H_{p,w}(a,b)$. For example, for $(1.98, 1.92) \in E_3$ and $(4,8) \in E_4$, we know that

$$h_{1.98,1.92}(1) = 0.0767\cdots > 0, \qquad h_{1.98,1.92}(59) = -0.0852\cdots < 0$$

and

$$h_{4,8}(1.01) = -0.999799 < 0, \qquad h_{4,8}(2) = 3 > 0,$$

where $h_{p,w}(u)$ is defined by (2.2). Thus it follows from (3.3) that the sign of $S(a,b;H_{p,w})$ is changed.

Remark 4. By combining Theorems 1 and 2, one finds from Figures 1 and 2 that if we let

$$\begin{aligned} E_{12} &:= \{(p,w) : 2 < p \quad \text{and} \quad 2 < w \leqq 2(p-1)\} \\ &\cup \{(p,w) : 1 < p \leqq 2 \quad \text{and} \quad w = 0\}, \end{aligned} \tag{3.6}$$

$$E_{22} := \left\{ (p, w) : 1 < p < \frac{3}{2} \quad \text{and} \quad 2(p-1) \leqq w < 1 \right\} \tag{3.7}$$

and

$$E_{23} := \left\{ (p, w) : \frac{3}{2} \leqq p < 2 \quad \text{and} \quad 2(p-1) \leqq w < 2 \right\}, \tag{3.8}$$

then we find that

$$E_1 = E_{11} \cup E_{12}$$

and

$$E_2 = E_{21} \cup E_{22} \cup E_{23}.$$

Thus, obviously, Theorem 1 is only to put forward a sufficient condition of the Schur-convexity of the generalized Heronian means $H_{p,w}(a, b)$.

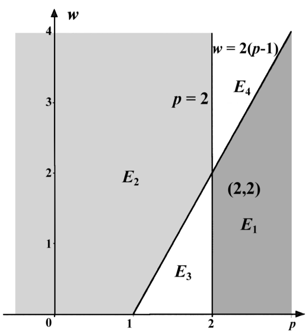

Fig. 1.

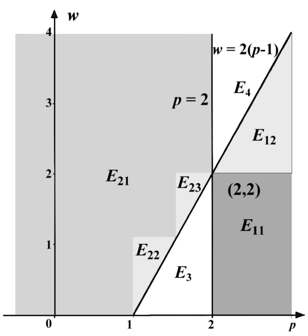

Fig. 2.

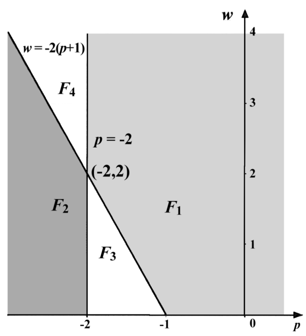

Fig. 3.

Similarly, the assertion of Theorem 3 below can be shown to hold true by applying Lemmas 2, 3 and 5.

Theorem 3. *The generalized Heronian means* $H_{p,w}(a, b)$ *is Schur-harmonic-convex if and only if* $(p, w) \in F_1$, *and is also Schur-harmonic-concave if and only if* $(p, w) \in F_2$, *where* F_1 *and* F_2 *are given, as in Lemma 5, by* (2.11) *and* (2.12), *respectively.*

Remark 5. Given (see Figure 3)

$$F_3 := \left\{ (p, w) : -2 < p < -1 \quad \text{and} \quad 0 < \frac{w}{2} < -(p+1) \right\} \tag{3.9}$$

and

$$F_4 := \left\{ (p, w) : p < -2 \quad \text{and} \quad -(p+1) < \frac{w}{2} \right\}, \tag{3.10}$$

we can deduce that

$$\mathbb{R} \times \mathbb{R}_0 = F_1 \cup F_2 \cup F_3 \cup F_4$$

and

$$F_1 \cap F_3 = F_2 \cap F_3 = F_1 \cap F_4 = F_2 \cap F_4 = \phi.$$

Similar to the observations made in Remark 3, we also cannot determine the Schur-harmonic-convexity of $H_{p,w}(a,b)$ with $(p,w) \in F_3 \cup F_4$.

As simple applications of Theorems 2 and 3, we are led to the following two interesting corollaries.

Corollary 1. *Let the* p*-th power mean of* $(a,b) \in \mathbb{R}_+^2$ *be defined by*

$$M_p(a,b) := H_{p,0}(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{\frac{1}{p}} & (p \neq 0) \\ \sqrt{ab} & (p = 0). \end{cases} \tag{3.11}$$

Then $M_p(a,b)$ *is Schur-convex if and only if* $p \geqq 1$ *and Schur-concave if and only if* $p \leqq 1$*, and is also Schur-harmonic-convex if and only if* $p \geqq -1$ *and Schur-harmonic-concave if and only if* $p \leqq -1$.

Corollary 2. *For*

$$\alpha = (\alpha_1, \alpha_2), \ \ \beta = (\beta_1, \ \ \beta_2) \in \mathbb{R}_0^2 \quad \textit{and} \quad (\frac{1}{2}, \frac{1}{2}) \prec \beta \prec \alpha \prec (1,0),$$

$$\begin{aligned} H_{p_1,w_1}(a,b) &\geqq H_{p_1,w_1}\Big(A_\alpha(a,b)\Big) \geqq H_{p_1,w_1}\Big(A_\beta(a,b)\Big) \geqq A(a,b) \\ &\geqq H_{p_2,w_2}\Big(A_\beta(a,b)\Big) \geqq H_{p_2,w_2}\Big(A_\alpha(a,b)\Big) \geqq H_{p_2,w_2}(a,b) \\ &\geqq H_{p_2,w_2}\Big(G_\alpha(a,b)\Big) \geqq H_{p_2,w_2}\Big(G_\beta(a,b)\Big) \geqq G(a,b) \\ &\geqq H_{p_3,w_3}\Big(G_\beta(a,b)\Big) \geqq H_{p_3,w_3}\Big(G_\alpha(a,b)\Big) \geqq H_{p_3,w_3}(a,b) \\ &\geqq H_{p_3,w_3}\Big(H_\alpha(a,b)\Big) \geqq H_{p_3,w_3}\Big(H_\beta(a,b)\Big) \geqq H(a,b) \\ &\geqq H_{p_4,w_4}\Big(H_\beta(a,b)\Big) \geqq H_{p_4,w_4}\Big(H_\alpha(a,b)\Big) \geqq H_{p_4,w_4}(a,b), \end{aligned} \tag{3.12}$$

if

$$\begin{aligned} &(p_1, w_1) \in E_1, \\ (p_2, w_2) \in &E_2 \cap \{(p,w) : 0 \leqq p \quad \textit{and} \quad 0 \leqq w\} \\ &= \{(p,w) : \max\{0, 2(p-1)\} \leqq w \quad \textit{and} \quad 0 < p \leqq 2\}, \end{aligned}$$

$$\begin{aligned}(p_3, w_3) \in &\{(p, w) : p < 0 \quad \textit{and} \quad 0 \leqq w\} \cap F_1 \\ =&\{(p, w) : -2 \leqq p < 0 \quad \textit{and} \quad \max\{0, -2(p+1)\} \leqq w\}\end{aligned}$$

and

$$(p_4, w_4) \in F_2,$$

where

$$A_\alpha(a, b) := (\alpha_1 a + \alpha_2 b, \alpha_2 a + \alpha_1 b), \quad G_\alpha(a, b) := (a^{\alpha_1} b^{\alpha_2}, a^{\alpha_2} b^{\alpha_1}),$$

$$H_\alpha(a, b) := \left(\frac{1}{\frac{\alpha_1}{a} + \frac{\alpha_2}{b}}, \frac{1}{\frac{\alpha_2}{a} + \frac{\alpha_1}{b}} \right)$$

and

$$A(a, b) := \frac{a+b}{2}, \; G(a, b) := \sqrt{ab} \quad \textit{and} \quad H(a, b) := \frac{2ab}{a+b}.$$

Remark 6. The inequalities (3.12) include a new refinement of the well-known arithmetic-geometric-harmonic means inequalities with $(a, b) \in \mathbb{R}^2_+$.

ACKNOWLEDGEMENTS

The present investigation was supported, in part, by the *National Natural Science Foundation of the People's Republic of China* under Grant Number 10962004.

REFERENCES

1. H. Alzer and W. Janous, Solution of problem 8*, *Crux Math.*, **13** (1987), 173-178.
2. P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, Boston and London, 1988.
3. K.-Z. Guan and H.-T. Zhu, The generalized Heronian mean and its inequalities, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*, **17** (2006), 60-75.
4. W. Janous, A note on generalized Heronian means. *Math. Inequal. Appl.*, **4** (2001), 369-375.
5. G. Jia and J.-D. Cao, A new upper bound of the logarithmic mean, *J. Inequal. Pure Appl. Math.*, **4** (2003), Article 80, 1-4 (electronic).
6. D.-M. Li, C. Gu and H.-N. Shi, Schur convexity of the power-type generalization of Heronian mean, *Math. Practice and Theory*, **36** (2006), 387-390 (in Chinese).
7. Q.-J. Mao, Dual means, logarithmic and Heronian dual means of two positive numbers, *J. Suzhou Coll. Ed.*, **16** (1999), 82-85 (in Chinese).

8. A. W. Marshall and I. Olkin, *Inequalities*: *Theory of Majorization and Its Applications*, Mathematics in Science and Engineering, Vol. **143**, Academic Press, New York and London, 1979.

9. H.-N. Shi, M. Bencze, S.-H. Wu and D.-M. Li, Schur convexity of generalized Heronian means involving two parameters, *J. Inequal. Appl.*, **2008** (2008), Article ID 879273, 1-9.

10. H.-N. Shi, S.-H. Wu and F. Qi, An alternative note on the Schur-convexity of the extended mean values, *Math. Inequal. Appl.*, **9** (2006), 219-224.

11. W.-F. Xia and Y.-M. Chu, Schur-convexity for a class of symmetric functions and its applications. *J. Inequal. Appl.*, (2009), Article ID 493759, 1-15.

12. Z.-H. Zhang and Y.-D. Wu, The generalized Heron mean and its dual form, *Appl. Math. E-Notes*, **5** (2005), 16-23 (electronic).

13. Z.-H. Zhang, Y.-D. Wu and H. M. Srivastava, Generalized Vandermonde determinants and mean values, *Appl. Math. Comput.*, **202** (2008), 300-310.

14. Z.-H. Zhang, Y.-D. Wu and A.-P. Zhao, The properties of the generalized Heron means and its dual form, *RGMIA Res. Rep. Collect.*, **7** (2004), Article 1.

15. N.-G. Zheng, Z.-H. Zhang and X.-M. Zhang, Schur-convexity of two types of one-parameter mean values in n variables, *J. Inequal. Appl.*, **2007** (2007), Article ID 78175, 1-10.

Li-Li Fu and Bo-Yan Xi
College of Mathematics
Inner Mongolia University for Nationalities
Inner Mongolia Autonomous Region
Tongliao City 028043
P. R. China
E-mail: fulili0209@163.com
baoyintu68@sohu.com

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
E-mail: harimsri@math.uvic.ca