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WEIGHTED LIPSCHITZ ESTIMATES FOR COMMUTATORS OF FRACTIONAL INTEGRALS WITH HOMOGENEOUS KERNELS

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Abstract. In this paper the authors give a sufficient condition such that the commutator generated by the weighted Lipschitz function and the fractional integral operator with homogeneous kernel satisfying certain Dini condition is bounded on weighted Lebesgue spaces.

1. INTRODUCTION

Assume that $\Omega \in L^s(S^{n-1})$ for $1 \leq s \leq \infty$, where S^{n-1} denotes the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ and Ω is a homogeneous function with degree zero on $\mathbb{R}^n$, i.e., for any $\lambda > 0$ and $x \in \mathbb{R}^n$,

$$\Omega(\lambda x) = \Omega(x).$$

The fractional integral operator with rough kernel is defined by

$$T_{\Omega,\alpha} f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy, \tag{1.1}$$

where $0 < \alpha < n$.

The commutator generated by a suitable function b and the fractional integral operator $T_{\Omega,\alpha}$ is defined by

$$[b, T_{\Omega,\alpha}]f(x) = b(x)T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(bf)(x).$$

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We say a function Ω satisfying $L^s - Dini(s \geq 1)$ condition, if Ω is homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^s(S^{n-1})$, and

$$\int_0^1 \frac{\omega_s(\delta)}{\delta} \ln \frac{1}{\delta}\, d\delta < \infty,$$

where $\omega_s(\delta)$ is the integral modulus of continuity of order s of Ω, that is,

$$\omega_s(\delta) = \sup_{|\rho| \leq \delta} \left(\int_{S^{n-1}} |\Omega(x') - \Omega(\rho x')|^s d\sigma(x') \right)^{\frac{1}{s}},$$

here ρ is a rotation in $\mathbb{R}^n$ and

$$|\rho| = \sup_{x' \in S^{n-1}} |\rho x' - x'|.$$

Chanillo [1] proved that the commutator $[b, I_\alpha]$ generated by a function $b \in BMO$ and the classical fractional integral operator I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then Paluszyński [11] showed that $b \in Lip_\beta$(the homogeneous Lipschitz space) if and only if the commutator $[b, T]$ generated by b and the singular integral operator T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < q < \infty$, $0 < \beta < 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. Also Paluszyński [11] obtained that $b \in Lip_\beta$ if and only if the commutator $[b, I_\alpha]$ generated by b and the classical fractional integral operator I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, where $1 < p < r < \infty$, $0 < \beta < 1$ and $\frac{1}{r} = \frac{1}{p} - \frac{\beta+\alpha}{n}$. Recently Hu and Gu [9] obtained that b is in the weighted Lipschitz space $Lip_{\beta,\mu}$, whose definition will be given in the next section, if and only if the commutator $[b, I_\alpha]$ is bounded from $L^p(\mu)$ to $L^r(\mu^{1-(1-\frac{\alpha}{n})r})$, where $1 < p < \frac{n}{\alpha+\beta}$, $0 < \alpha + \beta < n$, $\frac{1}{r} = \frac{1}{p} - \frac{\alpha+\beta}{n}$ and $\mu \in A_1(\mathbb{R}^n)$.

Since the kernel of the fractional integral operator with homogeneous kernel is more general than that of the classical fractional integral operator and the weighted Lipschitz space can be regarded as a generalization of the classical Lipschitz space, inspired by the above results, a question arises naturally. If we consider the commutator generated by the fractional integral operator with homogeneous kernel and the weighted Lipschitz function, what boundedness properties does this kind of operators have? In this paper, we will focus on its boundedness on weighted Lebesgue spaces.

2. Some Preliminaries and Notations

A non-negative function μ defined on $\mathbb{R}^n$ is called a weight if it is locally integrable. A weight μ is said to belong to the Muckenhoupt class $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|} \int_B \mu(x) dx \right) \left(\frac{1}{|B|} \int_B \mu(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C$$

for every ball $B \subset \mathbb{R}^n$. The class $A_1(\mathbb{R}^n)$ is defined by replacing the above inequality by

$$\frac{1}{|B|}\int_B \mu(x)dx \le C\mu(x), \quad a.e. x \in \mathbb{R}^n$$

for every ball $B \subset \mathbb{R}^n$; see[7].

Suppose that ω is a nonnegative locally integrable function on $\mathbb{R}^n$. Define $\omega \in A(p,q)(1 < p, q < \infty)$, if there exists a constant $C > 0$, such that for any ball B in $\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega(x)^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B \omega(x)^{-p'} dx\right)^{\frac{1}{p'}} \le C.$$

Some usual properties of weights we need in this paper are the following

Lemma 2.1. **[5].** *If* $\mu \in A_1(\mathbb{R}^n)$, *then there exists a* $\varepsilon > 0$ *such that* $\mu^{1+\varepsilon} \in A_1(\mathbb{R}^n)$.

Lemma 2.2. **[8].** *If* $\omega_1, \omega_2 \in A_p(\mathbb{R}^n)(1 \le p < \infty)$, *then for any* $0 \le \alpha \le 1$, $\omega_1^{\alpha}\omega_2^{1-\alpha} \in A_p(\mathbb{R}^n)$.

Lemma 2.3. **[3].** *Suppose that* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then*

$$\omega \in A(p,q) \Leftrightarrow \omega^q \in A_{1+\frac{q}{p'}}.$$

Standard real analysis tools as the Hardy-Littlewood maximal function Mf, the sharp maximal function $M^\sharp f$, naturally carries over to this context, namely,

$$\begin{aligned} Mf(x) &= \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|dy, \\ M^\sharp f(x) &= \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y) - f_B|dy \sim \sup_{B \ni x}\inf_{c \in \mathbb{C}} \frac{1}{|B|}\int_B |f(y) - c|dy, \end{aligned}$$

where $f_B = \frac{1}{|B|}\int_B f(y)dy$.

And following [6], we will say that a locally integrable function f belongs to the weighted Lipschitz space $Lip^p_{\beta,\mu}$ for $1 \le p \le \infty$, $0 < \beta < 1$, and $\mu \in A_\infty(\mathbb{R}^n)$, if there exists a constant $C > 0$, such that for any ball B in $\mathbb{R}^n$,

$$\frac{1}{\mu(B)^{\frac{\beta}{n}}}\left[\frac{1}{\mu(B)}\int_B |f(x) - f_B|^p \mu(x)^{1-p} dx\right]^{\frac{1}{p}} \le C.$$

Modulo constants, the Banach space of such functions is denoted by $Lip^p_{\beta,\mu}$. The smallest bound C satisfying the above condition is taken to be the norm of f in this space, and is denoted by $\|f\|_{Lip^p_{\beta,\mu}}$. Put $Lip_{\beta,\mu} = Lip^1_{\beta,\mu}$. Obviously, for the case $\mu \equiv 1$, the space $Lip_{\beta,\mu}$ is the classical Lipschitz space Lip_β. Thus, weighted Lipschitz spaces are generalization of classical Lipschitz spaces.

Let $\mu \in A_1(\mathbb{R}^n)$, García-Cuerva in [6] proved that the space $Lip^p_{\beta,\mu}$ coincide, and the norm of $\|\cdot\|_{Lip^p_{\beta,\mu}}$ are equivalent with respect to different values of p provided that $1 \le p \le \infty$. That is, $\|\cdot\|_{Lip^p_{\beta,\mu}} \sim \|\cdot\|_{Lip_{\beta,\mu}}$ for $1 \le p \le \infty$.

A variant of the maximal operator and the sharp maximal operator $M_\delta f(x) = M(|f|^\delta)^{\frac{1}{\delta}}(x)$ and $M^\sharp_\delta f(x) = M^\sharp(|f|^\delta)^{\frac{1}{\delta}}(x)$, will become the main tools in our paper.

3. Main Results

Our main results are as follows.

Theorem 3.1. *Let $0 < \beta < 1$, $0 < \alpha + \beta < n$, $1 < p < \frac{n}{\alpha+\beta}$, $\frac{1}{r} = \frac{1}{p} - \frac{\alpha+\beta}{n}$, $\mu^{\frac{r}{p}} \in A_1(\mathbb{R}^n)$ and $\frac{(\varepsilon+1)p}{\varepsilon(p-1)} < s < \infty$, where ε is the constant in Lemma 2.1. Let $T_{\Omega,\alpha}$ be the fractional integral operator defined by (1.1) with the kernel Ω satisfying*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta} \ln \frac{1}{\delta} d\delta < \infty.$$

If $b \in Lip_{\beta,\mu}$, then the commutator $[b, T_{\Omega,\alpha}]$ is bounded from $L^p(\mu)$ to $L^r(\mu^{1-(1-\frac{\alpha}{n})r})$.

If we hope that the index s in the Dini condition of the kernel is independent of the constant ε of $A_1(\mathbb{R}^n)$ in Lemma 2.1, then we can obtain that

Theorem 3.2. *Let $0 < \beta < 1$, $0 < \alpha + \beta < n$, $1 < p < \frac{n}{\alpha+\beta}$, $\frac{1}{r} = \frac{1}{p} - \frac{\alpha+\beta}{n}$, $\mu^{\frac{r}{p}} \in A_1(\mathbb{R}^n)$ and $\frac{pr}{(p-1)(r-p)} < s < \infty$. Let $T_{\Omega,\alpha}$ be the fractional integral operator defined by (1.1) with the kernel Ω satisfying*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta} \ln \frac{1}{\delta} d\delta < \infty.$$

If $b \in Lip_{\beta,\mu}$, then the commutator $[b, T_{\Omega,\alpha}]$ is bounded from $L^p(\mu)$ to $L^r(\mu^{1-(1-\frac{\alpha}{n})r})$.

4. Some Lemmas

First we need some known results about maximal operators and the fractional integral operator with rough kernel.

Lemma 4.1. ([3]). *Suppose that $0<\alpha<n$, $1\le s' < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $\Omega \in L^s(S^{n-1})$ and $\omega^{s'} \in A(\frac{p}{s'}, \frac{q}{s'})$, then there exists a constant C independent of f such that*

$$\left(\int_{\mathbb{R}^n}|T_{\Omega,\alpha}f(x)\omega(x)|^q dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p dx\right)^{\frac{1}{p}}.$$

Lemma 4.2. ([3]). *Suppose that* $0<\alpha<n$, $1\le s'<p<\frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *If* $\omega^{s'}\in A(\frac{p}{s'},\frac{q}{s'})$, *then there exists a constant* C *independent of* f *such that*

$$\left(\int_{\mathbb{R}^n}[M_{\alpha,s'}f(x)\omega(x)]^q dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p dx\right)^{\frac{1}{p}},$$

where $\frac{1}{s}+\frac{1}{s'}=1$ *and*

$$M_{\alpha,s'}(f)(x)=\sup_{B\ni x}\left(\frac{1}{|B|^{1-\frac{\alpha s'}{n}}}\int_B |f(x)|^{s'}dy\right)^{\frac{1}{s'}}.$$

Lemma 4.3. ([10]). *Suppose that* $0<\alpha<n$, $1<p<\frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *If* $\omega^q\in A_1(\mathbb{R}^n)$, *then there exists a constant* C *independent of* f *such that*

$$\left(\int_{\mathbb{R}^n}[M_\alpha f(x)\omega(x)]^q dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p dx\right)^{\frac{1}{p}},$$

where

$$M_\alpha(f)(x)=\sup_{B\ni x}\left(\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B |f(y)|dy\right).$$

Lemma 4.4. ([2]). *Suppose that* $0<\alpha<n$, $\Omega\in L^{\frac{n}{n-\alpha}}(S^{n-1})$ *and* Ω *is a homogeneous function with degree zero on* $\mathbb{R}^n$. *If* $f\in L^1(\mathbb{R}^n)$, *then for any* $\lambda>0$,

$$\left|\{x\in\mathbb{R}^n:\left|T_{\Omega,\alpha}f(x)\right|>\lambda\}\right|\le C\left(\frac{1}{\lambda}\|f\|_{L^1}\right)^{\frac{n}{n-\alpha}}.$$

Lemma 4.5. ([4]). *Suppose that* $0<\alpha<n$, Ω *is homogeneous of degree zero and satisfies the* $L^s-Dini(s\ge 1)$ *condition. If there exists a constant* a_0 *with* $0<a_0<1$ *such that* $|x|<a_0R$, *where* $x\in\mathbb{R}^n$ *and* $R>0$, *then there exists a constant* $C>0$ *such that*

$$\left(\int_{R<|y|\le 2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\alpha}}-\frac{\Omega(y)}{|y|^{n-\alpha}}\right|^s dy\right)^{\frac{1}{s}} \le CR^{\alpha-\frac{n}{s'}}\left\{\frac{|x|}{R}+\int_{\frac{|x|}{2R}}^{\frac{|x|}{R}}\frac{\omega_s(\delta)}{\delta}d\delta\right\},$$

where $\frac{1}{s}+\frac{1}{s'}=1$.

Lemma 4.6. ([12]). *Let* $0<p,\delta<\infty$ *and* $\mu\in A_\infty(\mathbb{R}^n)$. *There exists a positive constant* C *such that*

$$\int_{\mathbb{R}^n} M_\delta f(x)^p \mu(x)dx \le C \int_{\mathbb{R}^n} M_\delta^\sharp f(x)^p \mu(x)dx,$$

for any smooth function f for which the left-hand side is finite.

In order to obtain the main results, we still need to state some technical lemmas step by step.

Lemma 4.7. *Let $\mu \in A_1(\mathbb{R}^n)$, $0 < \beta < 1$ and the integer $k \ge 1$. If $b \in Lip_{\beta,\mu}$, then*

$$\left|b_{\bar{B}} - b_{B(x,2^{k+1}R)}\right| \le Ck\mu(x)\mu(B(x,2^{k+1}R))^{\frac{\beta}{n}}\|b\|_{Lip_{\beta,\mu}},$$

where $\bar{B} = B(x,2R)$.

Proof. Write

$$\begin{aligned}
&\left|b_{\bar{B}} - b_{B(x,2^{k+1}R)}\right| \\
&\le \sum_{j=1}^{k} \left|b_{B(x,2^jR)} - b_{B(x,2^{j+1}R)}\right| \\
&\le C\|b\|_{Lip_{\beta,\mu}} \sum_{j=1}^{k} \frac{1}{|B(x,2^{j+1}R)|}\mu(B(x,2^{j+1}R))^{1+\frac{\beta}{n}} \\
&\le Ck\mu(x)\mu(B(x,2^{k+1}R))^{\frac{\beta}{n}}\|b\|_{Lip_{\beta,\mu}}.
\end{aligned}$$

Lemma 4.8. *Let $\mu \in A_1(\mathbb{R}^n)$, ε be the constant in Lemma 2.1, $0 < \beta < 1$, $0 < \alpha+\beta < n$, $s > \frac{n\varepsilon+n}{n\varepsilon-\alpha\varepsilon-\beta\varepsilon}$ and $\frac{s\varepsilon}{s\varepsilon-\varepsilon-1} < r_0 < \frac{n}{\alpha+\beta}$. If $b \in Lip_{\beta,\mu}$, then there exists a constant $C > 0$ such that*

$$|B|^{\frac{\alpha}{n}}\left(\frac{1}{|B|}\int_B |b(w)-b_B|^{s'}|f(w)|^{s'}dw\right)^{\frac{1}{s'}} \le C\mu(x)^{1-\frac{\alpha}{n}}\|b\|_{Lip_{\beta,\mu}} M_{\alpha+\beta,\mu,r_0}(f)(x),$$

where $\frac{1}{s} + \frac{1}{s'} = 1$ and

$$M_{\alpha+\beta,\mu,r_0}(f)(x) = \sup_{B\ni x}\left(\frac{1}{\mu(B)^{1-\frac{r_0(\alpha+\beta)}{n}}}\int_B |f(y)|^{r_0}\mu(y)dy\right)^{\frac{1}{r_0}}.$$

Proof. Let $r_2 = \frac{r_0}{s'}$, $r_3 = \frac{\varepsilon}{s'-1}$ and $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$. Since $r_0 > \frac{s\varepsilon}{s\varepsilon-\varepsilon-1} > \frac{s}{s-1} = s'$ and $s > \frac{n\varepsilon+n}{n\varepsilon-\alpha\varepsilon-\beta\varepsilon} > \frac{n(1+\varepsilon)}{(n-\alpha)\varepsilon} > 1+\frac{1}{\varepsilon}$, then $r_1, r_2, r_3 > 1$ and $\frac{1}{s'} - \frac{1}{r_3 s'} - \frac{\alpha}{n} > 0$. By Hölder's inequality, $\mu \in A_1(\mathbb{R}^n)$ and $\mu^{1+\varepsilon} \in A_1(\mathbb{R}^n)$, we have

$$\begin{aligned}
&|B|^{\frac{\alpha}{n}}\left(\frac{1}{|B|}\int_B |b(w)-b_B|^{s'}|f(w)|^{s'}dw\right)^{\frac{1}{s'}}\\
&=|B|^{\frac{\alpha}{n}}\left(\frac{1}{|B|}\right)^{\frac{1}{s'}}\left(\int_B |b(w)-b_B|^{s'}\mu(w)^{\frac{1}{r_1}-s'}|f(w)|^{s'}\mu(w)^{\frac{1}{r_2}}\mu(w)^{s'-\frac{1}{r_1}-\frac{1}{r_2}}dw\right)^{\frac{1}{s'}}\\
&\le |B|^{\frac{\alpha}{n}}\left(\frac{1}{|B|}\right)^{\frac{1}{s'}}\left(\int_B |b(w)-b_B|^{r_1 s'}\mu(w)^{1-r_1 s'}dw\right)^{\frac{1}{r_1 s'}}\left(\int_B |f(w)|^{r_2 s'}\mu(w)dw\right)^{\frac{1}{r_2 s'}}\\
&\quad\cdot\left(\int_B \mu(w)^{1+r_3(s'-1)}dw\right)^{\frac{1}{r_3 s'}}\\
&=|B|^{\frac{\alpha}{n}}\left(\frac{1}{|B|}\right)^{\frac{1}{s'}}\frac{1}{\mu(B)^{\frac{\beta}{n}}}\left(\frac{1}{\mu(B)}\int_B |b(w)-b_B|^{r_1 s'}\mu(w)^{1-r_1 s'}dw\right)^{\frac{1}{r_1 s'}}\mu(B)^{\frac{1}{r_1 s'}+\frac{1}{r_2 s'}-\frac{\alpha}{n}}\\
&\quad\cdot|B|^{\frac{1}{r_3 s'}}\left(\frac{1}{\mu(B)^{1-\frac{r_2 s'(\alpha+\beta)}{n}}}\int_B |f(w)|^{r_2 s'}\mu(w)dw\right)^{\frac{1}{r_2 s'}}\left(\frac{1}{|B|}\int_B \mu(w)^{1+\varepsilon}dw\right)^{\frac{1}{r_3 s'}}\\
&\le C\left(\frac{\mu(B)}{|B|}\right)^{\frac{1}{s'}-\frac{1}{r_3 s'}-\frac{\alpha}{n}}\left(\mu(x)^{1+\varepsilon}\right)^{\frac{1}{r_3 s'}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,\mu,r_2 s'}(f)(x)\\
&\le C\mu(x)^{1-\frac{\alpha}{n}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,\mu,r_0}(f)(x).
\end{aligned}$$

Lemma 4.9. *Let* $0<\beta<1$*,* $0<\alpha+\beta<n$*,* $1<p<\frac{n}{\alpha+\beta}$*,* $\frac{1}{r}=\frac{1}{p}-\frac{\alpha+\beta}{n}$*,* $s>\frac{rn}{(r-p)(n-\alpha-\beta)}$*,* $\frac{s(r-p)}{s(r-p)-r}<r_0<\frac{n}{\alpha+\beta}$ *and* $\mu^{\frac{r}{p}}\in A_1(\mathbb{R}^n)$*. If* $b\in Lip_{\beta,\mu}$*, then there exists a constant* $C>0$ *such that*

$$\left|B\right|^{\frac{\alpha}{n}}\left(\frac{1}{|B|}\int_B |b(w)-b_B|^{s'}|f(w)|^{s'}dw\right)^{\frac{1}{s'}}\le C\mu(x)^{1-\frac{\alpha}{n}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,\mu,r_0}(f)(x),$$

where $\frac{1}{s}+\frac{1}{s'}=1$.

Proof. Let $r_2=\frac{r_0}{s'}$, $r_3=\frac{r-p}{p(s'-1)}$ and $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=1$. Since $r_0>\frac{s(r-p)}{s(r-p)-r}=\frac{s'(r-p)}{r-ps'}>s'$, $s>\frac{rn}{(r-p)(n-\alpha-\beta)}>\frac{rn}{(r-p)(n-\alpha)}>\frac{r}{r-p}$, then $r_1,r_2,r_3>1$ and $\frac{1}{s'}-\frac{1}{r_3 s'}-\frac{\alpha}{n}>0$. By the similar computation techniques used in Lemma 4.8, we have

$$\left|B\right|^{\frac{\alpha}{n}}\left(\frac{1}{|B|}\int_B |b(w)-b_B|^{s'}|f(w)|^{s'}dw\right)^{\frac{1}{s'}}\le C\mu(x)^{1-\frac{\alpha}{n}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,\mu,r_0}(f)(x).$$

Lemma 4.10. *Let* $\mu\in A_1(\mathbb{R}^n)$*,* $0<\beta<1$*,* $0<\alpha+\beta<n$*,* $s>\frac{n}{n-\alpha-\beta}$ *and the integer* $k\ge 1$*. If* $b\in Lip_{\beta,\mu}$*, then there exists a constant* $C>0$ *such that*

$$\begin{aligned}
&\left|B(x,2^{k+1}R)\right|^{\frac{\alpha}{n}}\left|b_{\bar{B}}-b_{B(x,2^{k+1}R)}\right|\left(\frac{1}{|B(x,2^{k+1}R)|}\int_{B(x,2^{k+1}R)}|f(w)|^{s'}dw\right)^{\frac{1}{s'}}\\
&\le Ck\mu(x)^{1+\frac{\beta}{n}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,s'}(f)(x),
\end{aligned}$$

where $\frac{1}{s}+\frac{1}{s'}=1$ *and* $\bar{B}=B(x,2R)$.

Proof. By Lemma 4.7 and $\mu\in A_1(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\big|B(x,2^{k+1}R)\big|^{\frac{\alpha}{n}}\big|b_{\bar{B}}-b_{B(x,2^{k+1}R)}\big|\left(\frac{1}{|B(x,2^{k+1}R)|}\int_{B(x,2^{k+1}R)}|f(w)|^{s'}dw\right)^{\frac{1}{s'}}\\
&\le C\big|B(x,2^{k+1}R)\big|^{\frac{\alpha}{n}}k\mu(x)\|b\|_{Lip_{\beta,\mu}}\mu\big(B(x,2^{k+1}R)\big)^{\frac{\beta}{n}}\big|B(x,2^{k+1}R)\big|^{-\frac{1}{s'}}\\
&\quad\cdot\left(\int_{B(x,2^{k+1}R)}|f(w)|^{s'}dw\right)^{\frac{1}{s'}}\\
&\le Ck\mu(x)^{1+\frac{\beta}{n}}\|b\|_{Lip_{\beta,\mu}}\left(\frac{1}{\big|B(x,2^{k+1}R)\big|^{1-\frac{(\alpha+\beta)s'}{n}}}\int_{B(x,2^{k+1}R)}|f(w)|^{s'}dw\right)^{\frac{1}{s'}}\\
&\le Ck\mu(x)^{1+\frac{\beta}{n}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,s'}(f)(x).
\end{aligned}$$

Lemma 4.11. *Let* $\mu\in A_1(\mathbb{R}^n)$, $0<\beta<1$, $b\in Lip_{\beta,\mu}$, $0<\alpha+\beta<n$, $s>\frac{n\varepsilon+n}{n\varepsilon-\alpha\varepsilon-\beta\varepsilon}$, $0<\delta\le\frac{1}{2}<\frac{s\varepsilon}{s\varepsilon-\varepsilon-1}<r_0<\frac{n}{\alpha+\beta}$, *where* ε *is the constant in Lemma 2.1. If* $T_{\Omega,\alpha}$ *is the fractional integral operator defined by (1.1) with the kernel* Ω *satisfying*

$$\int_0^1\frac{\omega_s(\delta)}{\delta}\ln\frac{1}{\delta}d\delta<\infty,$$

then there exists a constant $C>0$ *such that*

$$\begin{aligned}
&M_\delta^\sharp\big([b,T_{\Omega,\alpha}]f\big)(x)\\
&\le C\|b\|_{Lip_{\beta,\mu}}\bigg(\mu(x)^{1+\frac{\beta}{n}}M_\beta(T_{\Omega,\alpha}f)(x)+\mu(x)^{1-\frac{\alpha}{n}}M_{\alpha+\beta,\mu,r_0}(f)(x)\\
&\quad+\mu(x)^{1+\frac{\beta}{n}}M_{\alpha+\beta,s'}(f)(x)\bigg).
\end{aligned}$$

Proof. Taking $\lambda=b_{\bar{B}}$, the average of b on $\bar{B}$, where $\bar{B}=B(x,2R)$, we have

$$[b,T_{\Omega,\alpha}]f(x)=\big(b(x)-b_{\bar{B}}\big)T_{\Omega,\alpha}f(x)-T_{\Omega,\alpha}\big((b-b_{\bar{B}})f\big)(x).$$

We fix $x\in\mathbb{R}^n$ and denote $B=B(x,R)$ with $R>0$. Decompose $f=f_1+f_2$, where $f_1=f\chi_{\bar{B}}$. Let c be a constant to be fixed along the proof.

Since $0<\delta\le\frac{1}{2}$, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B\Big|\big|[b,T_{\Omega,\alpha}]f(y)\big|^\delta-|c|^\delta\Big|dy\right)^{\frac{1}{\delta}}\\
&\le\left(\frac{1}{|B|}\int_B\big|[b,T_{\Omega,\alpha}]f(y)-c\big|^\delta dy\right)^{\frac{1}{\delta}}
\end{aligned}$$

$$= \left(\frac{1}{|B|}\int_B \left|\big(b(y)-b_{\bar B}\big)T_{\Omega,\alpha}f(y) - T_{\Omega,\alpha}\big((b-b_{\bar B})f\big)(y) - c\right|^\delta dy\right)^{\frac{1}{\delta}}$$
$$\le C\left(\frac{1}{|B|}\int_B \left|\big(b(y)-b_{\bar B}\big)T_{\Omega,\alpha}f(y)\right|^\delta dy\right)^{\frac{1}{\delta}}$$
$$+C\left(\frac{1}{|B|}\int_B \left|T_{\Omega,\alpha}\big((b-b_{\bar B})f_1\big)(y)\right|^\delta dy\right)^{\frac{1}{\delta}}$$
$$+C\left(\frac{1}{|B|}\int_B \left|T_{\Omega,\alpha}\big((b-b_{\bar B})f_2\big)(y)+c\right|^\delta dy\right)^{\frac{1}{\delta}}$$
$$:= I + II + III.$$

To deal with I first, for $0<\delta\le\frac{1}{2}$, we can take a l such that $\frac{1}{1-\delta}\le l\le\frac{1}{\delta}$, so $\frac{1}{\delta l}\ge 1$ and $\frac{1}{\delta l'}\ge 1$. By Hölder's inequality we have

$$I \le C\left(\frac{1}{|\bar B|}\int_{\bar B}\left|b(y)-b_{\bar B}\right|^{\delta l}dy\right)^{\frac{1}{\delta l}}\left(\frac{1}{|B|}\int_B\left|T_{\Omega,\alpha}f(y)\right|^{\delta l'}dy\right)^{\frac{1}{\delta l'}}$$
$$\le C\left(\frac{1}{|\bar B|}\int_{\bar B}|b(y)-b_{\bar B}|dy\right)\left(\frac{1}{|B|}\int_B|T_{\Omega,\alpha}(f)(y)|dy\right)$$
$$\le C\Big(\frac{\mu(\bar B)}{|\bar B|}\Big)^{1+\frac{\beta}{n}}\|b\|_{Lip_{\beta,\mu}}M_\beta(T_{\Omega,\alpha}f)(x)$$
$$\le C\mu(x)^{1+\frac{\beta}{n}}\|b\|_{Lip_{\beta,\mu}}M_\beta(T_{\Omega,\alpha}f)(x),$$

where $\frac{1}{l'}+\frac{1}{l}=1$.

For II, we make use of Lemma 4.4 and Kolmogorov's inequality, then

$$II \le C\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B\left|\big((b-b_{\bar B})f_1\big)(y)\right|dy$$
$$\le C\frac{1}{|B|^{1-\frac{\alpha}{n}}}\left(\int_{\bar B}\left|b(y)-b_{\bar B}\right|^{r_0'}\mu(y)^{1-r_0'}dy\right)^{\frac{1}{r_0'}}\left(\int_{\bar B}\left|f(y)\right|^{r_0}\mu(y)dy\right)^{\frac{1}{r_0}}$$
$$= C\frac{1}{|B|^{1-\frac{\alpha}{n}}}\frac{1}{\mu(\bar B)^{\frac{\beta}{n}}}\left(\frac{1}{\mu(\bar B)}\int_{\bar B}\left|b(y)-b_{\bar B}\right|^{r_0'}\mu(y)^{1-r_0'}dy\right)^{\frac{1}{r_0'}}\mu(\bar B)^{\frac{\beta}{n}}\mu(\bar B)^{\frac{1}{r_0'}}$$
$$\cdot\left(\frac{1}{\mu(\bar B)^{1-\frac{(\alpha+\beta)r_0}{n}}}\int_B\left|f(y)\right|^{r_0}\mu(y)dy\right)^{\frac{1}{r_0}}\mu(\bar B)^{(1-\frac{(\alpha+\beta)r_0}{n})\frac{1}{r_0}}$$
$$\le C\Big(\frac{\mu(\bar B)}{\bar B}\Big)^{1-\frac{\alpha}{n}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,\mu,r_0}(f)(x)$$
$$\le C\mu(x)^{1-\frac{\alpha}{n}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,\mu,r_0}(f)(x),$$

where $\frac{1}{r_0}+\frac{1}{r_0'}=1$.

Finally, for III we first fix the value of c by taking $c = -T_{\Omega,\alpha}\big((b-b_{\bar{B}})f_2\big)(x)$, then by Hölder's inequality and Lemma 4.5, 4.8, 4.10, we have

$$
\begin{aligned}
III &\le \frac{C}{|B|}\int_B \big|T_{\Omega,\alpha}\big((b-b_{\bar{B}})f_2\big)(y) - T_{\Omega,\alpha}\big((b-b_{\bar{B}})f_2\big)(x)\big|dy \\
&\le \frac{C}{|B|}\int_B\int_{\mathbb{R}^n\setminus\bar{B}}\left|\frac{\Omega(y-w)}{|y-w|^{n-\alpha}} - \frac{\Omega(x-w)}{|x-w|^{n-\alpha}}\right|\big|b(w)-b_{\bar{B}}\big|\big|f(w)\big|dwdy \\
&= \frac{C}{|B|}\int_B\sum_{k=1}^{\infty}\int_{2^kR\le|x-w|<2^{k+1}R} \\
&\qquad \left|\frac{\Omega(y-w)}{|y-w|^{n-\alpha}} - \frac{\Omega(x-w)}{|x-w|^{n-\alpha}}\right|\big|b(w)-b_{\bar{B}}\big|\big|f(w)\big|dwdy \\
&\le \frac{C}{|B|}\int_B\sum_{k=1}^{\infty}\left(\int_{2^kR\le|x-w|<2^{k+1}R}\left|\frac{\Omega(y-w)}{|y-w|^{n-\alpha}} - \frac{\Omega(x-w)}{|x-w|^{n-\alpha}}\right|^s dw\right)^{\frac{1}{s}} \\
&\qquad \cdot\left(\int_{|x-w|\le 2^{k+1}R}\big|b(w)-b_{\bar{B}}\big|^{s'}\big|f(w)\big|^{s'}dw\right)^{\frac{1}{s'}}dy \\
&\le \frac{C}{|B|}\int_B\sum_{k=1}^{\infty}\left(2^{-k} + \int_{\frac{|x-y|}{2^{k+1}R}}^{\frac{|x-y|}{2^kR}}\frac{\omega_s(\delta)}{\delta}d\delta\right) \\
&\qquad \cdot\Bigg[\big|B(x,2^{k+1}R)\big|^{\frac{\alpha}{n}}\left(\frac{1}{|B(x,2^{k+1}R)|}\int_{B(x,2^{k+1}R)}\big|b(w)-b_{B(x,2^{k+1}R)}\big|^{s'}\big|f(w)\big|^{s'}dw\right)^{\frac{1}{s'}} \\
&\qquad +\big|B(x,2^{k+1}R)\big|^{\frac{\alpha}{n}}\big|b_{\bar{B}}-b_{B(x,2^{k+1}R)}\big|\left(\frac{1}{|B(x,2^{k+1}R)|}\int_{B(x,2^{k+1}R)}|f(w)|^{s'}dw\right)^{\frac{1}{s'}}\Bigg]dy \\
&\le \frac{C}{|B|}\int_B\sum_{k=1}^{\infty}\left(2^{-k} + \int_{\frac{|x-y|}{2^{k+1}R}}^{\frac{|x-y|}{2^kR}}\frac{\omega_s(\delta)}{\delta}d\delta\right)dy \\
&\qquad \cdot\big[\mu(x)^{1-\frac{\alpha}{n}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,\mu,r_0}(f)(x) + k\mu(x)^{1+\frac{\beta}{n}}\|b\|_{Lip_{\beta,\mu}}M_{\alpha+\beta,s'}(f)(x)\big] \\
&\le C\left(\sum_{k=1}^{\infty}k2^{-k} + \int_0^1\frac{\omega_s(\delta)}{\delta}\ln\frac{1}{\delta}d\delta\right)\|b\|_{Lip_{\beta,\mu}} \\
&\qquad \cdot\left(\mu(x)^{1-\frac{\alpha}{n}}M_{\alpha+\beta,\mu,r_0}(f)(x) + \mu(x)^{1+\frac{\beta}{n}}M_{\alpha+\beta,s'}(f)(x)\right) \\
&\le C\|b\|_{Lip_{\beta,\mu}}\left(\mu(x)^{1-\frac{\alpha}{n}}M_{\alpha+\beta,\mu,r_0}(f)(x) + \mu(x)^{1+\frac{\beta}{n}}M_{\alpha+\beta,s'}(f)(x)\right).
\end{aligned}
$$

Lemma 4.12. *Let* $0<\beta<1$, $0<\alpha+\beta<n$, $1<p<\frac{n}{\alpha+\beta}$, $\frac{1}{r}=\frac{1}{p}-\frac{\alpha+\beta}{n}$, $\mu^{\frac{r}{p}}\in A_1(\mathbb{R}^n)$, $b\in Lip_{\beta,\mu}$, $s>\frac{rn}{(r-p)(n-\alpha-\beta)}$ *and* $0<\delta\le\frac{1}{2}<\frac{s(r-p)}{s(r-p)-r}<r_0<\frac{n}{\alpha+\beta}$. *If* $T_{\Omega,\alpha}$ *is the fractional integral operator defined by (1.1) with the kernel* Ω *satisfying*

$$\int_0^1\frac{\omega_s(\delta)}{\delta}\ln\frac{1}{\delta}d\delta<\infty,$$

then there exists a constant $C>0$ *such that*

$$\begin{aligned}
&M_\delta^\sharp\big([b,T_{\Omega,\alpha}]f\big)(x)\\
&\le C\|b\|_{Lip_{\beta,\mu}}\bigg(\mu(x)^{1+\frac{\beta}{n}}M_\beta(T_{\Omega,\alpha}f)(x)+\mu(x)^{1-\frac{\alpha}{n}}M_{\alpha+\beta,\mu,r_0}(f)(x)\\
&\quad+\mu(x)^{1+\frac{\beta}{n}}M_{\alpha+\beta,s'}(f)(x)\bigg).
\end{aligned}$$

Proof. Since $\mu^{\frac{r}{p}}\in A_1(\mathbb{R}^n)$, by Lemma 2.2 we have $\mu\in A_1(\mathbb{R}^n)$. Making use of Lemma 4.5, 4.9, 4.10, similar to the proof of Lemma 4.11, we have

$$\begin{aligned}
&M_\delta^\sharp\big([b,T_{\Omega,\alpha}]f\big)(x)\\
&\le C\|b\|_{Lip_{\beta,\mu}}\bigg(\mu(x)^{1+\frac{\beta}{n}}M_\beta(T_{\Omega,\alpha}f)(x)+\mu(x)^{1-\frac{\alpha}{n}}M_{\alpha+\beta,\mu,r_0}(f)(x)\\
&\quad+\mu(x)^{1+\frac{\beta}{n}}M_{\alpha+\beta,s'}(f)(x)\bigg).
\end{aligned}$$

5. Proof of Theorems

Proof of Theorem 3.1. Since $\mu^{\frac{r}{p}}\in A_1(\mathbb{R}^n)$, by Lemma 2.2 we have $\mu\in A_1(\mathbb{R}^n)$. By $s>\frac{p(\varepsilon+1)}{(p-1)\varepsilon}$, we get $\frac{s\varepsilon}{s\varepsilon-\varepsilon-1}<p$. Thus there exist δ and r_0 such that $0<\delta\le\frac{1}{2}<\frac{s\varepsilon}{s\varepsilon-\varepsilon-1}<r_0<p<\frac{n}{\alpha+\beta}$. Since $p>\frac{s\varepsilon}{s\varepsilon-\varepsilon-1}>\frac{s}{s-1}=s'$, we get $\mu^{\frac{r}{p}}\in A_1(\mathbb{R}^n)\subseteq A_{1+\frac{r(p-s')}{ps'}}(\mathbb{R}^n)$. By Lemma 2.2 we have $\mu^{\frac{q}{p}}\in A_1(\mathbb{R}^n)\subseteq A_{1+\frac{q(p-s')}{ps'}}(\mathbb{R}^n)$. By Lemma 2.3 we have $(\mu^{\frac{1}{p}})^{s'}\in A(\frac{p}{s'},\frac{r}{s'})$ and $(\mu^{\frac{1}{p}})^{s'}\in A(\frac{p}{s'},\frac{q}{s'})$. Making use of Lemma 4.1, 4.2, 4.3, we get

$$\begin{aligned}
&\|M_\beta(T_{\Omega,\alpha}f)\|_{L^r(\mu^{\frac{r}{p}})}\le C\|T_{\Omega,\alpha}f\|_{L^q(\mu^{\frac{q}{p}})},\\
&\|T_{\Omega,\alpha}f\|_{L^q(\mu^{\frac{q}{p}})}\le C\|f\|_{L^p(\mu)},\\
&\|M_{\alpha+\beta,s'}\|_{L^r(\mu^{\frac{r}{p}})}\le C\|f\|_{L^p(\mu)},
\end{aligned}$$

where C is a constant independent of f and $T_{\Omega,\alpha}$, and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $\frac{1}{r}=\frac{1}{p}-\frac{\alpha+\beta}{n}$. Thus by Lemma 4.6 and Lemma 4.11 we obtain

$$\begin{aligned}
&\big\|[b,T_{\Omega,\alpha}]f\big\|_{L^r(\mu^{1-(1-\frac{\alpha}{n})r})}\\
&\le\big\|M_\delta\big([b,T_{\Omega,\alpha}]f\big)\big\|_{L^r(\mu^{1-(1-\frac{\alpha}{n})r})}\\
&\le\big\|M_\delta^\sharp\big([b,T_{\Omega,\alpha}]f\big)\big\|_{L^r(\mu^{1-(1-\frac{\alpha}{n})r})}\\
&\le C\|b\|_{Lip_{\beta,\mu}}\Big(\big\|M_{\alpha+\beta,\mu,r_0}(f)\big\|_{L^r(\mu)}+\big\|M_\beta(T_{\Omega,\alpha}f)\big\|_{L^r(\mu^{\frac{r}{p}})}+\big\|M_{\alpha+\beta,s'}(f)\big\|_{L^r(\mu^{\frac{r}{p}})}\Big)\\
&\le C\|b\|_{Lip_{\beta,\mu}}\|f\|_{L^p(\mu)}.
\end{aligned}$$

Proof of Theorem 3.2. Since $s>\frac{pr}{(p-1)(r-p)}>\frac{r}{(r-p)}\frac{n}{(n-\alpha-\beta)}$, we get $\frac{s(r-p)}{s(r-p)-r}<p$. Thus there exist δ and r_0 such that $0<\delta\le\frac{1}{2}<\frac{s(r-p)}{s(r-p)-r}<r_0<p<\frac{n}{\alpha+\beta}$. By

Lemma 4.6 and Lemma 4.12, similar to the proof of Theorem 3.1, we get $[b, T_{\Omega,\alpha}]$ is bounded from $L^p(\mu)$ to $L^r(\mu^{1-(1-\frac{\alpha}{n})r})$.

References

1. S. Chanillo, A note on commutators, *Indiana Univ. Math. J.*, **31** (1982), 7-16.
2. S. Chanillo, D. Watson and R. L. Wheeden, Some integral and maximal operators related to star like sets, *Studia Math.*, **107** (1993), 223-255.
3. Y. Ding and S. Z. Lu, Weighted norm inequalities for fractional integral operators with rough kernel, *Can. J. Math.*, **50** (1998), 29-39.
4. Y. Ding and S. Z. Lu, Homogeneous fractional integrals on Hardy spaces, *Tohoku Math. J.*, **52** (2000), 153-162.
5. J. Duoandikoetxea, *Fourier analysis*, American Mathematical Society Providence, Rnode Isiana, USA, 1995.
6. J. Garcîa-Cuerva, Weighted H^p space, *Dissert. Math.*, **162** (1979).
7. J. García-Cuerva and J. L. Rubio de Francia, *Weighed norm inequalities and related topics*, North-Holland, Amsterdam, The Netherlands, 1985.
8. Y. S. Han, *Methods of modern harmonic analysis and their applications*, Science Press, Beijing, P. R. China, 1988.
9. B. Hu and J. J. Gu, Necessary and sufficient conditions for boundedness of some commutators with weighted Lipschitz functions, *J. Math. Anal. Appl.*, **340** (2008), 598-605.
10. B. Muckenhoupt and R. L. Wheeden, Weighted norm inequalities for fractional integrals, *Trans. Amer. Math. Soc.*, **192** (1974), 261-274.
11. M. Paluszyński, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, *Indiana Univ. Math. J.*, **44** (1995), 1-17.
12. A. Torchinsky, *Real-variable methods in harmonic analysis*, Academic Press, New York, USA, 1986.

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