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DERIVATIONS ON MATRIX ALGEBRAS WITH APPLICATIONS TO HARMONIC ANALYSIS

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Abstract. In this paper, the derivations between ideals of the Banach algebra $\mathfrak{E}_\infty(I)$ are characterized. Necessary and sufficient conditions for weak amenability of Banach algebras $\mathfrak{E}_p(I)$, $1 \leq p \leq \infty$, are found. Also, some applications to compact groups and hypergroups are given.

1. Introduction

The Banach algebras $\mathfrak{E}_p(I)$, where $p \in [1, \infty] \cup \{0\}$, were introduced and extensively studied in Section 28 of [5]. For a compact group G with dual $\widehat{G}$, the Banach algebras $\mathfrak{E}_p(\widehat{G})$, where $p \in [1, \infty] \cup \{0\}$, and multipliers on these Banach algebras were introduced and extensively studied in [5]. The present paper continues of the study of these algebras, and investigate multipliers and derivations on ideals of $\mathfrak{E}_\infty(I)$ with applications to compact groups and hypergroups.

The organization of this paper is as follows. The preliminaries and notations are given in section 1. Section 2 is devoted to derivations between ideals of $\mathfrak{E}_\infty(I)$. In this paper, the set of all $M \in \mathfrak{E}(I)$ such that $MA, AM \in \mathfrak{B}$ $(A \in \mathfrak{A})$, and $M_i = 0$ $(i \in I, d_i = 1)$ is denoted by $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$. It is shown that if $\mathfrak{A}$ and $\mathfrak{B}$ are ideals of $\mathfrak{E}_\infty(I)$, and $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$, and moreover there exist norms $\|.\|_{\mathfrak{A}}$ on $\mathfrak{A}$, and $\|.\|_{\mathfrak{B}}$ on $\mathfrak{B}$ such that with these norms $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras, then $\mathfrak{B}$ is a Banach $\mathfrak{A}$-bimodule with the product of $\mathfrak{E}(I)$ giving the two module multiplications. It is shown that if D is a derivation from $\mathfrak{A}$ into $\mathfrak{B}$, then D is continuous. Furthermore, if at least one of the spaces $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{B}$ is a dual Banach $\mathfrak{E}_\infty(I)$-bimodule, then there exists $M \in \mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ such that $D(A) = AM - MA$ $(A \in \mathfrak{A})$. In section 3, the Banach space $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are any of Banach spaces $\mathfrak{E}_p(I)$ $(1 \leq p \leq \infty)$, is formulated. Indeed, Theorem 35.4 of [5] is generalized from ideals of $\mathfrak{E}_\infty(\widehat{G})$, where G is a compact group with dual $\widehat{G}$, to ideals of $\mathfrak{E}_\infty(I)$. In section

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4 a number of results on derivations between Banach algebras of $\mathfrak{E}_p(I)$ $(1 \leq p \leq \infty)$ are stated and proved, and applied in investigating the weakly amenability of Banach algebras $\mathfrak{E}_p(I)$ $(1 \leq p \leq \infty)$. It is proved that $\mathcal{H}^1(\mathfrak{E}_\infty(I), \mathfrak{E}_p(I)) = 0$ for each $1 \leq p \leq \infty$. Also it is shown that for $1 \leq p, q \lneqq \infty$, $\mathcal{H}^1(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) = 0$ if and only if the set $\{i \in I : d_i \gneqq 1\}$ is finite. Moreover it is proved that for $1 \leq p \lneqq \infty$, $\mathcal{H}^1(\mathfrak{E}_p(I), \mathfrak{E}_\infty(I)) = 0$ if and only if $\sup\{a_i : i \in I,\ d_i \gneqq 1\} < \infty$. Applications of these results enables one to prove that for each $1 < p < \infty$, $\mathfrak{E}_p(I)$ is is weakly amenable if and only if the set $\{i \in I : d_i \gneqq 1\}$ is finite. Also $\mathfrak{E}_1(I)$ is weakly amenable if and only if $\sup\{a_i : i \in I,\ d_i \gneqq 1\} < \infty$. However it is well-known that $\mathfrak{E}_\infty(I)$ is weakly amenable. In section 5 some applications of the previous sections in compact groups and hypergroups are given. Among other results, it is proved that if G is a compact group, then the convolution Banach algebra $A(G)$ is weakly amenable if and only if $\sup_{\pi \in \widehat{G}} d_\pi < \infty$, where $\widehat{G}$ is the dual of G and for each $\pi \in \widehat{G}$, $d_\pi = \textit{dim}\ \pi$. Also, a necessary and sufficient condition for weak amenability of the convolution Banach algebra $A(K)$, for a compact hypergroup K, is proved.

2. Preliminaries

Let H be an n-dimensional Hilbert space and suppose that $\mathcal{B}(H)$ be the space of all linear operators on H. Clearly $\mathcal{B}(H)$ can be identified with $\mathbb{M}_n(\mathbb{C})$ (the space of all $n \times n$-matrices on $\mathbb{C}$) as vector spaces. For $A \in \mathbb{M}_n(\mathbb{C})$, let $A^* \in \mathbb{M}_n(\mathbb{C})$ by $(A^*)_{ij} = \overline{A_{ji}}$ $(1 \leq i, j \leq n)$, and let $|A|$ denote the unique positive-definite square root of AA^*. A is called *unitary* if $A^*A = AA^* = I$, where I is the $n \times n$-identity matrix. For $E \in \mathcal{B}(H)$, let $(\lambda_1, \ldots, \lambda_n)$ be the sequence of eigenvalues of operator $|E|$, written in any order. Define $\|E\|_{\varphi_\infty} = \max_{1 \leq i \leq n} |\lambda_i|$, and $\|E\|_{\varphi_p} = (\sum_{i=1}^n |\lambda_i|^p)^{\frac{1}{p}}$ $(1 \leq p < \infty)$. For more details see Definition D.37 and Theorem D.40 of [5].

Let I be an arbitrary index set. For each $i \in I$, let H_i be a finite dimensional Hilbert space of dimension d_i, and let a_i be a real number ≥ 1. These notations will remain in place throughout the paper. The $*$-algebra $\prod_{i \in I} B(H_i)$ will denoted by $\mathfrak{E}(I)$; scalar multiplication, addition, multiplication, and the adjoint of an element are defined coordinate-wise. Let $E = (E_i)$ be an element of $\mathfrak{E}(I)$. Define $\|E\|_p := \left(\sum_{i \in I} a_i \|E_i\|_{\varphi_p}^p\right)^{\frac{1}{p}}$ $(1 \leq p < \infty)$, and $\|E\|_\infty = \sup_{i \in I} \|E_i\|_{\varphi_\infty}$. For $1 \leq p \leq \infty$, $\mathfrak{E}_p(I)$ is defined as the set of all $E \in \mathfrak{E}(I)$ for which $\|E\|_p < \infty$, and $\mathfrak{E}_0(I)$ is defined as the set of all $E \in \mathfrak{E}(I)$ such that $\{i \in I : \|E_i\|_{\varphi_\infty} \geq \epsilon\}$ is finite for all $\epsilon > 0$. The set of all $E \in \mathfrak{E}(I)$ such that $\{i \in I : \|E_i\|_{\varphi_\infty} \neq 0\}$ is finite is denoted by $\mathfrak{E}_{00}(I)$. By Theorems 28.25, 28.27, and 28.32(v) of [5], both $(\mathfrak{E}_p(I), \|.\|_p)$ $(1 \leq p \leq \infty)$, and $(\mathfrak{E}_0(I), \|.\|_\infty)$ are Banach algebras.

For a Banach algebra A, an A-bimodule will always refer to a *Banach A-bimodule* X, that is a Banach space which is algebraically an A-bimodule, and for which there is a constant $C_{A,X} \geq 0$ such that

$$\|a.x\|_X, \|x.a\|_X \leq C_{A,X}\|a\|_A\|x\|_X \quad (a \in A, x \in X).$$

A linear map $D : A \to X$ is called an X-*derivation*, if

$$D(ab) = D(a).b + a.D(b) \quad (a, b \in A).$$

For every $x \in X$, ad_x is defined by $ad_x(a) = a.x - x.a$ $(a \in A)$. It is easily seen that ad_x is a derivation. Derivations of this form are called *inner derivations*. The set of all derivations from A into X is denoted by $Z^1(A, X)$, and the set of all inner X-derivations is denoted by $B^1(A, X)$. Clearly, $Z^1(A, X)$ is a linear subspace of the space of all linear operators of A into X and $B^1(A, X)$ is a linear subspace of $Z^1(A, X)$. The difference space of $Z^1(A, X)$ modulo $B^1(A, X)$ is denote by $H^1(A, X)$. The set of all continuous derivations from A into X is denoted by $\mathcal{Z}^1(A, X)$, and the set of all (continuous) X-derivations is denoted by $\mathcal{B}^1(A, X)$. Clearly, $\mathcal{Z}^1(A, X)$ is a linear subspace of the space of all bounded linear operators of A into X and $\mathcal{B}^1(A, X)$ is a linear subspace of $\mathcal{Z}^1(A, X)$. Let $\mathcal{H}^1(A, X)$ be the difference space of $\mathcal{Z}^1(A, X)$ modulo $\mathcal{B}^1(A, X)$.

The Banach space A^* with the *dual* module multiplications defined by

$$(f.a)(b) = f(ab), (a.f)(b) = f(ba) \quad (a, b \in A, f \in A^*),$$

is a Banach A-bimodule called the *dual* Banach A-bimodule A^*. A Banach algebra A is called *weakly amenable* if $\mathcal{H}^1(A, A^*) = 0$.

For a locally compact group G and a function $f : G \to \mathbb{C}$, $\check{f}$ is defined by $\check{f}(x) = f(x^{-1})$ $(x \in G)$. Let $A(G)$ (or with the notation $\mathfrak{K}(G)$ defined in 35.16 of [5]) consist of all functions h in $C_0(G)$ that can be written in at least one way as $\sum_{n=1}^{\infty} f_n * \check{g}_n$, where $f_n, g_n \in L^2(G)$, and $\sum_{n=1}^{\infty} \|f_n\|_2\|g_n\|_2 < \infty$. For $h \in A(G)$, define

$$\|h\|_{A(G)} = \inf\left\{\sum_{n=1}^{\infty} \|f_n\|_2\|g_n\|_2 : h = \sum_{n=1}^{\infty} f_n * \check{g}_n\right\}.$$

With this norm $A(G)$ is a Banach space. For more details see 35.16 of [5]. In the case where G is a compact group, $A(G)$ with convolution and the norm $\|.\|_{A(G)}$ is a Banach algebra (see 34.35 of [5]).

Throughout this paper K is a compact hypergroup as defined by Jewett ([6]). By Theorem 1.3.28 of [1], K admits a left Haar measure. Throughout the present paper the normalized Haar measure ω_K on the compact hypergroup K (i.e. $\omega_K(K) = 1$) is used. If $\pi \in \widehat{K}$, (where $\widehat{K}$ is the set of equivalence classes of continuous irreducible representations of K, c.f. [1], 11.3 of [6], and [10]), then by Theorem 2.2 of [10], π is finite dimensional. Furthermore by the proof of Theorem 2.2 of [10], there exists a constant c_π such that for each $\xi \in H_\pi$ with $\|\xi\| = 1$

$$\int_K |\langle \pi(x)\xi, \xi\rangle|^2 \, d\omega_K(x) = c_\pi.$$

Let $k_\pi = c_\pi^{-1}$. By Theorem 2.6 of [10], $k_\pi \geq d_\pi$. Moreover if K is a group then $k_\pi = d_\pi$. For each $\pi \in \widehat{K}$, let H_π be the representation space of π and $d_\pi = \dim\ H_\pi$. The algebras $\mathfrak{C}(\widehat{K})$ and $\mathfrak{C}_p(\widehat{K})$ for $p \in [1, \infty] \cup \{0\}$, are defined as above with each $a_\pi = k_\pi$. Let $\mu \in M(K)$. The *Fourier transform* of μ at $\pi \in \widehat{K}$ is denoted by $\widehat{\mu}(\pi)$ and defined as the operator $\widehat{\mu}(\pi) = \int_K \pi(\bar{x}) \, d\mu(x)$ on H_π. Define $\widehat{\mu} \in \mathfrak{C}(\widehat{K})$ by $\widehat{\mu}_\pi = \widehat{\mu}(\pi) \in \mathcal{B}(H_\pi)$ (for more details see Theorem 3.2 of [10]). If $f \in L^1(K)$, and $\sum_{\pi \in \widehat{K}} k_\pi \|\widehat{f}(\pi)\|_{\varphi_1} < \infty$, then f is said to have an *absolutely convergent Fourier series*. The set of all functions with absolutely convergent Fourier series is denoted by $A(K)$ and called *the Fourier space* of K. For $f \in A(K)$, define $\|f\|_{A(K)} = \|\widehat{f}\|_1$. By Proposition 4.2 of [10], $A(K)$ with the convolution product is a Banach algebra and isometrically isomorphic with $\mathfrak{C}_1(\widehat{K})$. Note that the two definitions of $A(G)$ and $A(K)$ agree when $K = G$.

3. Derivations Between Ideals of $\mathfrak{C}_\infty(I)$

Throughout the paper for $A \in B(H_i)$, define A^i as an element of $\mathfrak{C}(I)$ given by

$$\left(A^i\right)_j = \begin{cases} A & \text{for } j = i \\ 0 & \text{otherwise.} \end{cases}$$

We denote the identity $d_i \times d_i$-matrix (i.e. the identity operator in $\mathcal{B}(H_i)$) by I_i.

Proposition 3.1. *Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{C}(I)$ such that $\mathfrak{C}_{00}(I) \subseteq \mathfrak{A}$, and $\mathfrak{B}$ be a subspace of $\mathfrak{C}(I)$. Suppose that $(\mathfrak{A}, \|.\|_{\mathfrak{A}})$ is a Banach algebra and $(\mathfrak{B}, \|.\|_{\mathfrak{B}})$ is a Banach space. Then each linear mapping $\Theta : \mathfrak{A} \to \mathfrak{B}$ that satisfies*

$$\Theta(AI_i^i) = \Theta(A)I_i^i \quad (A \in \mathfrak{A}, i \in I),$$

is continuous.

Proof. Let (A_n) be a sequence in $\mathfrak{A}$ such that $\|A_n\|_{\mathfrak{A}} \to 0$ and $\|\Theta(A_n) - B\|_{\mathfrak{B}} \to 0$, where $B \in \mathfrak{B}$. Let $i \in I$. Since $\mathcal{B}(H_i)$ is finite dimensional, so by Lemma 1.20 of [8] the linear mapping $\Theta_i : \mathcal{B}(H_i) \to \mathfrak{B} : A_i \mapsto \Theta(A_i^i)$ is continuous. On the other hand since $\mathfrak{A}$ is a Banach algebra, so for each $i \in I$

$$\|A_n I_i^i\|_{\mathfrak{A}} \leq \|A_n\|_{\mathfrak{A}} \|I_i^i\|_{\mathfrak{A}} \longrightarrow 0.$$

Therefore for each $i \in I$

$$\begin{aligned} BI_i^i &= \lim_{n\longrightarrow\infty} \Theta(A_n)I_i^i = \lim_{n\longrightarrow\infty} \Theta(A_nI_i^i) \\ &= \lim_{n\longrightarrow\infty} \Theta_i\left((A_n)_i\right) = \Theta_i\left(\lim_{n\longrightarrow\infty} A_nI_i^i\right) \\ &= \Theta_i(0) = 0. \end{aligned}$$

Hence $B = 0$. By the Closed Graph Theorem Θ is continuous. ∎

Corollary 3.2. *Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{E}(I)$ such that $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$, and $\mathfrak{B}$ be a subspace of $\mathfrak{E}(I)$. Suppose that $(\mathfrak{A}, \|.\|_{\mathfrak{A}})$ is a Banach algebra and $(\mathfrak{B}, \|.\|_{\mathfrak{B}})$ is a Banach $\mathfrak{A}$-bimodule. Then $Z^1(\mathfrak{A}, \mathfrak{B}) = \mathcal{Z}(\mathfrak{A}, \mathfrak{B})$. That is each derivation D from $\mathfrak{A}$ into $\mathfrak{B}$ is continuous.*

Proof. Let $i \in I$. By Proposition 1.8.2 of [3], $D(I_i^i) = 0$. Hence for each $A \in \mathfrak{A}$

$$D(AI_i^i) = D(A)I_i^i + AD(I_i^i) = D(A)I_i^i.$$

So by Proposition 3.1, D is continuous. ∎

Example 3.3. Let I be an infinite set. Fix $i_0 \in I$, and suppose that $\{i_n : n \in \mathbb{N}\}$ be an infinite countable subset of distinct elements of $I \setminus \{i_0\}$. Moreover suppose that for each $n \in \mathbb{N}$, *dim*$(H_{i_n}) \geq 2$. Define

$$\mathfrak{A} = \left\{A \in \mathfrak{E}_0(I) : A_{i_n} \in \mathbb{C}\mathcal{E}_{12}^{i_n} \textit{ for } n \in \mathbb{N}, \textit{ and } A_i = 0 \textit{ for all other } i\textit{'s}\right\},$$

with the norm $\|A\|_{\mathfrak{A}} = \|A\|_\infty$ $(A \in \mathfrak{A})$. Then $\mathfrak{A}$ is a Banach subalgebra of $\mathfrak{E}_\infty(I)$. Clearly $\{\mathcal{E}_{12}^{i_n} : n \in \mathbb{N}\}$ is a linearly independent subspace of the vector space $\mathfrak{A}$. Let $\mathcal{B}$ be a basis for $\mathfrak{A}$ such that $\{\mathcal{E}_{12}^{i_n} : n \in \mathbb{N}\} \subseteq \mathfrak{A}$. Let $D : \mathfrak{A} \to \mathfrak{A}$ be the linear mapping given by $D(\mathcal{E}_{12}^{i_n}) = n\mathcal{E}_{11}^{i_0}$, where $n \in \mathbb{N}$, and $D(E) = 0$, where $E \in \mathcal{B} \setminus \{\mathcal{E}_{12}^{i_n} : n \in \mathbb{N}\}$. Let $A, B \in \mathfrak{A}$. Then $AB = 0$, and so $D(AB) = 0$. Clearly $D(A)B = AD(B) = 0$ for each $A, B \in \mathfrak{A}$. Hence D is a derivation from $\mathfrak{A}$ into $\mathfrak{A}$. Clearly D is not continuous (indeed, for each $n \in \mathbb{N}$, $\|D\| \geq \left\|D(\mathcal{E}_{12}^{i_n})\right\|_\infty = n$). So the condition $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$, can not be omitted in Proposition 3.2.

Definition 3.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be subsets of $\mathfrak{E}(I)$. An element E in $\mathfrak{E}(I)$ is said to be a left (right, respectively) $(\mathfrak{A}, \mathfrak{B})$-*multiplier* if $EA \in \mathfrak{B}$ ($AE \in \mathfrak{B}$, respectively) for all $A \in \mathfrak{A}$. The set of all left (right, respectively) $(\mathfrak{A}, \mathfrak{B})$-multipliers will be denoted by $\mathcal{M}(\mathfrak{A}, \mathfrak{B})$ ($\mathcal{RM}(\mathfrak{A}, \mathfrak{B})$, respectively). The set of all $E \in \mathcal{M}(\mathfrak{A}, \mathfrak{B}) \cap \mathcal{RM}(\mathfrak{A}, \mathfrak{B})$ such that $E_i = 0$ whenever $d_i = 1$, will be denoted by $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$.

Lemma 3.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ideals of $\mathfrak{E}_\infty(I)$. Then $\mathfrak{B}$ is an algebraic $\mathfrak{A}$-bimodule with the product of $\mathfrak{E}(I)$ giving the two module multiplications. Also $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ is a $\mathfrak{E}_\infty(I)$-bimodule.*

Proof. Clearly $\mathfrak{B}$ is an algebraic $\mathfrak{A}$-bimodule, and $\mathcal{M}_1(\mathfrak{A},\mathfrak{B})$ is a subspace of $\mathfrak{E}(I)$. Let $L \in \mathcal{M}_1(\mathfrak{A},\mathfrak{B})$ and $E \in \mathfrak{E}_\infty(I)$. Since $\mathfrak{B}$ is an ideal of $\mathfrak{E}_\infty(I)$, so if $A \in \mathfrak{A}$, then $(EL)A = E(LA) \in \mathfrak{B}$. Hence $EL \in \mathcal{M}_1(\mathfrak{A},\mathfrak{B})$. Similarly since $\mathfrak{A}$ is an ideal of $\mathfrak{E}_\infty(I)$, so $LE \in \mathcal{M}_1(\mathfrak{A},\mathfrak{B})$. Therefore $\mathcal{M}_1(\mathfrak{A},\mathfrak{B})$ is a $\mathfrak{E}_\infty(I)$-bimodule. ∎

Proposition 3.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ideals of $\mathfrak{E}_\infty(I)$, and $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$. Then $\mathfrak{B}$ is an algebraic $\mathfrak{A}$-bimodule with the product of $\mathfrak{E}(I)$ giving the two module multiplications. Moreover, if D is a derivation from $\mathfrak{A}$ into $\mathfrak{B}$, then there exists a derivation $\widetilde{D}$ from $\mathfrak{E}_\infty(I)$ into $\mathcal{M}_1(\mathfrak{A},\mathfrak{B})$ such that $\widetilde{D}(A) = D(A)$ $(A \in \mathfrak{A})$.*

Proof. Suppose D is a derivation from $\mathfrak{A}$ into $\mathfrak{B}$. By Corollary 3.2 D is continuous. By Lemma 3.5, $\mathcal{M}_1(\mathfrak{A},\mathfrak{B})$ is a $\mathfrak{E}_\infty(I)$-bimodule.

Define $\widetilde{D} : \mathfrak{E}_\infty(I) \to \mathcal{M}_1(\mathfrak{A},\mathfrak{B})$ by

$$\left(\widetilde{D}(E)\right)_i = \left(D(EI_i^i)\right)_i \quad (E \in \mathfrak{E}_\infty(I), i \in I).$$

$\widetilde{D}$ is a well-defined continuous derivation. To see this, let $E \in \mathfrak{E}_{00}(I)$. Since $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$, so $EI_i^i \in \mathfrak{A}$ for each $i \in I$. Hence $D(EI_i^i)$ is well-defined. Let $A \in \mathfrak{A}$, and $i \in I$ be such that $d_i \gneqq 1$. Since $EA \in \mathfrak{A}$, so

$$\begin{aligned}\left(\widetilde{D}(E)A\right)_i &= \left(D(EI_i^i)A\right)_i = \left(D(EI_i^iA) - EI_i^iD(A)\right)_i \\ &= \left(D(EA)I_i^i - EI_i^iD(A)\right)_i = \left(D(EA) - ED(A)\right)_i.\end{aligned}$$

Also if $i \in I$, and $d_i = 1$, then $AI_i^i = A_iI_i^i$, and $EI_i^i = E_iI_i^i$, where $A_i, E_i \in \mathbb{C}$. Hence

$$\begin{aligned}(D(EA) - ED(A))I_i^i &= D(EA)I_i^i - E(D(A)I_i^i) = D(EAI_i^i) - ED(AI_i^i) \\ &= E_iA_iD(I_i^i) - EA_iD(I_i^i) = 0,\end{aligned}$$

and

$$(D(E)A)I_i^i = D(E)(A_iI_i^i) = A_i(D(E)I_i^i) = A_iD(EI_i^i) = A_iE_iD(I_i^i) = 0.$$

The above equations show that $\widetilde{D}(E)A = D(EA) - ED(A)$. But, $\mathfrak{B}$ is an ideal of $\mathfrak{E}_\infty(I)$, and so $\widetilde{D}(E)A = D(EA) - ED(A) \in \mathfrak{B}$. Therefore $\widetilde{D}(E) \in \mathcal{M}(\mathfrak{A},\mathfrak{B})$. Similarly one can prove that $A\widetilde{D}(E) = D(AE) - D(A)E \in \mathfrak{B}$, and so $\widetilde{D}(E) \in \mathcal{RM}(\mathfrak{A},\mathfrak{B})$. Hence by definition of $\mathcal{M}_1(\mathfrak{A},\mathfrak{B})$, $\widetilde{D}(E) \in \mathcal{M}_1(\mathfrak{A},\mathfrak{B})$.

Now, if $E, F \in \mathfrak{E}_\infty(I)$, and $i \in I$, then

$$\begin{aligned}\left(\widetilde{D}(EF)\right)_i &= \left(D\left((EF)I_i^i\right)\right)_i = \left(D\left((EI_i^i)(FI_i^i)\right)\right)_i \\ &= \left(D(EI_i^i)FI_i^i + EI_i^iD(FI_i^i)\right)_i = \left(D(EI_i^i)\right)_i F_i + E_i\left(D(FI_i^i)\right)_i \\ &= \left(\widetilde{D}(E)F + E\widetilde{D}(F)\right)_i.\end{aligned}$$

Hence $\widetilde{D}$ is a derivation. It is clear that if $A \in \mathfrak{A}$, then $\widetilde{D}(A) = D(A)$. ∎

Proposition 3.7. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be ideals of* $\mathfrak{E}_\infty(I)$*, and* $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$*. Suppose that there exist a norm* $\|.\|_{\mathfrak{A}}$ *on* $\mathfrak{A}$*, and a norm* $\|.\|_{\mathfrak{B}}$ *on* $\mathfrak{B}$ *such that with these norms* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are Banach* $\mathfrak{E}_\infty(I)$*-bimodules. Then* $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ *is a Banach* $\mathfrak{E}_\infty(I)$*-bimodule with the product of* $\mathfrak{E}(I)$ *giving the two module multiplications, and with the norm*

$$\|L\|_{\mathfrak{A},\mathfrak{B}} = \sup_{A\in\mathfrak{A}, \|A\|_{\mathfrak{A}}=1} (\|LA\|_{\mathfrak{B}} + \|AL\|_{\mathfrak{B}}) \quad (L \in \mathcal{M}_1(\mathfrak{A}, \mathfrak{B})).$$

Proof. Firstly, it is proved that $\|.\|_{\mathfrak{A},\mathfrak{B}}$ is a well defined norm on $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$. It is easy to see that $\mathfrak{B} \times \mathfrak{B}$ is a Banach space under the norm

$$\|(b_1, b_2)\|_{\mathfrak{B}\times\mathfrak{B}} = \|b_1\|_{\mathfrak{B}} + \|b_2\|_{\mathfrak{B}} \quad ((b_1, b_2) \in \mathfrak{B} \times \mathfrak{B}).$$

For $M \in \mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$, define $\widehat{M} : \mathfrak{A} \to \mathfrak{B} \times \mathfrak{B}$ by $\widehat{M}(A) = (MA, AM)$ $(A \in \mathfrak{A})$. By definition $\|\widehat{M}\| = \|M\|_{\mathfrak{A},\mathfrak{B}}$. But, by Proposition 3.1, the mappings $A \mapsto MA, AM : \mathfrak{A} \to \mathfrak{B}$ are continuous, and so $\|M\|_{\mathfrak{A},\mathfrak{B}} < \infty$. Let $\|M\|_{\mathfrak{A},\mathfrak{B}} = 0$. Then $\|MI_i^i\|_{\mathfrak{B}} \le \|M\|_{\mathfrak{A},\mathfrak{B}}\|I_i^i\|_{\mathfrak{A}} = 0$ (note that $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$). It follows that $MI_i^i = 0$ for each $i \in I$, and so $M = 0$. Therefore $\|.\|_{\mathfrak{A},\mathfrak{B}}$ is a norm on $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$.

Suppose that $(M_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$. By completeness of $\mathcal{B}(\mathfrak{A}, \mathfrak{B} \times \mathfrak{B})$ (the set of all continuous linear maps from $\mathfrak{A}$ into $\mathfrak{B} \times \mathfrak{B}$), there exists $\Theta \in \mathcal{B}(\mathfrak{A}, \mathfrak{B} \times \mathfrak{B})$ such that $\lim_{n\to\infty} \widehat{M_n} = \Theta$. Let $\pi_1, \pi_2 : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ be the natural projections $\pi_1 : (b_1, b_2) \mapsto b_1$, $\pi_2 : (b_1, b_2) \mapsto b_2$. Define $M \in \mathfrak{E}(I)$ by $MI_i^i = \pi_1(\Theta(I_i^i))I_i^i$. Then for $A \in \mathfrak{A}$

$$\begin{aligned}(MA)I_i^i &= MI_i^iAI_i^i = \pi_1\left(\Theta(I_i^i)\right)AI_i^i = \lim_{n\longrightarrow\infty} \pi_1\left(\widehat{M_n}(I_i^i)\right)AI_i^i \\ &= \lim_{n\longrightarrow\infty}(M_nI_i^i)AI_i^i = \lim_{n\longrightarrow\infty}(M_nA)I_i^i \\ &= \lim_{n\longrightarrow\infty} \pi_1\left(\widehat{M_n}(A)\right)I_i^i = \pi_1\left(\Theta(A)\right)I_i^i.\end{aligned}$$

But

$$\begin{aligned}MI_i^i &= \pi_1(\Theta(I_i^i))I_i^i = \pi_1\left(\widehat{M_n}(I_i^i)\right)I_i^i \\ &= \pi_1\left(M_nI_i^i, I_i^iM_n\right)I_i^i = \pi_2\left(M_nI_i^i, I_i^iM_n\right)I_i^i \\ &= \pi_2\left(\widehat{M_n}(I_i^i)\right)I_i^i = \pi_2(\Theta(I_i^i))I_i^i,\end{aligned}$$

and so by a similar method it can be proved that $(AM)I_i^i = \pi_2\left(\Theta(A)\right)I_i^i$. It follows that $\Theta = \widehat{M}$, and $M \in \mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$. Therefore $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ is a Banach space.

Let $L \in \mathcal{M}_1(\mathfrak{A},\mathfrak{B})$ and $E \in \mathfrak{E}_\infty(I)$. Since $\mathfrak{B}$ is an ideal of $\mathfrak{E}_\infty(I)$, so if $A \in \mathfrak{A}$, then $(EL)A = E(LA) \in \mathfrak{B}$. Similarly since $\mathfrak{A}$ is an ideal of $\mathfrak{E}_\infty(I)$, so $A(EL) = (AE)L \in \mathfrak{B}$. Clearly if $d_i = 1$, then $(LE)_i = 0$. Therefore $LE \in \mathcal{M}_1(\mathfrak{A},\mathfrak{B})$. Similarly $EL \in \mathcal{M}_1(\mathfrak{A},\mathfrak{B})$. Now,

$$\begin{aligned}\|EL\|_{\mathfrak{A},\mathfrak{B}} &= \sup_{\|A\|_{\mathfrak{A}}=1}(\|(EL)A\|_{\mathfrak{B}} + \|A(EL)\|_{\mathfrak{B}})\\ &\le \sup_{\|A\|_{\mathfrak{A}}=1}\|E(LA)\|_{\mathfrak{B}} + \sup_{\|A\|_{\mathfrak{A}}=1}\|(AE)L\|_{\mathfrak{B}}\\ &\le C_{\mathfrak{E}_\infty(I),\mathfrak{B}}\|E\|_\infty \sup_{\|A\|_{\mathfrak{A}}=1}\|LA\|_{\mathfrak{B}} + \|L\|_{\mathfrak{A},\mathfrak{B}}\sup_{\|A\|_{\mathfrak{A}}=1}\|AE\|_{\mathfrak{A}}\\ &\le C_{\mathfrak{E}_\infty(I),\mathfrak{B}}\|E\|_\infty \sup_{\|A\|_{\mathfrak{A}}=1}\|LA\|_{\mathfrak{B}} + C_{\mathfrak{E}_\infty(I),\mathfrak{A}}\|L\|_{\mathfrak{A},\mathfrak{B}}\|E\|_\infty\\ &\le \max\left(C_{\mathfrak{E}_\infty(I),\mathfrak{A}}, C_{\mathfrak{E}_\infty(I),\mathfrak{B}}\right)\|E\|_\infty\|L\|_{\mathfrak{A},\mathfrak{B}}.\end{aligned}$$

Similarly

$$\|LE\|_{\mathfrak{A},\mathfrak{B}} \le \max\left(C_{\mathfrak{E}_\infty(I),\mathfrak{A}}, C_{\mathfrak{E}_\infty(I),\mathfrak{B}}\right)\|E\|_\infty\|L\|_{\mathfrak{A},\mathfrak{B}}.$$

Hence $\mathcal{M}_1(\mathfrak{A},\mathfrak{B})$ is a Banach $\mathfrak{E}_\infty(I)$-bimodule. ■

Lemma 3.8. *Let I be a finite set, and X be a Banach $\mathfrak{E}_\infty(I)$-bimodule. If $D : \mathfrak{E}_\infty(I) \to X$ is a derivation, then there exists $x \in X$ such that $\|x\|_X \le \|D\|$, and*

$$D(A) = A.x - x.A \quad (A \in \mathfrak{E}_\infty(I)).$$

Proof. Clearly $\mathfrak{E}_\infty(I)$ can be identified with $\ell^\infty - \bigoplus_{i\in I}\mathbb{M}_{d_i}(\mathbb{C})$. Let G be the set of all elements E of $\ell^\infty - \bigoplus_{i\in I}\mathbb{M}_{d_i}(\mathbb{C})$ such that $(E_i)_{kl} \in \{-1,0,1\}$ $(i \in I, 1 \le k,l \le d_i)$ and each column and each row of E_i $(i \in I)$ contains exactly one non-zero term. By a similar method as the proof of Proposition 1.9.20, it is proved that $\frac{1}{card(G)}\sum_{E\in G} E\otimes E^{-1}$ whenever $(E^{-1})_i = E_i^{-1}$ $(i \in I)$, is a diagonal for $\ell^\infty - \bigoplus_{i\in I}\mathbb{M}_{d_i}(\mathbb{C})$, and so if

$$x = \frac{1}{card(G)}\sum_{E\in G} E.D(E^{-1}),$$

then $D = ad_x$ (see the proof of Theorem 1.9.21((b)⇒(a)) of [3], or the proof of Theorem 2.2.4((ii)⇒(i)) of [9]). Clearly for each $E \in G$, $\|E\|_{\varphi_\infty} = \|E^{-1}\|_{\varphi_\infty} = 1$. Hence

$$\begin{aligned}\|x\|_X &= \left\|\frac{1}{card(G)}\sum_{E\in G} E.D(E^{-1})\right\|_X \le \frac{1}{card(G)}\sum_{E\in G}\|E.D(E^{-1})\|_X\\ &\le \frac{1}{card(G)}\sum_{E\in G}\|E\|_{\varphi_\infty}\|D\|\|E^{-1}\|_{\varphi_\infty} = \|D\|.\end{aligned}$$

■

Theorem 3.9. *Let $\mathfrak{A}$ be a subspace of $\mathfrak{E}(I)$, and there exists a norm $\|.\|_{\mathfrak{A}}$ such that with this norm $\mathfrak{A}$ is a dual Banach $\mathfrak{E}_{\infty}(I)$-bimodule. Then $Z^1(\mathfrak{E}_\infty(I), \mathfrak{A}) = \mathcal{Z}^1(\mathfrak{E}_\infty(I), \mathfrak{A}) = 0$. I. e. each derivation D from $\mathfrak{E}_\infty(I)$ into $\mathfrak{A}$ is continuous and inner.*

Proof. Let D be a derivation from $\mathfrak{E}_\infty(I)$ into $\mathfrak{A}$. By Corollary 3.2, D is continuous. For each finite subset F of I, let

$$\mathfrak{E}_\infty^F(I) = \{E \in \mathfrak{E}_\infty(I) : E_i = 0 \ (i \notin F)\},$$

and define $D_F : \mathfrak{E}_\infty^F(I) \to \mathfrak{A}$ by $D_F(A) = D(A) \ (A \in \mathfrak{E}_\infty^F(I))$. By Lemma 3.8, there exists $E_F \in \mathfrak{A}$ such that $\|E_F\|_{\mathfrak{A}} \leq \|D_F\| \leq \|D\|$, and $D(A) = AE_F - E_F A$ $(A \in \mathfrak{E}_\infty^F(I))$. Since $\mathfrak{A}$ is a dual Banach space, by Banach-Alaoglue's Theorem there exist $E \in \mathfrak{A}$, and a subnet $(E_{F_\alpha})_\alpha$ of $(E_F)_F$ such that weak*-$\lim_\alpha E_{F_\alpha} = E$. Let $\mathfrak{A}_*$ be a predual of $\mathfrak{A}$ (i.e. $\mathfrak{A}_*^* = \mathfrak{A}$). For each $A \in \mathfrak{E}_\infty(I)$, $i \in I$, and $x \in \mathfrak{A}_*$

$$\begin{aligned}\langle x, (AE - EA)I_i^i\rangle &= \langle x.AI_i^i - AI_i^i.x, E\rangle \\ &= \lim_{\alpha, i\in F_\alpha} \langle x.AI_i^i - AI_i^i.x, E_{F_\alpha}\rangle \\ &= \lim_{\alpha, i\in F_\alpha} \langle x, (AI_i^i.E_{F_\alpha} - E_{F_\alpha}.AI_i^i)\rangle \\ &= \lim_{\alpha, i\in F_\alpha} \langle x, D(AI_i^i)\rangle = \langle x, D(A)I_i^i\rangle.\end{aligned}$$

Hence $D(A) = AE - EA$, and so D is inner. ∎

The following is the main theorem of this paper.

Theorem 3.10. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ideals of $\mathfrak{E}_\infty(I)$, and $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$. Suppose that there exist norms $\|.\|_{\mathfrak{A}}$ on $\mathfrak{A}$, and $\|.\|_{\mathfrak{B}}$ on $\mathfrak{B}$ such that with these norms $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras. Suppose one of the following statements are valid:*

(i) *$\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ is a dual Banach $\mathfrak{E}_\infty(I)$-bimodule,*
(ii) *$\mathfrak{B}$ is a dual Banach $\mathfrak{E}_\infty(I)$-bimodule.*

If D is a derivation from $\mathfrak{A}$ into $\mathfrak{B}$, then D is continuous and there exists $M \in \mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ such that $D(A) = AM - MA \ (A \in \mathfrak{A})$.

Proof. By Proposition 3.6, there exists a derivation $\widetilde{D}$ from $\mathfrak{E}_\infty(I)$ into $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ such that $\widetilde{D}(A) = D(A) \ (A \in \mathfrak{A})$.

Suppose (i) is valid. By Theorem 3.9, $\widetilde{D}$ is inner. Hence there exists $M \in \mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ such that $D(A) = AM - MA \ (A \in \mathfrak{A})$.

Now, suppose that (ii) is valid. By the proof of Theorem 3.9, for each finite subset F of I, there exists $M_F \in \mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ such that $\widetilde{D}(A) = AM_F - M_F A$ $(A \in \mathfrak{E}_\infty^F(I))$. Let M be a cluster point of (M_F) in the weak*-operator topology (note that since $\mathfrak{B}$ is a dual Banach space, so the weak*-operator topology is well-defined, see also Remark 3.4 of [4]). Then by a method as the proof of the Theorem 3.9, $\widetilde{D}(A) = AM - MA$ $(A \in \mathfrak{E}_\infty(I))$. Hence $D(A) = AM - MA$ $(A \in \mathfrak{A})$. ■

From the above theorem, one can obtain the following result.

Proposition 3.11. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ideals of $\mathfrak{E}_\infty(I)$, and $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$. Suppose that there exist norms $\|.\|_{\mathfrak{A}}$ on $\mathfrak{A}$, and $\|.\|_{\mathfrak{B}}$ on $\mathfrak{B}$ such that with these norms $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras. Then $\mathfrak{B}$ is a Banach $\mathfrak{A}$-bimodule with the product of $\mathfrak{E}(I)$ giving the two module multiplications. Moreover if at least one of the spaces $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{B}$ is a dual Banach $\mathfrak{E}_\infty(I)$-bimodule, then*

$$Z^1(\mathfrak{A}, \mathfrak{B}) = \mathcal{Z}^1(\mathfrak{A}, \mathfrak{B}) = \{D_E : E \in \mathcal{M}_1(\mathfrak{A}, \mathfrak{B}\},$$

where $D_E(A) = AE - EA$ $(A \in \mathfrak{A})$.

The following elementary result is needed.

Lemma 3.12. *Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{E}(I)$ such that $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$. If $E \in \mathfrak{E}(I)$ is such that for each $A \in \mathfrak{A}$, $AE = EA$, then there exists a set $\{\lambda_i : i \in I\} \subseteqq \mathbb{C}$ such that for each $i \in I$, $E_i = \lambda_i I_i$.*

Proof. Let $i \in I$. For each $d_i \times d_i$-matrix A,

$$AE_i = \left(A^i E\right)_i = \left(EA^i\right)_i = E_i A,$$

and hence by Corollary 27.10 of [5], there exists $\lambda_i \in \mathbb{C}$ such that $E_i = \lambda_i I_i$. ■

Notation. Throughout the paper the set of all $E \in \mathfrak{E}(I)$ such that $E_i = \lambda_i I_i$ $(i \in I)$, for a set $\{\lambda_i : i \in I\} \subseteqq \mathbb{C}$, is denoted by $C(\mathfrak{E}(I))$.

Proposition 3.13. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ideals of $\mathfrak{E}_\infty(I)$, and $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$. Suppose that there exist a norm $\|.\|_{\mathfrak{A}}$ on $\mathfrak{A}$, and $\|.\|_{\mathfrak{B}}$ on $\mathfrak{B}$ such that with these norms $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras. Then $\mathfrak{B}$ is a Banach $\mathfrak{A}$-bimodule with the product of $\mathfrak{E}(I)$ giving the two module multiplications. Moreover if at least one of the spaces $\mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{B}$ is a dual Banach $\mathfrak{E}_\infty(I)$-bimodule, then*

$$\mathcal{H}^1(\mathfrak{A}, \mathfrak{B}) = H^1(\mathfrak{A}, \mathfrak{B}) \cong \frac{\mathcal{M}_1(\mathfrak{A}, \mathfrak{B}) + C(\mathfrak{E}(I))}{\mathfrak{B} + C(\mathfrak{E}(I))},$$

where $\cong$ denoted vector isomorphism.

Proof. Define

$$\Theta : \mathcal{M}_1(\mathfrak{A}, \mathfrak{B}) + C(\mathfrak{E}(I)) \to Z^1(\mathfrak{A}, \mathfrak{B}); E \mapsto D_E,$$

where $D_E(A) = AE - EA$ $(A \in \mathfrak{A})$. By Proposition 3.11 Θ is onto. By Lemma 3.12 $\ker\Theta = C(\mathfrak{E}(I)$. Therefore

$$\frac{\mathcal{M}_1(\mathfrak{A},\mathfrak{B}) + C(\mathfrak{E}(I))}{C(\mathfrak{E}(I))} \cong Z^1(\mathfrak{A},\mathfrak{B}),$$

through the mapping

$$\widetilde{\Theta} : E + C(\mathfrak{E}(I)) \mapsto \Theta(E) = D_E \quad (E \in \mathcal{M}_1(\mathfrak{A},\mathfrak{B}) + C(\mathfrak{E}(I))).$$

It is easy to show that

$$\widetilde{\Theta}\left(\frac{\mathfrak{B} + C(\mathfrak{E}(I))}{C(\mathfrak{E}(I)}\right) = \{D_E : E \in \mathfrak{B}\} = B^1(\mathfrak{A},\mathfrak{B}).$$

Hence

$$H^1(\mathfrak{A},\mathfrak{B}) = \frac{Z^1(\mathfrak{A},\mathfrak{B})}{B^1(\mathfrak{A},\mathfrak{B})} \cong \frac{\mathcal{M}_1(\mathfrak{A},\mathfrak{B}) + C(\mathfrak{E}(I))}{\mathfrak{B} + C(\mathfrak{E}(I))}.$$

By Proposition 3.2 $\mathcal{H}^1(\mathfrak{A},\mathfrak{B}) = H^1(\mathfrak{A},\mathfrak{B})$. ■

Corollary 3.14. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ideals of $\mathfrak{E}_\infty(I)$, and $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$. Suppose that there exist a norm $\|.\|_\mathfrak{A}$ on $\mathfrak{A}$, and $\|.\|_\mathfrak{B}$ on $\mathfrak{B}$ such that with these norms $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras. Moreover if at least one of the spaces $\mathcal{M}_1(\mathfrak{A},\mathfrak{B})$ and $\mathfrak{B}$ is a dual Banach $\mathfrak{E}_\infty(I)$-bimodule. Then $\mathcal{H}^1(\mathfrak{A},\mathfrak{B}) = 0$ if and only if $\mathcal{M}_1(\mathfrak{A},\mathfrak{B}) \subseteq \mathfrak{B} + C(\mathfrak{E}(I))$.*

4. General Results about the Banach Algebras $\mathfrak{E}_p(I)$ $(1 \le p \le \infty)$

For each $i \in I$, and $1 \le m, n \le d_i$, let $\mathcal{E}^i_{mn}$ be the elementary $d_i \times d_i$-matrix such that for $1 \le k, l \le d_i$,

$$\left(\mathcal{E}^i_{mn}\right)_{kl} = \begin{cases} 1 & \text{if } k = m, l = n \\ 0 & \textit{otherwise.} \end{cases}$$

The following lemma is indeed a generalization of Theorem D.54 of [5].

Lemma 4.1. *Let H be a finite-dimensional Hilbert space and $A \in \mathcal{B}(H)$, and $1 \le p \le \infty$. Then there exists $B \in \mathcal{B}(H)$ with $\|B\|_{\varphi_p} = 1$ such that $\|A\|_{\varphi_\infty} = \|AB\|_{\varphi_\infty}$. Moreover*

$$\|A\|_{\varphi_\infty} = \sup\left\{\|AB\|_{\varphi_\infty} : B \in \mathcal{B}(H) \textit{ and } \|B\|_{\varphi_p} = 1\right\}.$$

Proof. By Theorem D.30 of [5], there exists a unitary operator $U_0 \in \mathcal{B}(H)$ such that $AU_0 = |A|$. Let $(\lambda_1, \ldots, \lambda_n)$ be the sequence of eigenvalues of the operator $|A|$, written in any order. By Spectral Theorem (see for example Theorem 6.4.4 of [7], or Corollary 5.4 of section of section II of [2]) there exists a unitary matrix $U \in \mathcal{B}(H)$ such that $U^{-1}|A|U = \sum_{i=1}^{n} \lambda_i \mathcal{E}_{ii}$. Let $\lambda_{i_0} = \|A\|_{\varphi_\infty}$. If $B = U_0 U \mathcal{E}_{i_0 i_0}$, then by Theorem D.41 of [5], $\|B\|_{\varphi_p} = \|\mathcal{E}_{i_0 i_0}\|_{\varphi_p} = 1$. On one hand since U is a unitary matrix, so is U^{-1}. Therefore by Theorem D.41 of [5]

$$\begin{aligned}\|AB\|_{\varphi_\infty} &= \|A\left(U_0 U \mathcal{E}_{i_0 i_0}\right)\|_{\varphi_\infty} = \||A| U \mathcal{E}_{i_0 i_0}\|_{\varphi_\infty} = \left\|\left(U^{-1}|A|U\right)\mathcal{E}_{i_0 i_0}\right\|_{\varphi_\infty} \\ &= \left\|\left(\sum_{i=1}^{n} \lambda_i \mathcal{E}_{ii}\right)\mathcal{E}_{i_0 i_0}\right\|_{\varphi_\infty} = \|\lambda_{i_0}\mathcal{E}_{i_0 i_0}\|_{\varphi_\infty} = \lambda_{i_0} = \|A\|_{\varphi_\infty}.\end{aligned}$$

Hence $\|A\|_{\varphi_\infty} \leq \sup\{\|AB\|_{\varphi_\infty} : \|B\|_{\varphi_p} = 1\}$. On the other hand if $\|B\|_{\varphi_p} = 1$, then by Theorems D.51 and D.52 of [5],

$$\|AB\|_{\varphi_\infty} \leq \|A\|_{\varphi_\infty}\|B\|_{\varphi_\infty} \leq \|A\|_{\varphi_\infty}\|B\|_{\varphi_p} = \|A\|_{\varphi_\infty}.$$

Therefore $\|A\|_{\varphi_\infty} = \sup\{\|AB\|_{\varphi_\infty} : \|B\|_{\varphi_p} = 1\}$. ■

The following theorem is a generalization of parts **IV** and **V** of Theorem 35.4 of [5].

Proposition 4.2. *Let* $1 \leq p < q \leq \infty$. *Then* $\mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) = \mathfrak{E}_\infty(I)$, *if and only if sup*$_{i\in I} a_i < \infty$.

Proof. Since $p < q$, so by Theorem 28.32(iii),(iv) of [5], $\mathfrak{E}_\infty(I)\mathfrak{E}_p(I) \subseteq \mathfrak{E}_p(I) \subseteq \mathfrak{E}_q(I)$. Hence $\mathfrak{E}_\infty(I) \subseteq \mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_q(I))$.

Suppose $\sup_{i\in I} a_i < \infty$. We modify the proof of part IV Theorem 35.4 of [5], using Lemma 4.1. Let $E \in \mathfrak{E}(I) \setminus \mathfrak{E}_\infty(I)$. For each $n \in \mathbb{N}$, there exists $i_n \in I$ with $\|E_{i_n}\|_{\varphi_\infty} > n^3$ and such that $i_n \neq i_m$ for $n \neq m$. By applying Lemma 4.1, there exists $B_{i_n} \in B(H_{i_n})$ such that $\|B_{i_n}\|_{\varphi_p} = 1$ and $\|E_{i_n} B_{i_n}\|_{\varphi_\infty} = \|E_{i_n}\|_{\varphi_\infty} > n^3$. Define A_{i_n} as $n^{-2} B_{i_n}$ for each n and $A_i = 0$ for all other i's. Since

$$\|A\|_p = \left(\sum_{i\in I} a_i \|A_i\|_{\varphi_p}^p\right)^{\frac{1}{p}} = \left(\sum_{n\in\mathbb{N}} a_{i_n} n^{-2p}\right)^{\frac{1}{p}} \leq \left(\sup_{i\in I} a_i\right)^{\frac{1}{p}} \left(\sum_{n\in\mathbb{N}} n^{-2p}\right)^{\frac{1}{p}} < \infty,$$

so $A \in \mathfrak{E}_p(I)$. Since for each $n \in \mathbb{N}$, $\|E_{i_n} A_{i_n}\|_{\varphi_\infty} > n$, so $EA \notin \mathfrak{E}_\infty(I)$. Hence $EA \notin \mathfrak{E}_q(I)$, and so $E \notin \mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_q(I))$. Therefore $\mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) = \mathfrak{E}_\infty(I)$.

Suppose $\sup_{i\in I} a_i = \infty$. Define $E \in \mathfrak{E}(I)$ by $E_i = a_i^{\frac{1}{p} - \frac{1}{q}} I_i$ for all $i \in I$. Clearly $E \notin \mathfrak{E}_\infty(I)$. For $A \in \mathfrak{E}_p(I)$, by the same method of the proof of part **V** of Theorem 35.4 of [5], one can prove that $\|EA\|_\infty \le \|EA\|_q \le \|A\|_p < \infty$, and hence $E \in \mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_q(I))$. So $\mathfrak{E}_\infty(I) \subsetneqq \mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_q(I))$. ■

Proposition 4.3. *If* $1 \le p \le \infty$, *then* $\mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_p(I)) = \mathfrak{E}_\infty(I)$.

Proof. By 28.32(iii),(iv) of [5], $\mathfrak{E}_\infty(I) \subseteq \mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_p(I))$. Let $E \in \mathfrak{E}(I) \setminus \mathfrak{E}_\infty(I)$. As in the proof of Theorem 4.2, for each $n \in \mathbb{N}$, there exists $i_n \in I$ such that $\|E_{i_n}\|_{\varphi_\infty} > n$ and such that $i_n \neq i_m$ for $n \neq m$. Also there exists $B_{i_n} \in B(H_{i_n})$ such that $\|B_{i_n}\|_{\varphi_p} = 1$ and $\|E_{i_n} B_{i_n}\|_{\varphi_\infty} \ge n$. Define A_{i_n} as $(a_{i_n} n^2)^{-\frac{1}{p}} B_{i_n}$ for each n, and $A_i = 0$ for all other i's. By the same method of the proof of part **II** of Theorem 35.4 of [5], one can prove that $A \in \mathfrak{E}_p(I)$ and $EA \notin \mathfrak{E}_p(I)$. Therefore $\mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_p(I)) = \mathfrak{E}_\infty(I)$. ■

Proposition 4.4. *For* $1 \le q < p \le \infty$, $\mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) = \mathfrak{E}_r(I)$, *where* r *is defined by* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, *with the convention* $\frac{1}{\infty} = 0$.

Proof. By the same method of the proof of parts **VI** and **VII** of Theorem 35.4 of [5], $\mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) = \mathfrak{E}_r(I)$. ■

Theorem 4.5. *Let* $1 \le p < q \le \infty$, *and* $I_1 = \{i \in I : d_i \gneqq 1\}$. *Then the following assertions are equivalent:*

(*i*) $\sup_{i \in I_1} a_i < \infty$.

(*ii*) $\mathcal{M}_1(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) = \{E \in \mathfrak{E}_\infty(I) : E_i = 0 \ (i \notin I_1)\}$.

(*iii*) $\mathcal{M}_1(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) \subseteq \mathfrak{E}_\infty(I) + C(\mathfrak{E}(I))$.

Proof. (i)⇒(ii): On one hand by Theorem 4.2 $\mathcal{M}(\mathfrak{E}_p(I_1), \mathfrak{E}_q(I_1)) = \mathfrak{E}_\infty(I_1)$. On the other hand, since $p < q$, by Theorem 28.32(iii),(iv) of [5],

$$\mathfrak{E}_p(I_1)\mathfrak{E}_\infty(I_1) \cup \mathfrak{E}_\infty(I)\mathfrak{E}_p(I) \subseteq \mathfrak{E}_p(I) \subseteq \mathfrak{E}_q(I_1).$$

Therefore $\mathcal{M}(\mathfrak{E}_p(I_1), \mathfrak{E}_q(I_1)) \cap \mathcal{RM}(\mathfrak{E}_p(I_1), \mathfrak{E}_q(I_1)) = \mathfrak{E}_\infty(I_1)$. By regarding $\mathcal{M}_1(\mathfrak{E}_p(I), \mathfrak{E}_q(I))$ as a subspace of $\mathfrak{E}(I_1)$, it follows that $\mathcal{M}_1(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) = \{E \in \mathfrak{E}_\infty(I) : E_i = 0 \ (i \notin I_1)\}$.

(ii)⇒(iii) is obvious.

(iii)⇒(i): Suppose that $\sup_{i \in I_1} a_i = \infty$. Define $E \in \mathfrak{E}(I)$ by $E_i = a_i^{\frac{1}{p} - \frac{1}{q}} \mathcal{E}_{11}^i$ for all $i \in I_1$, and $E_i = 0$ for all $i \notin I_1$. Note that $\|E_i\|_{\varphi_q} = a_i^{\frac{1}{p} - \frac{1}{q}}$. For $A \in \mathfrak{E}_p(I)$, use (D.51.1) and (D.52.iii) of [5] and the same method of the proof of part **V** of Theorem 35.4 of [5] to write

$$\begin{aligned}&\|EA\|_\infty\\ &\le \|EA\|_q = \left(\sum_{i\in I}\left(a_i^{\frac{1}{q}}\|E_iA_i\|_{\varphi_q}\right)^q\right)^{\frac{1}{q}} \le \left(\sum_{i\in I}\left(a_i^{\frac{1}{q}}\|E_iA_i\|_{\varphi_q}\right)^p\right)^{\frac{1}{p}}\\ &\le \left(\sum_{i\in I}\left(a_i^{\frac{1}{q}}\|E_i\|_{\varphi_q}\|A_i\|_{\varphi_q}\right)^p\right)^{\frac{1}{p}} = \left(\sum_{i\in I} a_i\|A_i\|_{\varphi_q}^p\right)^{\frac{1}{p}}\\ &\le \left(\sum_{i\in I} a_i\|A_i\|_{\varphi_p}^p\right)^{\frac{1}{p}} = \|A\|_p < \infty.\end{aligned}$$

Therefore $E \in \mathcal{M}(\mathfrak{E}_p(I), \mathfrak{E}_q(I))$. Similarly one can prove that $E \in \mathcal{RM}(\mathfrak{E}_p(I), \mathfrak{E}_q(I))$. Hence $E \in \mathcal{M}_1(\mathfrak{E}_p(I), \mathfrak{E}_q(I))$. It can be proved that $E \notin \mathfrak{E}_q(I)+C(\mathfrak{E}(I))$. Suppose to the contrary that $E \in \mathfrak{E}_q(I) + C(\mathfrak{E}(I))$. Then there exists $E' \in \mathfrak{E}_q(I)$ and a set $\{\lambda_i : i \in I\} \subseteq \mathbb{C}$ such that for each $i \in I$, $E_i = E'_i + \lambda_i I_i$. Since $\sup_{i\in I_1} a_i = \infty$, there exists a subset $\{i_n : n \in \mathbb{N}\}$ of I_1 such that $i_m \ne i_n$ for $m \ne n$ and $\lim_n a_{i_n} = \infty$. The eigenvalues of $|E_{i_n} - \lambda_{i_n} I_{i_n}|$ are $|\lambda_{i_n}|$ with multiplicity $d_{i_n} - 1$ and $\left|a_{i_n}^{\frac{1}{p}-\frac{1}{q}} - \lambda_{i_n}\right|$ with multiplicity 1. Therefore

$$\begin{aligned}\|{E'}_{i_n}\|_{\varphi_q} &\ge \|{E'}_{i_n}\|_{\varphi_\infty} = \|E_{i_n} - \lambda_{i_n} I_{i_n}\|_{\varphi_\infty}\\ &= \max\left(|\lambda_{i_n}|, \left|a_{i_n}^{\frac{1}{p}-\frac{1}{q}} - \lambda_{i_n}\right|\right) \ge \frac{1}{2}a_{i_n}^{\frac{1}{p}-\frac{1}{q}},\end{aligned}$$

and hence

$$\|E'\|_q \ge \|E'\|_\infty \ge \sup_{n\in\mathbb{N}}\|E'_{i_n}\|_{\varphi_\infty} \ge \frac{1}{2}\sup_{n\in\mathbb{N}} a_{i_n}^{\frac{1}{p}-\frac{1}{q}} = \frac{1}{2}\left(\lim_n a_{i_n}\right)^{\frac{1}{p}-\frac{1}{q}} = \infty.$$

This contradiction shows that $E \notin \mathfrak{E}_q(I)+C(\mathfrak{E}(I))$. Therefore $\mathcal{M}_1(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) \nsubseteq \mathfrak{E}_\infty(I) + C(\mathfrak{E}(I))$. ■

By Propositions 4.3 and 4.4, the following results are obtained.

Proposition 4.6. *Let* $1 \le p \le \infty$, *and* $I_1 = \{i \in I : d_i \gneqq 1\}$. *Then* $\mathcal{M}_1(\mathfrak{E}_p(I), \mathfrak{E}_p(I)) = \{E \in \mathfrak{E}_\infty(I) : E_i = 0\ (i \notin I_1)\}$.

Proposition 4.7. *Let* $1 \le q < p \le \infty$, *and* $I_1 = \{i \in I : d_i \gneqq 1\}$. *Then* $\mathcal{M}_1(\mathfrak{E}_p(I), \mathfrak{E}_q(I)) = \{E \in \mathfrak{E}_r(I) : E_i = 0\ (i \notin I_1)\}$, *where* r *is defined by* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, *with the convention* $\frac{1}{\infty} = 0$.

5. Derivations Between the Banach Algebras $\mathfrak{E}_p(I)$ $(1 \le p \le \infty)$

By Theorem 28.32 of [5], the Banach algebra $\mathfrak{E}_p(I)$ ia an ideal of $\mathfrak{E}_\infty(I)$. In this chapter $\mathcal{H}^1(\mathfrak{E}_p(I), \mathfrak{E}_q(I))$ for $1 \le p, q \le \infty$ is calculated.

The following lemma is frequently used in the rest of paper.

Lemma 5.1. *If the set* $I_1 = \{i \in I : d_i \gneqq 1\}$ *is infinite, then for* $p, q \in [1, \infty]$,

$$\{E \in \mathfrak{E}_p(I) : E_i = 0 \ (i \notin I_1)\} \subseteq \mathfrak{E}_q(I) + C(\mathfrak{E}(I)),$$

if and only if $p \leq q$. *In particular* $\mathfrak{E}_p(I) \subseteq \mathfrak{E}_q(I) + C(\mathfrak{E}(I))$, *if and only if* $p \leq q$.

Proof. If $p \leq q$, then by Theorem 28.32(iv) of [5], $\mathfrak{E}_p(I) \subseteq \mathfrak{E}_q(I) \subseteq \mathfrak{E}_q(I) + C(\mathfrak{E}(I))$.

Let $p > q$. Since the set $I_1 = \{i \in I : d_i \gneqq 1\}$ is infinite, so there exists a countable infinite subset $\{i_n : n \in \mathbb{N}\}$ of distinct elements of I_1. Define $A_{i_n} = a_{i_n}^{-\frac{1}{p}} n^{-\frac{1}{q}} \mathcal{E}_{11}^{i_n}$ for each n, and $A_i = 0$ for all other i's. Since $\frac{p}{q} > 1$, so

$$\|A\|_p = \left(\sum_{i\in I} a_i \|A_i\|_{\varphi_p}^p\right)^{\frac{1}{p}} = \left(\sum_{n\in\mathbb{N}} a_{i_n} \|A_{i_n}\|_{\varphi_p}^p\right)^{\frac{1}{p}} = \left(\sum_{n\in\mathbb{N}} n^{-\frac{p}{q}}\right)^{\frac{1}{p}} < \infty,$$

and hence $A \in \{E \in \mathfrak{E}_p(I) : E_i = 0 \ (i \notin I_1)\}$. One can prove that $A \notin \mathfrak{E}_q(I) + C(\mathfrak{E}(I))$. Suppose to the contrary that $A \in \mathfrak{E}_q(I) + C(\mathfrak{E}(I))$. So there exist $A' \in \mathfrak{E}_q(I)$ and a set $\{\lambda_i : i \in I\} \subseteq \mathbb{C}$ such that for each $i \in I$, $A_i = A'_i + \lambda_i I_i$. Since the eigenvalues of $|A_{i_n} - \lambda_{i_n} I_{i_n}|$ are $|\lambda_{i_n}|$ with multiplicity $d_{i_n} - 1$, and $\left|a_{i_n}^{-\frac{1}{p}} n^{-\frac{1}{q}} - \lambda_{i_n}\right|$ with multiplicity 1, so

$$\begin{aligned}\|A'_{i_n}\|_{\varphi_q} &\geq \|A'_{i_n}\|_{\varphi_\infty} = \|A_{i_n} - \lambda_{i_n} I_{i_n}\|_{\varphi_\infty} \\ &= \max\left(|\lambda_{i_n}|, \left|a_{i_n}^{-\frac{1}{p}} n^{-\frac{1}{q}} - \lambda_{i_n}\right|\right) \geq \frac{1}{2} a_{i_n}^{-\frac{1}{p}} n^{-\frac{1}{q}}.\end{aligned}$$

It follows that

$$\begin{aligned}\|A'\|_q &= \left(\sum_{i\in I} a_i \|A'_i\|_{\varphi_q}^q\right)^{\frac{1}{q}} \geq \left(\sum_{n\in\mathbb{N}} a_{i_n} \|A'_{i_n}\|_{\varphi_q}^q\right)^{\frac{1}{q}} \\ &\geq \frac{1}{2}\left(\sum_{n\in\mathbb{N}} a_{i_n}^{(1-\frac{q}{p})} n^{-1}\right)^{\frac{1}{q}} \geq \frac{1}{2}\left(\sum_{n\in\mathbb{N}} n^{-1}\right)^{\frac{1}{q}} = \infty.\end{aligned}$$

This contradiction shows that $\{E \in \mathfrak{E}_p(I) : E_i = 0 \ (i \notin I_1)\} \nsubseteq \mathfrak{E}_q(I) + C(\mathfrak{E}(I))$. ∎

Notation: Throughout the rest of the paper for $1 < p < \infty$, let p' denote the exponent conjugate to p, that is $\frac{1}{p} + \frac{1}{p'} = 1$, for $p = 1$, let $p' = 0$ (not ∞), and for $p = \infty$, let $p' = 1$.

Proposition 5.2. *Let* $1 \leq p \lneqq \infty$. *Then the dual Banach* $\mathfrak{C}_p(I)$*-bimodule* $\mathfrak{C}_p(I)^*$ *can be identified with the Banach* $\mathfrak{C}_p(I)$*-bimodule* $\mathfrak{C}_{p'}(I)$ *with the product of* $\mathfrak{C}(I)$ *giving the two module multiplications.*

Proof. By Theorem 28.31 of [5], the mapping $T : \mathfrak{C}_{p'}(I) \to \mathfrak{C}_p(I)^*$ given by

$$\langle B, T(A)\rangle = \sum_{i\in I} a_i \mathbf{tr}(B_i A_i) \quad (A \in \mathfrak{C}_{p'}(I), B \in \mathfrak{C}_p(I)),$$

is an isometric Banach space isomorphism. Let $A, B \in \mathfrak{C}_p(I)$ and $X \in \mathfrak{C}_{p'}(I)$. For each $B \in \mathfrak{C}_{p'}(I)$,

$$\begin{aligned}\langle B, T(X).A\rangle &= \langle AB, T(X)\rangle = \sum_{i\in I} a_i \mathbf{tr}((AB)_i X_i) \\ &= \sum_{i\in I} a_i \mathbf{tr}(X_i (AB)_i)) = \sum_{i\in I} a_i \mathbf{tr}((XA)_i B_i) \\ &= \langle B, T(XA)\rangle.\end{aligned}$$

So $T(X).A = T(XA)$. Similarly $A.T(X) = T(AX)$. ∎

Proposition 5.3. *Let* $1 \leq p \lneqq \infty$ *and* $D : \mathfrak{C}_p(I) \to \mathfrak{C}_p(I)$ *be a derivation. Then* D *is continuous, and there is an element* $L \in \mathfrak{C}_\infty(I)$ *such that*

$$D(A) = AL - LA \quad (A \in \mathfrak{C}_p(I)).$$

Moreover $\mathcal{H}^1(\mathfrak{C}_p(I), \mathfrak{C}_p(I)) = 0$ *if and only if the set* $\{i \in I : d_i \gneqq 1\}$ *is finite.*

Proof. By Proposition 4.6, $\mathcal{M}_1(\mathfrak{C}_p(I), \mathfrak{C}_p(I)) = \{E \in \mathfrak{C}_\infty(I) : E_i = 0 \ (i \in I, d_i = 1)\}$. So by Theorem 3.10 and Proposition 5.2, D is continuous, and there exists $L \in \mathcal{M}_1(\mathfrak{C}_p(I), \mathfrak{C}_p(I)) \subseteq \mathfrak{C}_\infty(I)$ such that $D(A) = AL - LA \ (A \in \mathfrak{C}_p(I))$.

If $I_1 = \{i \in I : d_i \gneqq 1\}$ is finite, then

$$\mathcal{M}_1(\mathfrak{C}_p(I), \mathfrak{C}_p(I)) = \{E \in \mathfrak{C}_\infty(I) : E_i = 0 \ (i \notin I_1)\} \subseteq \mathfrak{C}_{00}(I) \subseteq \mathfrak{C}_p(I),$$

and so by Corollary 3.14, $\mathcal{H}^1(\mathfrak{C}_p(I), \mathfrak{C}_p(I)) = 0$.

Let I_1 be infinite. By Lemma 5.1, $\{E \in \mathfrak{C}_\infty(I) : E_i = 0 \ (i \notin I_1)\} \nsubseteqq \mathfrak{C}_p(I) + C(\mathfrak{C}(I))$, and hence by Corollary 3.14 $\mathcal{H}^1(\mathfrak{C}_p(I), \mathfrak{C}_p(I)) \neq 0$. ∎

Proposition 5.4. *Let* $1 \leq p \leq q \lneqq \infty$ *and suppose that* $D : \mathfrak{C}_p(I) \to \mathfrak{C}_q(I)$ *is a derivation. Then* D *is continuous, and there is an element* $L \in \mathcal{M}_1(\mathfrak{C}_p(I), \mathfrak{C}_q(I))$ *such that*

$$D(A) = AL - LA \quad (A \in \mathfrak{C}_p(I)).$$

Moreover each derivation from $\mathfrak{C}_p(I)$ *into* $\mathfrak{C}_q(I)$ *is inner if and only if the set* $\{i \in I : d_i \gneqq 1\}$ *is finite.*

Proof. Note that $\mathcal{M}_1(\mathfrak{C}_p(I), \mathfrak{C}_q(I)) \subseteq \mathfrak{C}(I)$. Hence by Theorem 3.10 and Proposition 5.2, D is continuous, and there exists $L \in \mathcal{M}_1(\mathfrak{C}_p(I), \mathfrak{C}_q(I))$ such that $D(A) = AL - LA$ $(A \in \mathfrak{C}_p(I))$.

If $\{i \in I : d_i \gneqq 1\}$ is finite, then $\mathcal{M}_1(\mathfrak{C}_p(I), \mathfrak{C}_q(I)) \subseteq \mathfrak{C}_{00}(I) \subseteq \mathfrak{C}_q(I)$, and so by Corollary 3.14 $\mathcal{H}^1(\mathfrak{C}_p(I), \mathfrak{C}_q(I)) = 0$.

Let $I_1 = \{i \in I : d_i \gneqq 1\}$ be infinite. Since $p \leq q$, so $\{E \in \mathfrak{C}_\infty(I) : E_i = 0\ (i \notin I_1)\} \subseteq \mathcal{M}_1(\mathfrak{C}_p(I), \mathfrak{C}_q(I))$. Hence by Lemma 5.1, $\{E \in \mathfrak{C}_\infty(I) : E_i = 0\ (i \notin I_1)\} \nsubseteq \mathfrak{C}_q(I) + C(\mathfrak{C}(I))$ and hence by Corollary 3.14, $\mathcal{H}^1(\mathfrak{C}_p(I), \mathfrak{C}_q(I)) \neq 0$. ∎

By Proposition 4.7, and a method similar to the proof of Proposition 5.3, one can prove the following result.

Proposition 5.5. *Let $1 \leq q < p \lneqq \infty$ and $D : \mathfrak{C}_p(I) \to \mathfrak{C}_q(I)$ be a derivation. Then D is continuous and there is an element $L \in \mathfrak{C}_r(I)$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, such that*

$$D(A) = AL - LA \quad (A \in \mathfrak{C}_p(I)).$$

Moreover $\mathcal{H}^1(\mathfrak{C}_p(I), \mathfrak{C}_q(I)) = 0$ if and only if the set $\{i \in I : d_i \gneqq 1\}$ is finite.

Proof. The proof is similar to the proof of Proposition 5.3. Also note that since $p \neq \infty$, so $r > q$. Hence by Lemma 5.1, if $I_1 = \{i \in I : d_i \gneqq 1\}$ is infinite, then $\{E \in \mathfrak{C}_r(I) : E_i = 0\ (i \notin I_1)\} \nsubseteq \mathfrak{C}_q(I) + C(\mathfrak{C}(I))$. ∎

By using a method similar to the proof of Proposition 5.3, one can obtained the following result as a consequence of Theorems 3.10 and 4.5, and Corollary 3.14.

Theorem 5.6. *Let $1 \leq p < q \leq \infty$. Then $\mathcal{Z}^1(\mathfrak{C}_p(I), \mathfrak{C}_q(I)) = \{D_L : L \in \mathfrak{C}_\infty(I)\}$, where $D_L(A) = AL - LA$ $(A \in \mathfrak{C}_p(I))$, if and only if $\sup_{i \in I_1} a_i < \infty$, where $I_1 = \{i \in I : d_i \gneqq 1\}$.*

Corollary 5.7. *Let $1 \leq p < \infty$. Then $\mathcal{H}^1(\mathfrak{C}_p(I), \mathfrak{C}_\infty(I)) = 0$ if and only if $\sup_{i \in I_1} a_i < \infty$, where $I_1 = \{i \in I : d_i \gneqq 1\}$.*

Theorem 3.9 yields the following result.

Proposition 5.8. *For each $1 \leq p \leq \infty$, and each $n \in \mathbb{N}$, $\mathcal{H}^1(\mathfrak{C}_\infty(I), \mathfrak{C}_p(I)) = 0$.*

A combination of Lemma 5.2, and Propositions 5.4 and 5.5 yields the following result.

Theorem 5.9. *For $1 < p < \infty$, $\mathfrak{C}_p(I)$ is weakly amenable if and only if the set $\{i \in I : d_i \gneqq 1\}$ is finite.*

Lemma 5.2 and Theorem 5.6 yields the following two corollaries.

Corollary 5.10. *The Banach algebra $\mathfrak{E}_1(I)$ is weakly amenable if and only if $\sup_{i\in I_1} a_i < \infty$, where $I_1 = \{i \in I : d_i \gneqq 1\}$.*

Remark 5.11. *By Theorem 28.26 of [5], $\mathfrak{E}_\infty(I)$ is a C^*-algebra. But by Theorem 4.2.4 of [9], each C^*-algebra is weakly amenable. Therefore $\mathfrak{E}_\infty(I)$ is weakly amenable.*

6. Applications to Compact Groups and Hypergroups

Let G be a compact group with dual $\widehat{G}$ (the set of all equivalence classes of irreducible representations of G). Let H_π be the representation space of π, for each $\pi \in \widehat{G}$. The algebras $\mathfrak{E}(\widehat{G})$ and $\mathfrak{E}_p(\widehat{G})$ for $p \in [1,\infty] \cup \{0\}$, are defined as in the preliminaries with each a_π equal to the dimension d_π of $\pi \in \widehat{G}$ (c.f Definition 28.34 of [5]).

Corollary 5.7 yields the following result. Note that by definition of $\mathfrak{E}_p(\widehat{G})$ $(p \in [1,\infty] \cup \{0\})$, $a_\pi = d_\pi$ $(\pi \in \widehat{G})$.

Theorem 6.1. *If G is a compact group, then each derivation from $\mathfrak{E}_p(\widehat{G})$ into $\mathfrak{E}_\infty(\widehat{G})$ is continuous. Moreover $\mathcal{H}^1(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_\infty(\widehat{G})) = 0$ if and only if $\sup_{\pi\in\widehat{G}} d_\pi < \infty$.*

By Theorem 34.35 of [5], the convolution Banach algebra $A(G)$ is isometrically algebra isomorphic with $\mathfrak{E}_1(\widehat{G})$. Hence the convolution Banach algebra $A(G)$ is weakly amenable if and only if $\mathfrak{E}_1(\widehat{G})$ is weakly amenable. Therefore as a consequence of Corollary 5.10, the following theorem is obtained.

Theorem 6.2. *If G is a compact group, then the convolution Banach algebra $A(G)$ is weakly amenable if and only if $\sup_{\pi\in\widehat{G}} d_\pi < \infty$.*

Proposition 6.3. *If G is an infinite non-abelian compact group, then the set $\{\pi \in \widehat{G} : dim\ \pi \gneqq 1\}$ is infinite.*

Proof. Suppose that the set $\{\pi \in \widehat{G} : dim\ \pi \gneqq 1\}$ is finite. Hence by Theorem 5.3, each derivation from $\mathfrak{E}_2(\widehat{G})$ into itself is inner. Now, by Peter-Weyl theorem [5], the convolution Banach algebra $L^2(G)$ is isometrically algebra isomorphic with $\mathfrak{E}_2(\widehat{G})$. So by Proposition 5.3, $\mathcal{H}^1(L^2(G), L^2(G)) = 0$. If G is infinite and non-abelian, then there exist $x, y \in G$ such that $xy \neq yx$. The mapping $D_x : L^2(G) \to L^2(G)$ defined by

$$D_x(f) = \delta_x * f - f * \delta_x \quad (f \in L^2(G)),$$

is a non-inner derivation. To see this, let $D_x = ad_g$ for some $g \in L^2(G)$. Then for each $f \in L^2(G)$, $f * (\delta_x - g) = (\delta_x - g) * f$. Since $L^2(G)$ is dense in $L^1(G)$, so for each $f \in L^1(G)$, $f * (\delta_x - g) = (\delta_x - g) * f$. Let (e_α) be a bounded approximate identity for $L^1(G)$. With the weak*-topology on $M(G)$

$$\begin{aligned}\delta_{xy} - \delta_{yx} &= \textit{weak}^* - \lim_\alpha (\delta_x * (e_\alpha * \delta_y) - (e_\alpha * \delta_y) * \delta_x)\\ &= \textit{weak}^* - \lim_\alpha D_x(e_\alpha * \delta_y) = \textit{weak}^* - \lim_\alpha ad_g(e_\alpha * \delta_y)\\ &= g * \delta_y - \delta_y * g \in L^2(G) \subseteq L^1(G).\end{aligned}$$

Since G is compact and infinite, it is not discrete and hence $\delta_{xy} - \delta_{yx} \notin L^1(G)$. This contradiction proves that G must be abelian or finite. ∎

A combination of Theorem 5.3, Theorem 5.9, and Proposition 6.3 yields the following result.

Corollary 6.4. *Let G be a compact group. Then*

(a) *For $1 \le p < \infty$, $\mathcal{H}^1(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_p(\widehat{G}) = 0$, if and only if G is finite or abelian.*

(b) *For $1 < p < \infty$, $\mathfrak{E}_p(\widehat{G})$ is weakly amenable, if and only if G is finite or abelian.*

Proposition 6.5. *Let G be a compact group and $1 \le p < q < \infty$. Then the following statements are equivalent:*

(i) $\mathcal{Z}^1(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})) = \{ad_L : L \in \mathfrak{E}_\infty(\widehat{G})\}$.

(ii) $\sup_{\pi \in \widehat{G}} d_\pi < \infty$.

Furthermore $\mathcal{H}^1(\mathfrak{E}_p(\widehat{G}), \mathfrak{E}_q(\widehat{G})) = 0$ if and only if G is finite or abelian.

Proof. By Theorem 5.6, the statements (i) and (ii) are equivalent. The remainder is a corollary of Proposition 5.4 and Proposition 6.3. ∎

Example 6.6. Let G be a compact group. Then $(A(G), *)$ is isometrically algebra isometric with $\mathfrak{E}_1(\widehat{G})$, and $(L^2(G), *)$ is isometrically algebra isometric with $\mathfrak{E}_2(\widehat{G})$.

(a) By Proposition 3.11, each derivation from the convolution Banach algebra $A(G)$ into the convolution Banach algebra $L^2(G)$ is continuous, i.e. $Z^1(A(G), L^2(G)) = \mathcal{Z}^1(A(G), L^2(G))$.

(b) If $\sup_{\pi \in \widehat{G}} d_\pi < \infty$, then by Proposition 6.5 $D \in \mathcal{Z}^1(A(G), L^2(G))$ if and only if there is an $T \in VN(G)$ such that $D(f) = f.T - T.f$ $(f \in A(G))$.

(c) If for each $D \in \mathcal{Z}^1(A(G), L^2(G))$ there is an $T \in VN(G)$ such that $D(f) = f.T - T.f$ $(f \in A(G))$, then $\sup_{\pi \in \widehat{G}} d_\pi < \infty$.

(d) $\mathcal{H}^1(A(G), L^2(G)) = 0$ if and only if G is finite or abelian.

The above results can be extended to compact hypergroups by the same way. Note that if K is a compact hypergroup, then by Theorem 2.6 of [10], for each $\pi \in \widehat{K}$, $k_\pi \geq d_\pi$. Hence $\sup_{\{\pi\in\widehat{K}: d_\pi \gneqq 1\}} k_\pi < \infty$ is equivalent to $\sup_{\pi\in\widehat{K}} k_\pi(d_\pi - 1) < \infty$.

Proposition 6.7. *If K is a compact hypergroup, then each derivation from $\mathfrak{E}_p(\widehat{K})$ into $\mathfrak{E}_\infty(\widehat{K})$ is continuous. Moreover $\mathcal{H}^1(\mathfrak{E}_p(\widehat{K}), \mathfrak{E}_\infty(\widehat{K})) = 0$ if and only if $\sup_{\pi\in\widehat{K}} k_\pi(d_\pi - 1) < \infty$.*

Theorem 6.8. *If K is a compact hypergroup, then the convolution Banach algebra $A(K)$ is weakly amenable if and only if $\sup_{\pi\in\widehat{G}} k_\pi(d_\pi - 1) < \infty$.*

Proposition 6.9. *Let K be a compact hypergroup and $1 \leq p < q < \infty$. Then the following statements are equivalent:*

(*i*) $\mathcal{Z}^1(\mathfrak{E}_p(\widehat{K}), \mathfrak{E}_q(\widehat{K})) = \{ad_L : L \in \mathfrak{E}_\infty(\widehat{K})\}$.

(*ii*) $\sup_{\pi\in\widehat{K}} k_\pi(d_\pi - 1) < \infty$.

Proposition 6.10. *Suppose K is an infinite non-abelian compact hypergroup such that for each $x, y \in K$, the set $x*y$ is finite. Then the set $\{\pi \in \widehat{K} : \text{dim}\,\pi \gneqq 1\}$ is infinite.*

Proof. By using the same method as the proof of Proposition 6.3, the proposition is proved. Note that since for each $x, y \in K$, the set $x * y$ is finite, so $\delta_{xy} - \delta_{yx} \in \ell^1(K)$. If K is compact and infinite, then $\delta_{xy} - \delta_{yx} \notin L^1(K)$. ∎

Corollary 6.11. *Suppose K is a compact hypergroup such that for each $x, y \in K$, the set $x * y$ is finite. Then*

(a) *For $1 \leq p < \infty$, $\mathcal{H}^1(\mathfrak{E}_p(\widehat{K}), \mathfrak{E}_p(\widehat{K})) = 0$, if and only if K is finite or abelian.*

(b) *For $1 < p < \infty$, $\mathfrak{E}_p(\widehat{K})$ is weakly amenable, if and only if K is finite or abelian.*

Corollary 6.12. *Suppose that K is a compact hypergroup such that for each $x, y \in K$, the set $x*y$ is finite. Let $1 \leq p < q < \infty$. Then $\mathcal{H}^1(\mathfrak{E}_p(\widehat{K}), \mathfrak{E}_q(\widehat{K})) = 0$ if and only if K is finite or abelian.*

We close the paper with the following open problem.

Open problem: Let $\mathfrak{A}$ and $\mathfrak{B}$ be ideals of $\mathfrak{E}_\infty(I)$, and $\mathfrak{E}_{00}(I) \subseteq \mathfrak{A}$. Suppose that there exist norms $\|.\|_{\mathfrak{A}}$ on $\mathfrak{A}$, and $\|.\|_{\mathfrak{B}}$ on $\mathfrak{B}$ such that with these norms $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras. Let D be a derivation from $\mathfrak{A}$ into $\mathfrak{B}$. Is there $M \in \mathcal{M}_1(\mathfrak{A}, \mathfrak{B})$ such that $D(A) = AM - MA$ $(A \in \mathfrak{A})$ (see Theorem 3.10 for a special case)?

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REFERENCES

1. W. R. Bloom and H. Heyer, *Harmonic analysis of probability measures on hypergroups*, Walter de Gruyter, Berlin, 1995.
2. J. B. Conway, *A course in functional analysis*, Springer Verlag, 1978.
3. H. G. Dales, *Banach algebras and automatic continuity*, Clarendon Press, Oxford, 2000.
4. F. Ghahramani and A. T. M. Lau, Approximate weak amenability, derivations and Arens regularity of Segal algebras, *Studia Mathematica*, **169(2)** (2005), 189-205.
5. E. Hewitt and K. A. Ross, *Abstract harmonic analysis, Vol. II*, Springer, Berlin, 1970.
6. R. I. Jewett, Spaces with an abstract convolution of measures, *Adv. Math.*, **18** (1975), 1-110.
7. S. J. Leon, *Linear algebra with applications*, 3nd ed., Macmillan Publishing Company, New York.
8. W. Rudin, *Functional analysis*, McGraw Hill, New York, 1991.
9. V. Runde, *Lectures on amenability*, Lecture Notes in Mathematics, Vol. **1774**, Springer, Berlin, 2002.
10. R. C. Vrem, Harmonic analysis on compact hypergroups, *Pacific J. Math.*, **85** (1979), 239-251.

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