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# ABSENCE OF POSITIVE ROOTS OF SEXTIC POLYNOMIALS

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**Abstract.** Given a general monic sextic polynomial with six real coefficients, necessary and sufficient conditions are found such that the polynomial does not have any positive roots. This 'nonlinear eigenvalue problem' is a relatively difficult one since we have 6 real parameters. Fortunately, we succeed in applying the Cheng-Lin envelope method in [1] together with several new ideas and techniques to express our criteria in terms of roots of quartic polynomials and explicit parametric curves and therefore our problem is completely solved. Several specific examples are also included to illustrate various applications including the seeking of periodic solutions of the logistic equation studied in chaos theory.

## 1. Introduction

It is well known that the real quadratic polynomial

$$\lambda^2 + \alpha\lambda + \beta, \ \lambda \in \mathbf{R},$$

has no real roots if, and only if, the discriminant $\alpha^2 - 4\beta$ is less than $0$. This condition is the same as requiring $(\alpha, \beta)$ to lie in the region strictly above the parabola $y = x^2/4$. Similar conditions have been obtained for the real cubic, quartic and quintic polynomials (see [1]). In this paper, we intend to consider the much more difficult general real sextic polynomial

$$Q(\lambda|a, b, c, d, \alpha, \beta) = \lambda^6 + a\lambda^5 + b\lambda^4 + c\lambda^3 + d\lambda^2 + \alpha\lambda + \beta, \tag{1}$$

and to find the exact geometric region containing the parameters $a, b, c, d, \alpha, \beta$ such that $Q$ does not have any positive roots. We remark that although a corresponding discriminant theory is available for arbitrary polynomials, an examination of the discriminants of the cubic and the quartic polynomials will reveal extremely difficult

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manipulations of multivariate symmetric polynomials or determinants of Sylvester's matrices. Hence our investigations are meaningful and may lead to new results in algebraic number theory and others.

The sextic polynomial was studied in great detail by Felix Klein and Robert Fricke in the 19th century and is directly related to the algebraic aspect of Hilbert's 13th Problem. There are now several approaches towards the solutions of sextic equations (see e.g. [2-5, 7, 8] and the references therein).

Sextic polynomials also appear in recent studies of chaos, anisotropic elastic materials, and others. For instance, in [6], the authors look for 3-periodic solutions of the logistic equation

$$x_{n+1} = \mu x_n(1 - x_n),\ n = 0, 1, 2, ...,$$

under the initial condition $x_0 = \lambda$. If we let $g_\mu(x) = \mu x(1 - x)$, then it suffices to solve the octic equation

$$g_\mu\left(g_\mu\left(g_\mu(\lambda)\right)\right) - \lambda = 0.$$

In case $\mu = -1$, we may consider the equivalent sextic equation (see Example 1 in the last section)

$$\frac{g(g(g(\lambda))) - \lambda}{\lambda(\lambda - 2)} = 0.$$

While in [9], the analysis of two-dimensional deformation of linear anisotropic elastic materials is reduced to the computation of certain eigenvalues that are the roots of a sextic algebraic equation whose coefficients depend only on the elastic constants.

In this paper, we are not directly involved in the location of the roots of sextic equations. Instead we are treating the sextic equation as one containing 6 real parameters. In such a context, our problem stated above is similar to the 'eigenvalue problem' in matrix theory, and therefore it is natural to call the set containing the required parameters the 'characteristic region' such that all roots fall outside the positive axis. More generally, given a function $G(\lambda) = G(\lambda|a_1, a_2, ..., a_n)$ depending on $n$ real parameters, the subset of all $(a_1, ..., a_n) \in \mathbf{R}^n$ such that none of the roots of the corresponding function $G$ are positive is called the $\mathbf{C}\backslash(0, +\infty)$-characteristic region of $G$.

The problem posed above is a relatively difficult one since we have 6 real parameters. Fortunately, an envelope method for handling the existence of real roots of functions involving finitely many real parameters is developed recently by Cheng and Lin. This method is used several times successfully (see e.g. [10]) and formalized recently in a book [1]. We will apply this method together with several new ideas and techniques to tackle our problem. Necessary as well as sufficient conditions are obtained. These conditions are expressed in terms of roots of quartic polynomials and explicit parametric curves, and therefore our problem is completely solved.

For a sample corollary of our main results, we show that the sextic polynomial

$$\lambda^6 + \alpha\lambda + \beta,\ \lambda \in \mathbf{R},$$

has no positive roots if, and only if, $\alpha \geq 0$ and $\beta \geq 0$, or, $\alpha < 0$ and $\beta > 5(\frac{\alpha}{6})^{6/5}$.

Some of the materials in this paper are, although elementary, quite technical. It is therefore advisable to first look at the other examples in the last section for further motivation.

## 2. Preparatory Results

To facilitate discussions, we first recall a few basic concepts and tools explained in [1]. Let $\mathbf{R}$, $\mathbf{R}^+$ and $\mathbf{C}$ be respectively the set of real, positive real and complex numbers and let the function $\Theta_0$ be the null function, that is $\Theta_0(x) = 0$ for $x \in \mathbf{R}$. Given an interval $I$ in $\mathbf{R}$, the chi-function $\chi_I : I \to \mathbf{R}$ is defined by

$$\chi_I(x) = 1,\ x \in I.$$

The restriction of a real function $f$ defined over an interval $J$, which is not disjoint from $I$, will be written as $f\chi_I$, so that $f\chi_I$ is now defined on $I \cap J$ and

$$(f\chi_I)(x) = f(x),\ x \in I \cap J.$$

Let $S$ be a plane curve and $L$ be a plane straight line. Let $d(A, B)$ denote the distance of two points $A$ and $B$, and let $d(A, L)$ be the distance between the point $A$ and the straight line $L$. Assume $S$ and $L$ have a common point $P$. According to the theory of contact (due to Langrange), the straight line $L$ is called the tangent of the curve $S$ at the point $P$ if

$$\lim_{A \to P, A \in S} \frac{d(A, L)}{d(A, P)} = 0. \tag{2}$$

In case $S$ is described by a pair of parametric functions, we have the following relatively easy result (see also [11]).

**Lemma 1.** *Let the plane curve $S$ be described by the parametric functions $x(t)$ and $y(t)$ on an interval $I$. Given $t_0 \in I$ such that $x(t) \neq x(t_0)$ for all $t \in I\backslash\{t_0\}$. For any $m \in \mathbf{R}$, let the straight line $L_m$ be defined by $L_m(x) = m(x - x(t_0)) + y(t_0)$ for $x \in \mathbf{R}$. Suppose the limit*

$$M := \lim_{t \to t_0} \frac{y(t) - y(t_0)}{x(t) - x(t_0)}$$

*exists. Then the straight line $L_M(x)$ is the unique tangent of the curve $S$ at $(x(t_0), y(t_0))$.*

We remark that the above definition is compatible with the concept of 'tangent lines' associated with the graph of a real smooth function $y = f(x)$ of a real variable. Indeed, let $S$ be the curve which is also described by the graph of a smooth function $f$ passing through $P = (x_0, y_0)$. By Lemma 1, it is easy to see that (2) holds if, and only if, the straight line $L$ is the tangent of the graph of the function $f$.

A point in the plane is said to be a *dual point* of order $m$ of the plane curve $S$, where $m$ is a nonnegative integer, if there exist exactly $m$ mutually distinct tangents of $S$ that also pass through it. The set of all dual points of order $m$ of $S$ in the plane is called the dual set of order $m$ of $S$. We remark that $m = 0$ is allowed. In this case, there are no tangents of $S$ that pass through the point in consideration.

Let $\{C_\lambda : \lambda \in I\}$, where $I$ is a real interval, be a family of plane curves. With each $C_\lambda$, suppose we can associate just one point $P_\lambda$ in each $C_\lambda$ such that the totality of these points form a curve $S$. Then $S$ is called an **envelope** of the family $\{C_\lambda | \, \lambda \in I\}$ if the curves $C_\lambda$ and $S$ share a common tangent line at the common point $P_\lambda$. Suppose we have a family of curves in the $x, y$-plane implicitly defined by

$$F(x, y, \lambda) = 0, \ \lambda \in I,$$

where $I$ is an interval of $\mathbf{R}$. Then it is well known that the envelope $S$ is described by a pair of parametric functions $(\psi(\lambda), \phi(\lambda))$ that satisfy

$$\begin{cases} F(\psi(\lambda), \phi(\lambda), \lambda) = 0, \\ F'_\lambda(\psi(\lambda), \phi(\lambda), \lambda) = 0, \end{cases}$$

for $\lambda \in I$, provided some "good conditions" are satisfied. In particular, let $f, g, h : I \to \mathbf{R}$. Then for each fixed $\lambda \in I$, the equation

$$L_\lambda : f(\lambda)x + g(\lambda)y = h(\lambda), \ (f(\lambda), g(\lambda)) \neq 0, \tag{3}$$

defines a straight line $L_\lambda$ in the $x, y$-plane, and we have a collection $\{L_\lambda : \lambda \in I\}$ of straight lines. For such a collection, we have the following result.

**Theorem 1.** (See [1, Theorems 2.3 and 2.5]). *Let $f, g, h$ be real differentiable functions defined on the interval $I$ such that $f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda) \neq 0$ and $g(\lambda) \neq 0$ for $\lambda \in I$. Let $\Phi$ be the family of straight lines of the form (3). Let the curve $S$ be defined by the functions $x = \psi(\lambda), y = \phi(\lambda)$:*

$$\psi(\lambda) = \frac{g'(\lambda)h(\lambda) - g(\lambda)h'(\lambda)}{f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda)}, \quad \phi(\lambda) = \frac{f(\lambda)h'(\lambda) - f'(\lambda)h(\lambda)}{f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda)}, \quad \lambda \in I. \tag{4}$$

*Suppose $\psi$ and $\phi$ are smooth functions over $I$ and one of the following cases holds: (i) $\psi'(\lambda) \neq 0$ for $\lambda \in I$; (ii) $\psi'(\lambda) \neq 0$ for $I \backslash \{d\}$ where $d \in I$ and $\lim_{\lambda \to d^-} \phi'(\lambda)/\psi'(\lambda)$ as well as $\lim_{\lambda \to d^+} \phi'(\lambda)/\psi'(\lambda)$ exist and are equal. Then $S$ is the envelope of the family $\Phi$.*

**Theorem 2.** (See [1, Theorem 2.6]). *Let $\Lambda$ be an interval in $\mathbf{R}$, and $f, g, h$ be real differentiable functions defined on $\Lambda$ such that $f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda) \neq 0$ for $\lambda \in \Lambda$. Let $\Phi$ be the family of straight lines of the form (3), where $\lambda \in \Lambda$, and let the curve $S$ be the envelope of the family $\Phi$. Then the point $(\alpha, \beta)$ in the plane is a dual point of order $m$ of $S$, if, and only if, the function $f(\lambda)\alpha + g(\lambda)\beta - h(\lambda)$, as a function of $\lambda$, has exactly $m$ mutually distinct roots in $\Lambda$.*

The above result states roughly that the roots of the function $F(\lambda|\alpha, \beta) = f(\lambda)\alpha + g(\lambda)\beta - h(\lambda)$ in the interval $\Lambda$ 'match' the tangents connecting the point $(\alpha, \beta)$ to the envelope of the family $\{L_\lambda|\lambda \in \Lambda\}$ of straight lines, where $L_\lambda$ is the straight line defined by $F(\lambda|x, y)$ for $x, y \in R$. Therefore we only need to count the number of such tangents for different pairs of $(\alpha, \beta)$, that is, to classify dual points of envelopes.

Plane curves can take on complicated forms. Fortunately, for some plane curves, their dual points can be described precisely. Indeed, a complete list of distribution maps of dual points of strictly convex and smooth (i.e. continuously differentiable) graphs of real functions of one variable defined on real intervals can be found in [1, Theorems 3.3-3.20.]. Based on such distribution maps, a partial list of distributions of dual points of piecewise convex-concave and smooth graphs is also available (see [1, Appendix A]). In this paper, we will need some of these distribution maps (see Lemmas 3, 4 and 5 below) and will build some new ones (see Lemmas 6 and 7 below) for use in later discussions.

In deriving the complete list of distribution maps in [1], strictly convex and smooth functions are classified by their monotonicity and behaviors near the boundary points of their domains. Some of these classifications are standard. A less familiar one is recalled here as follows. Let $g$ be a function defined on an interval $I$ with $c = \inf I$ and $d = \sup I$. Note that $c$ or $d$ may be infinite, or may be outside the interval $I$, and that $g(c^+)$, $g(d^-)$, $g'(c^+)$ or $g'(d^-)$ may not exist. For $\lambda \in (c, d)$, let

(5) $$L_{g|\lambda}(x) = g'(\lambda)(x - \lambda) + g(\lambda),\ x \in R.$$

In case $d$ is finite and $g(d^-)$, $g'(d^-)$ exist, we let

(6) $$L_{g|d}(x) = g'(d^-)(x - d) + g(d^-),\ x \in R,$$

and in case $c$ is finite and $g(c^+), g'(c^+)$ exist, we let

(7) $$L_{g|c}(x) = g'(c^+)(x - c) + g(c^+),\ x \in R.$$

When $d$ is finite, we say $g \sim H_{d^-}$ if $\lim_{\lambda \to d^-} L_{g|\lambda}(\alpha) = -\infty$ for any $\alpha < d$; and similarly when $c$ is finite, $g \sim H_{c^+}$ if $\lim_{\lambda \to c^+} L_{g|\lambda}(\alpha) = -\infty$ for any $\alpha > c$.

In case $d$ is infinite, we say $g \sim H_{+\infty}$ if $\lim_{\lambda \to +\infty} L_{g|\lambda}(\alpha) = -\infty$ for any $\alpha \in \mathbf{R}$; and similarly, when $c$ is infinite, we say $g \sim H_{-\infty}$ if $\lim_{\lambda \to -\infty} L_{g|\lambda}(\alpha) = -\infty$ for any $\alpha \in \mathbf{R}$.

There is a convenient criteria for the determination of functions with the above stated properties.

**Lemma 2.** ([1, Lemmas 3.1 and 3.5]). *Let $g : (c, d) \to \mathbf{R}$ is a smooth and strictly convex function. (i) Assume $d < +\infty$. If $g'(d^-) = +\infty$, then $g \sim H_{d^-}$. (ii) Assume $d = +\infty$. If $g'(+\infty) = +\infty$, or, $g'(+\infty) = 0$ and $g(+\infty) = -\infty$, then $g \sim H_{+\infty}$.*

The description of the distribution of dual points of a plane curve can be cumbersome. For this reason, it is convenient to introduce several notations. We say that a point $(a, b)$ in the plane is strictly above (above, strictly below, below) the graph of a function $g$ if $a$ belongs to the domain of $g$ and $g(a) < b$ (respectively $g(a) \le b$, $g(a) > b$ and $g(a) \ge b$). The notation is $(a, b) \in \vee(g)$ (respectively $(a, b) \in \overline{\nabla}(g)$, $(a, b) \in \wedge(g)$ and $(a, b) \in \underline{\triangle}(g)$). Suppose we now have two real functions $g_1$ and $g_2$ defined on real subsets $I_1$ and $I_2$ respectively. We say that $(a, b) \in \vee(g_1) \oplus \vee(g_2)$ if $a \in I_1 \cap I_2$ and $b > g_1(a)$ and $b > g_2(a)$, or, $a \in I_1 \backslash I_2$ and $b > g_1(a)$, or, $a \in I_2 \backslash I_1$ and $b > g_2(a)$. The notations $(a, b) \in \overline{\nabla}(g_1) \oplus \vee(g_2)$, $(a, b) \in \overline{\nabla}(g_1) \oplus \wedge(g_2)$, etc. are similarly defined. If we now have $n$ real functions $g_1, ..., g_n$ defined on intervals $I_1, ..., I_n$ respectively, we write $(a, b) \in \vee(g_1) \oplus \vee(g_2) \oplus \cdots \oplus \vee(g_n)$ if $a \in I_1 \cup I_2 \cup \cdots \cup I_n$, and if

$$a \in I_{i_1} \cup I_{i_2} \cup \cdots \cup I_{i_m} \Rightarrow b > g_{i_1}(a), b > g_{i_2}(a), ..., b > g_{i_m}(a),\ i_1, ..., i_m \in \{1, ..., n\}.$$

The notations $(a, b) \in \overline{\nabla}(g_1) \oplus \overline{\nabla}(g_2) \oplus \cdots \oplus \overline{\nabla}(g_n)$, etc. are similarly defined.

Equipped with the functions $L_{g|\lambda}$ defined by (5)-(7) and the ordering of points and graphs in the plane, we may now state the following distribution results for dual points.

**Lemma 3.** ([1, Theorem 3.20], see Figure 1). *Let $g : R \to R$ be a strictly convex and smooth function such that $g \sim H_{-\infty}$ and $g \sim H_{+\infty}$. Then $(\alpha, \beta)$ is a dual point of order $0, 1$ or $2$ of $g$ if, and only if, respectively $\beta > g(\alpha)$, $\beta = g(\alpha)$ or $\beta < g(\alpha)$.*

**Lemma 4.** ([1, Theorem 3.3], see Figure 2). *Let $g : (c, d) \to \mathbf{R}$ be a strictly convex and smooth function such that $g(a^+)$, $g(d^-)$, $g'(a^+)$ and $g'(d^-)$ exist. Then $(\alpha, \beta)$ in the plane is a dual point of order $0$ of $g$ if, and only if, $(\alpha, \beta) \in \vee(g) \oplus \overline{\nabla}(L_{g|c}) \oplus \overline{\nabla}(L_{g|d})$ or $(\alpha, \beta) \in \underline{\triangle}(L_{g|c}) \oplus \underline{\triangle}(L_{g|d})$.*

**Lemma 5.** ([1, Theorem 3.11], see Figure 3). *Let $g : (c, +\infty) \to \mathbf{R}$ be a strictly*

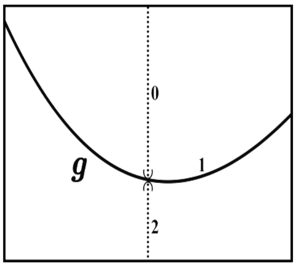



Fig. 1.

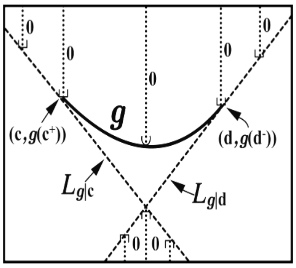



Fig. 2.

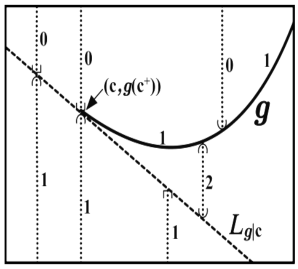



Fig. 3.

*convex and smooth function such that* $g(c^+)$ *and* $g'(c^+)$ *exist, and* $g \sim H_{+\infty}$. *Then the following statements hold:*

(1) *Dual set of order* $0$ *of* $g$ *is* $\vee(g) \oplus \overline{\vee}(L_{g|c})$.

(2) *Dual set of order* $1$ *of* $g$ *is the union of* $\wedge(L_{g|c}\chi_{(-\infty,c]}) \oplus \underline{\triangle}(L_{g|c}\chi_{(c,+\infty)})$ *and* $\{(a,b) \in R^2 : a > c \text{ and } g(a) = b\}$.

(3) *Dual set of order* $2$ *of* $g$ *is* $\wedge(g) \oplus \vee(L_{g|c}\chi_{(c,+\infty)})$.

(4) *Dual set of order greater than* $2$ *of* $g$ *is empty.*

As explained in [1, Appendix A], dual sets of order 0 of plane curves that are made up of several pieces of convex and concave functions can be obtained by intersections. In particular, the following result is easily deduced from Theorems 3.4 and A.3 in [1].

**Lemma 6.** (See Figure 4). *Let* $a, b, c \in \mathbf{R}$, $g_1 \in C^1(a, +\infty)$, $g_2 \in C^1[a, b)$ *and* $g_3 \in C^1[c, b]$. *Suppose the following hold:*

$(i)$ $g_1$ *is strictly convex on* $(a, +\infty)$ *such that* $g_1 \sim H_{+\infty}$;

$(ii)$ $g_2$ *is strictly concave on* $[a, b)$;

$(iii)$ $g_3$ *is strictly convex on* $[c, b]$;

$(iv)$ $g_1^{(v)}(a^+) = g_2^{(v)}(a^+)$ *and* $g_2^{(v)}(b^-) = g_3^{(v)}(b^-)$ *for* $v = 0, 1$.

*Then the intersection of dual set of order* $0$ *of* $g_1, g_2$ *and* $g_3$ *is* $\vee(g_1) \oplus \vee(g_3) \oplus \vee(L_{g_3|c})$.

The following result is deduced from Lemma 4 and Theorem 3.3 in [1].

**Lemma 7.** (See Figure 5). *Let* $a, b, c, d \in \mathbf{R}$, $g_1 \in C^1(a, +\infty)$, $g_2 \in C^1[a, b)$, $g_3 \in C^1[c, b]$, *and* $g_4 \in C^1(c, d)$. *Suppose the following hold:*

$(i)$ $g_1$ *is strictly convex on* $(a, +\infty)$ *such that* $g_1 \sim H_{+\infty}$;

$(ii)$ $g_2$ *is strictly concave on* $[a, b)$;

$(iii)$ *$g_3$ is strictly convex on $[c, b]$;*

$(iv)$ *$g_4$ is strictly concave on $[c, d)$ such that $g_4(d^-)$ and $g_4'(d^-)$ exist;*

$(v)$ *$g_1^{(v)}(a^+) = !g_2^{(v)}(a^+)$, $g_2^{(v)}(b^-) = g_3^{(v)}(b^-)$ and $g_3^{(v)}(c^+) = g_4^{(v)}(c^+)$ for $v = 0, 1$.*

*Then the intersection of dual set of order* $0$ *of $g_1, g_2, g_3$ and $g_4$ is* $\vee(g_1) \oplus \vee(g_3) \oplus \overline{\nabla}(L_{g_4|d})$ *(see Figure 5).*

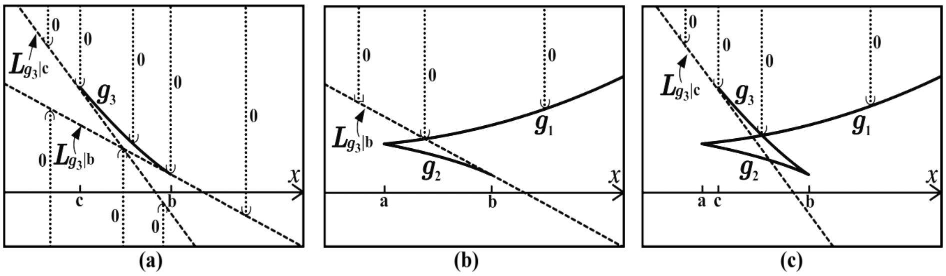



Fig. 4. Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorems 3.4 and A.3]) to yield (c).

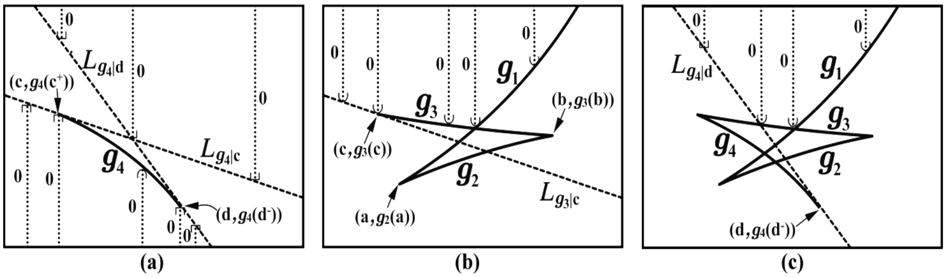



Fig. 5. Intersection of the dual sets of order 0 in (a) and (b) (see [1, Lemma 4 and Theorem 3.3]) to yield (c).

Given a pair of parametric functions $x = \psi(\lambda)$ and $y = \varphi(\lambda)$ defined on an interval $I$. We may sometimes be able to solve for $\lambda$ from $x = \psi(\lambda)$ and then substitute it into $\varphi(\lambda)$ to yield a function $y = f(x)$. The following simple result can be used to make sure that smooth graphs can be obtained from parametric curves in this manner.

**Lemma 8.** (See [1, Theorem 2.1]). *Let $G$ be the curve described by a pair of smooth functions $\psi(\lambda)$ and $\phi(\lambda)$ on an interval $I$ such that $\psi'(\lambda) > 0$ (or $\psi'(\lambda) < 0$) for $t \in I$ except at perhaps one point $r$. Suppose $q$ is a continuous function defined on $I$ such that $\phi'(\lambda)/\psi'(\lambda) = q(\lambda)$ for $\lambda \in I \backslash \{r\}$. Then $G$ is also the graph of a smooth function $y = S(x)$ defined on $\psi(I)$.*

For the sake of convenience, we will use the same notation to indicate a real function of a real variable and its graph. Therefore, in the above result, we may stay the conclusion in the form "Then the curve $G$ is the graph of a smooth function $y = G(x)$ defined over $\psi(I)$."

## 3. Positive Roots of Quartic Polynomials

Before we can actually discuss the $\mathbf{C}\backslash(0, +\infty)$-characteristic region of the polynomial $Q$ defined in (1), we need to first handle the positive roots of quartic polynomials of the form

$$P(\lambda|c, d) = 5\lambda^4 + c\lambda + d,\ c, d \in \mathbf{R},\ \lambda > 0 \tag{8}$$

and

$$T(\lambda|a, b, c, d) = \lambda^4 + \frac{2a}{3}\lambda^3 + \frac{2b}{5}\lambda^2 + \frac{c}{5}\lambda + \frac{d}{15},\ a, b, c, d \in \mathbf{R},\ \lambda > 0. \tag{9}$$

We first consider the positive roots of $P$. To this end, we break (cf. [1, Section 5.3.1]) the $xy$-plane into four mutually disjoint parts $\Gamma_0$, $\Gamma_1'$, $\Gamma_1''$ and $\Gamma_2$ (see Figure 6)

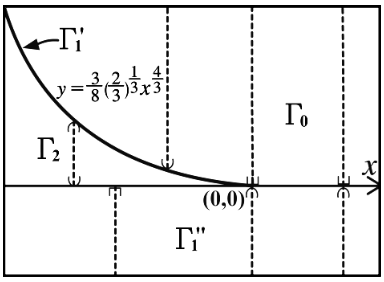



Fig. 6.

where

$$\Gamma_0 = \left\{ (x, y) \in \mathbf{R}^2 : x \geq 0 \text{ and } y \geq 0 \right\} \cup \left\{ (x, y) \in \mathbf{R}^2 : x < 0 \text{ and } y > \frac{3}{8}\left(\frac{2}{3}\right)^{1/3} x^{4/3} \right\}, \tag{10}$$

$$\Gamma_1' = \left\{ (x, y) \in \mathbf{R}^2 : x < 0 \text{ and } y = \frac{3}{8}\left(\frac{2}{3}\right)^{1/3} x^{4/3} \right\}, \tag{11}$$

$$\Gamma_1'' = \{(x, y) \in \mathbf{R}^2 : x < 0 \text{ and } y \leq 0\} \cup \{(x, y) \in \mathbf{R}^2 : x \geq 0 \text{ and } y < 0\}, \tag{12}$$

and

$$\Gamma_2 = \left\{ (x, y) \in \mathbf{R}^2 : x < 0 \text{ and } 0 < y < \frac{3}{8}\left(\frac{2}{3}\right)^{1/3} x^{4/3} \right\}. \tag{13}$$

**Lemma 9.** *Assume $c, d \in \mathbf{R}$. Let $P(\lambda|c, d)$ be defined by (8). The following results hold:*

($i$) *if $(c, d) \in \Gamma_0$, then $P(\lambda|c, d)$ has no positive roots and $P(\lambda|c, d) > 0$ for $\lambda > 0$;*

($ii$) *if $(c, d) \in \Gamma_2$, then $P(\lambda|c, d)$ has exactly two positive roots $\varsigma_1$ and $\varsigma_2$ such that $\varsigma_1 < \varsigma_2$, $P(\lambda|c, d) > 0$ for $\lambda \in \mathbf{R}^+ \backslash [\varsigma_1, \varsigma_2]$ and $P(\lambda|c, d) < 0$ for $\lambda \in (\varsigma_1, \varsigma_2)$;*

($iii$) *if $(c, d) \in \Gamma_1' \cup \Gamma_1''$, then $P(\lambda|c, d)$ has exactly one positive root $\varsigma$; furthermore, if $(c, d) \in \Gamma_1'$, then $P(\lambda|c, d) > 0$ for $\lambda \in \mathbf{R}^+ \backslash \{\varsigma\}$; while if $(c, d) \in \Gamma_1''$, then $P(\lambda|c, d) < 0$ for $\lambda \in (0, \varsigma)$ and $P(\lambda|c, d) > 0$ for $\lambda \in (\varsigma, +\infty)$.*

*Proof.* First note that $P(0^+|c, d) = d$ and $P(+\infty|c, d) = +\infty$. Furthermore, $P'(\lambda|c, d) = 20\lambda^3 + c$ and

$$P''(\lambda|c, d) = 60\lambda^2 > 0 \tag{14}$$

for $\lambda > 0$. Therefore $P$ is a strictly convex function over $(0, +\infty)$ and has a local minimum at $\lambda = (-c/20)^{1/3} > 0$ when, and only when, $c < 0$. To gain further information related to $P$, we consider the family $\{L_\lambda | \ \lambda > 0\}$ of straight lines defined by $L_\lambda : \lambda x + y = -5\lambda^4$. $L_\lambda$ is of the form (3) and $f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda) = -1 \neq 0$ and $g(\lambda) = 1$. By calculating the parametric functions $\psi$ and $\phi$ in (4), and then renaming $\psi(\lambda)$ and $\phi(\lambda)$ as $c(\lambda)$ and $d(\lambda)$ respectively, we see that

$$c(\lambda) = -4\lambda \text{ and } d(\lambda) = \lambda^4 \text{ for } \lambda > 0.$$

Since $c'(\lambda) \neq 0$ for $\lambda > 0$, by Theorem 1, the curve $W$ described by the parametric functions $c(\lambda)$ and $d(\lambda)$ is the envelope of the family $\{L_\lambda | \ \lambda > 0\}$. Solving $\lambda$ from $c(\lambda) = -4\lambda$ and substituting $\lambda$ into $d(\lambda)$, we see that the curve $W$ is the graph of the function

$$y = W(x) = \frac{3}{8}\left(\frac{2}{3}\right)^{1/3} x^{4/3}, \ x < 0.$$

It is easy to see that $W$ is strictly decreasing, strictly convex and smooth function such that $W'(0^-) = 0$. Since $W'(-\infty) = -\infty$, $W \sim H_{-\infty}$ by Lemma 2. Thus, by Lemma 5, $\Gamma_0$ is the dual set of order 0 of $W$, $\Gamma_1' \cup \Gamma_1''$ is the dual set of order 1 of $W$ and $\Gamma_2$ is the dual set of order 2 of $W$.

Assume $(c, d) \in \Gamma_0$. Then by Theorem 2, $P(\lambda|c, d)$ has no positive roots. Since $d \geq 0$, we see that $P(\lambda|c, d) > 0$ for $\lambda > 0$.

Assume $(c, d) \in \Gamma_2$. By Theorem 2, $P(\lambda|c, d)$ has exactly two positive roots $\varsigma_1$ and $\varsigma_2$ such that $\varsigma_1 < \varsigma_2$. Since $d > 0$, and since $P$ is strictly convex on $(0, \infty)$, we see further that $P(\lambda|c, d)$ has exactly one local minimum at some point in $(\varsigma_1, \varsigma_2)$, and $P(\lambda|c, d) > 0$ for $\lambda \in \mathbf{R}^+ \backslash [\varsigma_1, \varsigma_2]$ as well as $P(\lambda|c, d) < 0$ for $\lambda \in (\varsigma_1, \varsigma_2)$.

Assume $(c,d) \in \Gamma_1' \cup \Gamma_1''$. By theorem 2, then $P(\lambda|c,d)$ has exactly one positive root $\varsigma$. If $(c,d) \in \Gamma_1'$, then $d > 0$. Since $P(\lambda|c,d)$ is strictly convex on $(0,\infty)$ and since $P(+\infty|c,d) = +\infty$, $\varsigma$ must be the local minimal point and $P(\lambda|c,d) > 0$ for $\lambda \in \mathbf{R}^+ \backslash \{\varsigma\}$. If $(c,d) \in \Gamma_1''$, then $d \leq 0$. When $d < 0$, since $P$ is strictly convex on $(0,\infty)$ and since $P(+\infty|c,d) = +\infty$), it is easy to see that $P(\lambda|c,d) < 0$ for $\lambda \in (0,\varsigma)$ and $P(\lambda|c,d) > 0$ for $\lambda \in (\varsigma,+\infty)$. When $d = 0$, since $P$ is strictly convex on $(0,\infty)$ and since $P$ has exactly one positive root $\varsigma$, $P(\lambda|c,d) < 0$ for $\lambda \in (0,\varsigma)$ and $P(\lambda|c,d) > 0$ for $\lambda \in (\varsigma,+\infty)$. The proof is complete.

We now turn to the polynomial $T(\lambda|a,b,c,d)$. We will treat $(c,d)$ as a fixed pair of points and consider the positive roots of $T(\lambda|a,b,c,d)$ for different $(a,b)$. To this end, we first show that $T(\lambda|a,b,c,d)$, when $(c,d)$ is fixed, has exactly $m$ distinct positive roots if, and only if, $(a,b)$ is a dual point of order $m$ of the plane curve $S$ described by the pair of parametric functions $(\psi(\lambda),\varphi(\lambda))$ defined by

$$(15) \qquad \psi(\lambda) = -3\lambda + \frac{3c}{10\lambda^2} + \frac{d}{5\lambda^3} \text{ and } \varphi(\lambda) = \frac{5}{2}\lambda^2 - \frac{c}{\lambda} - \frac{d}{2\lambda^2} \text{ for } \lambda > 0.$$

Before doing so, let us observe that $\psi$ and $\varphi$ are smooth for $\lambda > 0$, that

$$(16) \qquad \lim_{\lambda\to+\infty} (\psi(\lambda),\varphi(\lambda)) = (-\infty,+\infty),$$

and that

$$(17) \qquad \psi'(\lambda) = \frac{-3}{5\lambda^4}P(\lambda|c,d),\ \varphi'(\lambda) = \frac{1}{\lambda^3}P(\lambda|c,d),$$

where $P(\lambda|c,d)$ is the polynomial defined by (8), which, in view of Lemma 9, can have at most two distinct positive roots. Therefore, for each $\lambda > 0$ which is not a root of $P(\lambda|c,d)$, we have

$$(18) \qquad \frac{\varphi'(\lambda)}{\psi'(\lambda)} = -\frac{5}{3}\lambda \quad\text{and}\quad \frac{\frac{d}{d\lambda}\frac{\varphi'(\lambda)}{\psi'(\lambda)}}{\psi'(\lambda)} = \frac{25\lambda^4}{9}\frac{1}{P(\lambda|c,d)}.$$

**Lemma 10.** *Let $c,d$ be fixed real numbers. Let $T(\lambda|a,b,c,d)$ be defined by (9) and let $S$ be the plane curve defined by (15). For any given $\alpha \in R$, let $y = \hbar(\lambda|\alpha)$ be the function*[1] *defined by*

$$(19) \qquad \hbar(\lambda|\alpha) = -\frac{5}{3}\lambda(\alpha - \psi(\lambda)) + \varphi(\lambda) \textit{ for } \lambda > 0.$$

*The following results hold:*

(i) $(a,b)$ is a dual point of order $m$ of $S$ if, and only if, $\hbar(\lambda|a) = b$ has exactly $m$ distinct positive solutions;

[1] This function is called the sweeping function in [1, p.24].

(ii) $T(\lambda|a,b,c,d)$ has exactly $m$ distinct positive roots if, and only if, $\hbar(\lambda|a) = b$ has exactly $m$ distinct positive solutions;

(iii) $\overline{\lambda}$ is a positive root and an extremal point of $T(\lambda|a,b,c,d)$ if, and only if, $\overline{\lambda}$ is a positive solution of $(\psi(\lambda), \varphi(\lambda)) = (a,b)$ and

$$\left(\psi(\overline{\lambda}-\delta) - \psi(\overline{\lambda})\right)\left(\psi(\overline{\lambda}+\delta) - \psi(\overline{\lambda})\right) < 0 \tag{20}$$

for all sufficiently small positive number $\delta$.

*Proof.* Since $P(\lambda|c,d)$ can have at most two positive roots by Lemma 9, we see from (17) that for each $\lambda_0 > 0$, $\psi(\lambda) \neq \psi(\lambda_0)$ for all $\lambda \in \mathbf{R}^+ \backslash \{\lambda_0\}$ which is also sufficiently close to $\lambda_0$. Since

$$\lim_{\lambda\to\lambda_0} \frac{\varphi(\lambda)-\varphi(\lambda_0)}{\psi(\lambda)-\psi(\lambda_0)} = \lim_{\lambda\to\lambda_0} \frac{\varphi'(\lambda)}{\psi'(\lambda)} = -\frac{5}{3}\lambda_0,$$

by Lemma 1, the straight line

$$L_{\lambda_0}(x) = \frac{-5}{3}\lambda_0(x-\psi(\lambda_0)) + \varphi(\lambda_0),\ x \in \mathbf{R},$$

is the tangent line of $S$ at the point $(\psi(\lambda_0), \varphi(\lambda_0))$. So by (19),

$$\hbar(\lambda|a) = L_\lambda(a) \text{ for any } a \in \mathbf{R}.$$

In other words, $\hbar(\lambda|a)$ can be interpreted as the $y$-coordinate of the point of intersection of the vertical straight line $x = a$ with the tangent line of $S$ at the point $(\psi(\lambda), \varphi(\lambda))$. Therefore, if there is a tangent line of $S$ at $(\psi(\lambda), \varphi(\lambda))$ that passes through the point $(a,b)$, then $\hbar(\lambda|a) = b$. Conversely, if $\hbar(\lambda|a) = b$ for some $\lambda > 0$, then there is a tangent line of the graph of $S$ at $(\psi(\lambda), \varphi(\lambda))$ that passes through the points $(a,b)$. The proof of the statement (i) is complete.

Next, by substituting $\psi(\lambda)$ and $\varphi(\lambda)$ into (19), we may easily obtain

$$\hbar(\lambda|a) = b - \frac{5}{2\lambda^2}T(\lambda|a,b,c,d). \tag{21}$$

Clearly, if $\overline{\lambda}$ is a positive root of $T(\lambda|a,b,c,d)$, then $\overline{\lambda}$ is a positive solution of $\hbar(\lambda|a) = b$. The converse is also true. The proof of statement (ii) is complete.

To prove statement (iii), we may assume without loss of generality that $\overline{\lambda}$ is a positive root as well as a local minimal point of $T(\lambda|a,b,c,d)$. We first assert that $\overline{\lambda}$ is a maximal point of $\hbar(\lambda|a)$. Indeed, since $T(\overline{\lambda} \pm \delta|a,b,c,d) \geq 0$ for all sufficiently small positive number $\delta$, by (21), $\hbar(\overline{\lambda}|a) = b$ and

$$\hbar(\overline{\lambda} \pm \delta|a) = b - \frac{5}{2\lambda^2}T(\overline{\lambda} \pm \delta|a,b,c,d) \leq b = \hbar(\overline{\lambda}|a)$$

for all sufficiently small positive number $\delta$. So $\overline{\lambda}$ is a local maximal point of $\hbar(\lambda|a)$. But since from (19), we have

$$\hbar'_\lambda(\lambda|a) = -\frac{5}{3}(a-\psi(\lambda)) + \frac{5}{3}\lambda\psi'(\lambda) + \varphi'(\lambda) = \frac{-5}{3}(a-\psi(\lambda)), \tag{22}$$

we see further from $0 = \hbar'_\lambda(\overline{\lambda}|a)$ that $\psi(\overline{\lambda}) = a$. In addition, by (19) and (21), we see that $\varphi(\overline{\lambda}) = b$.

Next we need to show that (20) holds for all sufficiently small positive $\delta$. To this end, since $\overline{\lambda}$ is the positive root which is a local minimal point of the polynomial $T(\lambda|a,b,c,d)$, we see that there is a natural number $n$ such that $T_\lambda^{(i)}(\overline{\lambda}|a,b,c,d) = 0$ and $T_\lambda^{(2n+1)}(\overline{\lambda}|a,b,c,d) \neq 0$ for $i = 0,1,...,2n$. By (21),

$$\hbar'_\lambda(\lambda|a) = -\frac{5}{2\lambda^3}G(\lambda|a,b,c,d), \tag{23}$$

where

$$G(\lambda|a,b,c,d) = \lambda T'_\lambda(\lambda|a,b,c,d) - 2T(\lambda|a,b,c,d).$$

Since

$$G_\lambda^{(j)}(\lambda|a,b,c,d) = \lambda T_\lambda^{(j+1)}(\lambda|a,b,c,d) + (j-2)T_\lambda^{(j)}(\lambda|a,b,c,d) \text{ for } j \geq 0,$$

we see that $G_\lambda^{(i)}(\overline{\lambda}|a,b,c,d) = 0$ and $G_\lambda^{(2n)}(\overline{\lambda}|a,b,c,d) \neq 0$ for $i = 0,1,...,2n-1$. Thus, $\overline{\lambda}$ is the positive root of $G(\lambda|a,b,c,d)$ with odd multiplicities. Since $G(\lambda|a,b,c,d)$ is also a polynomial in $\lambda$, $G(\overline{\lambda}+\delta|a,b,c,d)G(\overline{\lambda}-\delta|a,b,c,d) < 0$ for all sufficiently small positive number $\delta$. From (23), we may conclude that $\hbar'_\lambda(\overline{\lambda}+\delta|a)\hbar'_\lambda(\overline{\lambda}-\delta|a) < 0$ for all sufficiently small positive number $\delta$. By (22), the condition (20) holds for all sufficiently small positive $\delta$.

Conversely, assume $\overline{\lambda}$ is a positive solution of $(\psi(\overline{\lambda}), \varphi(\overline{\lambda})) = (a,b)$ and the condition (20) holds for any sufficiently small positive $\delta$. By (19) and (21), it is easy to see that $\overline{\lambda}$ is a positive root of $T(\lambda|a,b,c,d)$. We are going to prove that $\overline{\lambda}$ is an extremal point of $T(\lambda|a,b,c,d)$. Without loss of generality we may assume $\psi(\overline{\lambda}-\delta) - \psi(\overline{\lambda}) > 0$ and $\psi(\overline{\lambda}+\delta) - \psi(\overline{\lambda}) < 0$ for all sufficiently small positive number $\delta$. By (22), $\hbar(\lambda|a)$ is strictly increasing for $\lambda < \overline{\lambda}$ and $\lambda$ is sufficiently close to $\overline{\lambda}$, and $\hbar(\lambda|a)$ is strictly decreasing for $\lambda > \overline{\lambda}$ and $\lambda$ is sufficiently close to $\overline{\lambda}$. Thus, we see that $\overline{\lambda}$ is a local maximal point of $\hbar(\lambda|a)$. Then

$$\hbar(\overline{\lambda}|a) = b \geq \hbar(\overline{\lambda} \pm \delta|a),$$

and hence

$$T(\overline{\lambda} \pm \delta|a,b,c,d) = \frac{2\lambda^2}{5}(b - \hbar(\overline{\lambda} \pm \delta|a)) \geq 0 = T(\overline{\lambda}|a,b,c,d)$$

for all sufficiently small positive number $\delta$. In other words, $\overline{\lambda}$ is a local minimal point of $T$. The proof of statement (iii) is complete.

The statements (i) and (ii) in the above result asserts that $T$ has exactly $m$ distinct positive roots if, and only if, $(a, b)$ is a dual point of order $m$ of the curve $S$. The next thing we need to do is to investigate the distribution of dual points of $S$. To this end, **for each fixed pair** $(c, d)$, **let** $\Omega_m(c, d)$ **be the set of all dual points of order** $m$ **of** $S$.

As seen before, the behavior of $S$ defined by (15) depends on the properties of $P(\lambda|c, d)$, which, as seen in Lemma 4, in turn depends on the location of $(c, d)$. Therefore, we need to consider different but exhaustive cases:

(i) $(c, d) \in (\Gamma_1' \cup \Gamma_0)\backslash\{(0, 0)\}$,
(ii) $(c, d) = (0, 0)$,
(iii) $(c, d) \in \Gamma_1''$, and
(iv) $(c, d) \in \Gamma_2$.

Suppose $(c, d) \in (\Gamma_1' \cup \Gamma_0)\backslash\{(0, 0)\}$. First, we assume that $(c, d) \in \Gamma_0\backslash\{(0, 0)\}$. Then from (15), $(\psi(0^+), \varphi(0^+)) = (+\infty, -\infty)$. By Lemma 9, $P(\lambda|c, d) > 0$ for $\lambda > 0$. Hence $\psi(\lambda)$, in view of (17), is strictly decreasing on $(0, \infty)$ (see Figure 7(a)). Solving $\lambda$ from $\psi(\lambda) = x$ and then substituting it into $\varphi(\lambda)$, we may then see from Lemma 8 that $S$ is also the graph of a smooth function $y = S(x)$ over $\mathbf{R}$ (see Figure 7(b)). By the chain rule and other previously obtained information related to $\psi$ and $\varphi$, we may then see that $S$ is strictly decreasing and strictly convex on $\mathbf{R}$, that $S(+\infty) = -\infty$ and that $S'(-\infty) = -\infty$ as well as $S'(+\infty) = 0$. The latter two properties imply, by Lemma 2, that $S \sim H_{+\infty}$ and $S \sim H_{-\infty}$. Second, if $(c, d) \in \Gamma_1'$, then by Lemma 9, $P(\lambda|c, d)$ has exactly one positive root $\varsigma$ but $P(\lambda|c, d) > 0$ for $\lambda \in \mathbf{R}^+\backslash\{\varsigma\}$. Hence $\psi$ is strictly decreasing on $(0, \infty)$. Furthermore, in view of (18), we may then infer from Lemma 8 that $S$ is again the graph of a smooth function over $\mathbf{R}$. By similar arguments in the previous case, we may also see that $S$ is a strictly convex function over $\mathbf{R}$ such that $S \sim H_{+\infty}$ and $S \sim H_{-\infty}$. We may now invoke Lemma 3 to conclude that (see Figure 7(b)) if $(c, d) \in (\Gamma_1' \cup \Gamma_0)\backslash\{(0, 0)\}$, then

$$\Omega_0(c, d) = \{(a, b) \in \mathbf{R}^2 : b > S(a)\},$$

$$\Omega_1(c, d) = \{(a, b) \in \mathbf{R}^2 : b = S(a)\},$$

$$\Omega_2(c, d) = \{(a, b) \in \mathbf{R}^2 : b < S(a)\}.$$

Note that since $\Omega_0(c, d), \Omega_1(c, d)$ and $\Omega_2(c, d)$ together form a partition of the $a, b$-plane, all other dual sets of $S$ are empty. Furthermore, since $\psi(\lambda)$ is strictly decreasing on $\mathbf{R}$, we see that for each $\lambda > 0$,

$$(\psi(\lambda - \delta) - \psi(\lambda)) (\psi(\lambda + \delta) - \psi(\lambda)) < 0 \tag{24}$$

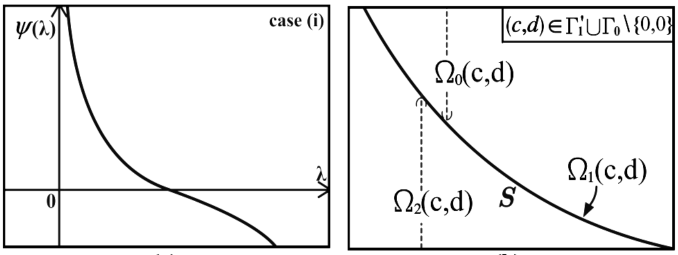



Fig. 7.

holds for any sufficiently small positive number $\delta$. By Lemma 10(iii), we may then state the additional conclusion:

(P1) Suppose $(c,d) \in (\Gamma_1' \cup \Gamma_0)\backslash\{(0,0)\}$. Then $T(\lambda|a,b,c,d)$ has a (unique) positive root which is also an extremal point if, and only if, $(a,b)$ lies in $\Omega_1(c,d)$

Next, suppose $(c,d) = (0,0)$. Then by (15), $(\psi(0^+),\varphi(0^+)) = (0,0)$. By Lemma 9, $P(\lambda|c,d) > 0$ for $\lambda > 0$. Hence, by (17), $\psi(\lambda)$ is strictly decreasing on $\mathbf{R}^+$ (see Figure 8(a)). By (15)-(18), we may then see that the curve $S$ is also the graph (see Figure 8(b)) of a function $y = S(x)$ which is strictly decreasing, strictly convex and smooth over $(-\infty, 0)$ such that $S'(0^-) = 0$ and $S \sim H_{-\infty}$. We may now invoke Lemma 5 to conclude that (see Figure 8(b))

$$\Omega_0(0,0) = \{(a,b) \in \mathbf{R}^2 : (a,b) \in \vee(S) \oplus \triangledown(\Theta_0)\},$$

$$\Omega_1(0,0) = \Omega_1^s(0,0) \cup \Omega_1^{sc}(0,0),$$

$$\Omega_2(0,0) = \{(a,b) \in \mathbf{R}^2 : (a,b) \in \wedge(S) \oplus \vee(\Theta_0\chi_{(-\infty,0)})\},$$

where

$$\Omega_1^s(0,0) = \{(a,b) \in \mathbf{R}^2 : b = S(a)\}$$

and

$$\Omega_1^{sc}(0,0) = \{(a,b) \in \mathbf{R}^2 : a < 0 \text{ and } b \le 0\} \cup \{(a,b) \in \mathbf{R}^2 : a \ge 0 \text{ and } b < 0\}.$$

Note that since $\Omega_0(0,0), \Omega_1(0,0)$ and $\Omega_2(0,0)$ together form a partition of the $a,b$-plane, all other dual sets of $S$ are empty. Furthermore, since $\psi(\lambda)$ is strictly decreasing on $\mathbf{R}$, we see that (24) holds for any sufficiently small positive number $\delta$. By Lemma 10(iii), we may then state the additional conclusion:

Next suppose $(c,d) \in \Gamma_1''$. Then from (15), $(\psi(0^+),\varphi(0^+)) = (-\infty,+\infty)$. By Lemma 9, $P(\lambda|c,d)$ has exactly one positive root $\varsigma$. Furthermore, $P(\lambda|c,d) < 0$

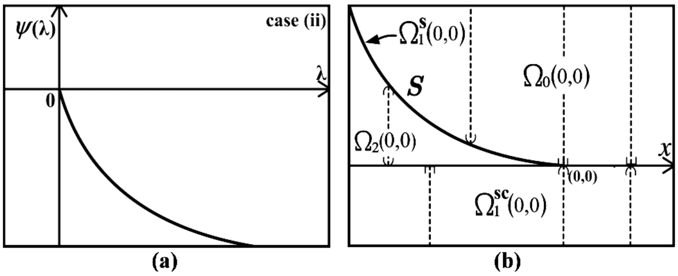



Fig. 8.

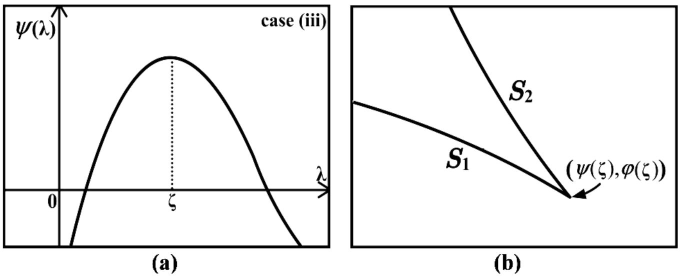



Fig. 9.

(P2) Suppose $(c, d) = (0, 0)$. Then $T(\lambda|a, b, c, d)$ has a positive root which is also a local extremal point if, and only if, $(a, b)$ lies in $\Omega_1^s(0, 0)$.

for $\lambda \in (0, \varsigma)$ and $P(\lambda|c, d) > 0$ for $\lambda \in (\varsigma, +\infty)$. Hence by (17), $\psi(\lambda)$ is strictly increasing on $(0, \varsigma)$ and strictly decreasing on $(\varsigma, +\infty)$ and $\psi(\varsigma)$ is the global maximum of $\psi$ over $(0, \infty)$. By means of these information together with (17)-(18), we may easily check that the curve $S$ is composed of two pieces $S_1$ and $S_2$ (see Figure 9(b)), the first piece $S_1$ corresponds to the case where $\lambda \in (0, \varsigma]$ and the second $S_2$ corresponds to the case where $\lambda \in (\varsigma, +\infty)$. Furthermore, $S_1$ is the graph of a function $y = S_1(x)$ which is strictly decreasing, strictly concave, and smooth over $(-\infty, \psi(\varsigma)]$ such that $-S_1 \sim H_{-\infty}$; and $S_2$ is the graph of a function $y = S_2(x)$ which is strictly decreasing, strictly convex, and smooth over $(-\infty, \psi(\varsigma))$ such that $S_2 \sim H_{-\infty}$.

As in the previous case, we may now invoke distribution maps for dual points of plane curves to make conclusions about the dual sets of $S$. Unfortunately, such maps are not available and we need to seek alternate means. To this end, recall the sweeping function $\hbar(\lambda|\alpha)$ defined by (19) for any $\alpha \in \mathbf{R}$. By Lemma 10, we may solve the equation $\hbar(\lambda|a) = b$ for each pair $(a, b)$ in order to determine whether $(a, b)$ is a dual point of order $m$. Let us therefore look into this mater. First, by (21),

$$\hbar(\lambda|a) = b - \frac{5}{2\lambda^2}\left(\lambda^4 + \frac{2a}{3}\lambda^3 + \frac{2b}{5}\lambda^2 + \frac{c}{5}\lambda + \frac{d}{15}\right),\ \lambda > 0.$$

Hence $\hbar(0^+|a) = +\infty$ and $\hbar(+\infty|a) = -\infty$.

Pick a pair $(a, b)$ in the plane such that $a < \psi(\varsigma)$. By means of the properties of $\psi(\lambda)$ mentioned above, the equation $\psi(\lambda) = a$ has exactly two positive solutions $\lambda_{\min} \in (0, \varsigma)$ and $\lambda_{\max} \in (\varsigma, +\infty)$ (see Figure 10(a)) so that

$$a < \psi(\lambda) \text{ for } \lambda \in (\lambda_{\min}, \lambda_{\max}) \text{ and } \psi(\lambda) < a \text{ for } \lambda \in \mathbf{R}^+\backslash[\lambda_{\min}, \lambda_{\max}].$$

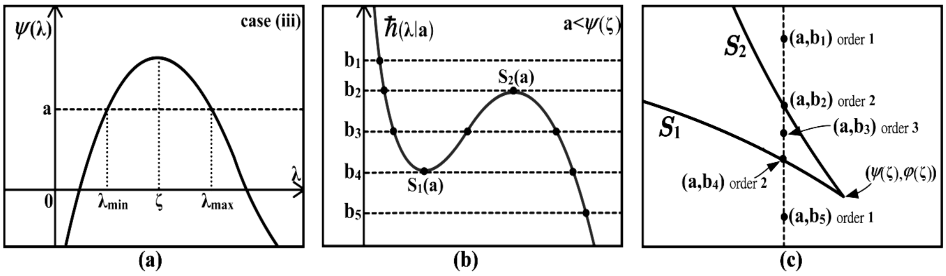



Fig. 10.

By (22), we see that $\hbar(\lambda|a)$ is strictly decreasing on $\mathbf{R}^+\backslash[\lambda_{\min}, \lambda_{\max}]$ and $\hbar(\lambda|a)$ is strictly increasing on $(\lambda_{\min}, \lambda_{\max})$. So $\lambda_{\min}$ is a local minimal point of $\hbar$ and $\lambda_{\max}$ is a local maximal point of $\hbar$. Furthermore,

$$\hbar(\lambda_{\min}|a) = \varphi(\lambda_{\min}) = S_1(\psi(\lambda_{\min})) = S_1(a)$$

and

$$\hbar(\lambda_{\max}|a) = \varphi(\lambda_{\max}) = S_2(\psi(\lambda_{\max})) = S_2(a).$$

If $b = b_1 > S_2(a)$ (see Figure 10(b)), then $\hbar(\lambda|a) = b_1$ has exactly one positive solution. Therefore $(a, b_1)$ is a dual point of order 1 of $S$. If $b = b_2 = S_2(a)$, then $\hbar(\lambda|a) = b_1$ has exactly two positive solutions. Therefore $(a, b_2)$ is a dual point of order 2. For similar reasons, $(a, b_3)$, where $b_3 \in (S_1(a), S_2(a))$, is a dual point of order 3; $(a, b_4)$, where $b_4 = S_1(a)$, is a dual point of order 2; and $(a, b_5)$, where $b_5 < S_1(a)$, is a dual point of order 1 of $S$. See Figure 10(c).

Pick another point $(a, b)$ in the plane such that $a \geq \psi(\varsigma)$. By (22), we see that $\hbar(\lambda|a)$ is strictly decreasing on $\mathbf{R}^+$ (see Figures 11(a) and 11(d)). For the same reason we have just explained, the equation $\hbar(\lambda|a) = b$ has exactly one positive solution. Hence $(a, b)$ is a dual point of order 1 of $S$ (see Figures 11(c) and 11(f)).

We may now conclude that (see Figure 12) if $(c, d) \in \Gamma_1''$, then

$$\Omega_1(c, d) = \big(\wedge(S_1\chi_{(-\infty,\psi(\varsigma))}) \oplus \vee(S_2)\big) \cup \{(a, b) \in \mathbf{R}^2 : a \geq \psi(\varsigma)\},$$

$$\Omega_2(c, d) = \{(a, b) \in \mathbf{R}^2 : a < \psi(\varsigma) \text{ and } b = S_1(a)\} \cup \{(a, b) \in \mathbf{R}^2 : a < \psi(\varsigma) \text{ and } b = S_2(a)\},$$

and

$$\Omega_3(c,d) = \{(a,b) \in \mathbf{R}^2 : S_1(a) < b < S_2(a)\}.$$

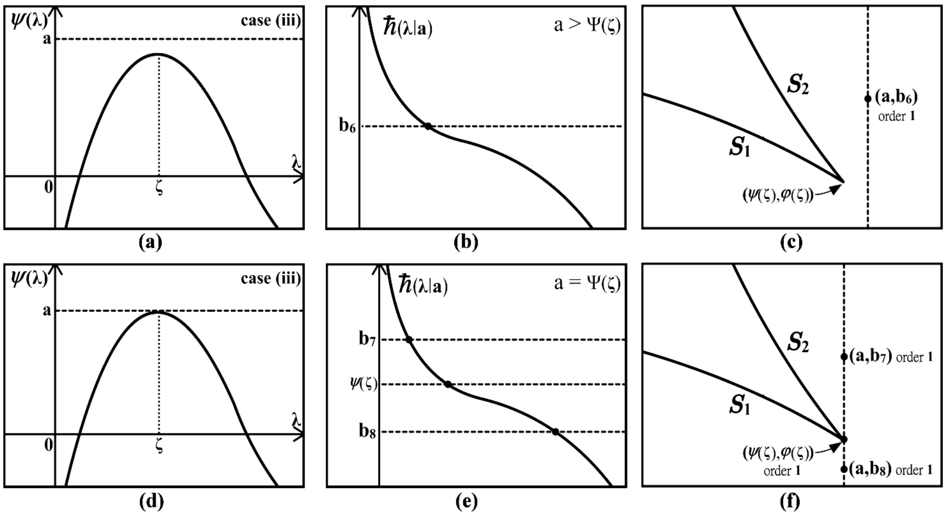



Fig. 11.

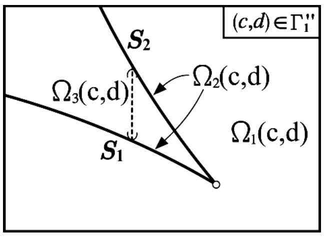



Fig. 12.

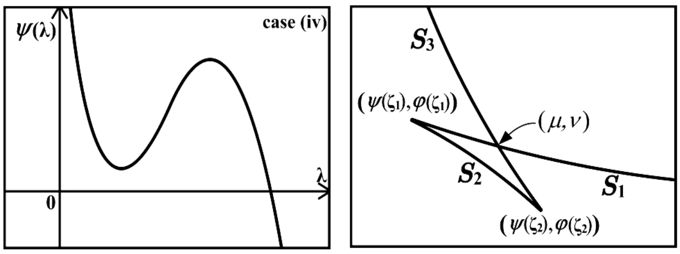
By means of the monotonic properties of $\psi(\lambda)$, the condition (24) is satisfied for all positive $\lambda$ except $\lambda = \varsigma$, we may make several additional conclusions:

(P3) Suppose $(c,d) \in \Gamma_1''$. Then $T(\lambda|a,b,c,d)$ has a positive root which is also an extremal point if, and only if, $(a,b) \in \Omega_2(c,d)$. Furthermore, when $a < \psi(\varsigma)$ and $b = S_1(a)$, the smallest positive root of $T(\lambda|a,b,c,d)$ is a local extremal point and the largest positive root of $T(\lambda|a,b,c,d)$ is not; and when $a < \psi(\varsigma)$ and $b = S_2(a)$, the largest positive root of $T(\lambda|a,b,c,d)$ is a local extremal point and the smallest positive root of $T(\lambda|a,b,c,d)$ is not.

Indeed, when $a < \psi(\varsigma)$ and $b = S_1(a)$, then $(a,b) \in \Omega_2(c,d)$ so that $T(\lambda|a,b,$ $c,d)$ has exactly two positive roots $r_1$ and $r_2$ with $r_1 < r_2$. Furthermore, $(a,b) =$

$(\psi(r_1), \varphi(r_1))$ and $(a, b) \neq (\psi(r_2), \varphi(r_2))$. Thus, $r_1$ is a local extremal point and $r_2$ is not by Lemma 10(iii). The other assertion is similarly proved.

Finally suppose $(c, d) \in \Gamma_2$. Then by (15), $(\psi(0^+), \varphi(0^+)) = (+\infty, -\infty)$. By Lemma 9, $P(\lambda|c, d)$ has exactly two positive roots $\varsigma_1$ and $\varsigma_2$ such that $P(\lambda|c, d) > 0$ for $\lambda \in \mathbf{R}^+ \backslash [\varsigma_1, \varsigma_2]$ and $P(\lambda|c, d) < 0$ for $\lambda \in (\varsigma_1, \varsigma_2)$. By (17), $\psi(\lambda)$ is strictly decreasing on $\mathbf{R}^+ \backslash [\varsigma_1, \varsigma_2]$ and strictly increasing on $(\varsigma_1, \varsigma_2)$. See Figure 13(a).




(a) (b)

Fig. 13.

By means of these information, we may easily see that the curve $S$ defined by (15) is composed of three pieces $S_1$, $S_2$ and $S_3$ (see Figure 13(b)). The first piece $S_1$ corresponds to the case where $\lambda \in (0, \varsigma_1]$; the second $S_2$ corresponding to the case where $\lambda \in (\varsigma_1, \varsigma_2)$; and the third $S_3$ corresponding to the case where $\lambda \in [\varsigma_2, +\infty)$. By the elementary properties of the functions $\psi(\lambda)$ and $\varphi(\lambda)$, it is easy to see that $S_1$ is the graph of a function $y = S_1(x)$ which is strictly decreasing, strictly convex, and smooth over $[\psi(\varsigma_1), +\infty)$ such that $S_1 \sim H_{+\infty}$; $S_2$ is the graph of a function $y = S_2(x)$ which is strictly decreasing, strictly concave, and smooth over $(\psi(\varsigma_1), \psi(\varsigma_2))$; and $S_3$ is the graph of a function $y = S_3(x)$ which is strictly decreasing, strictly convex, and smooth over $(-\infty, \psi(\varsigma_2)]$ such that $S_3 \sim H_{-\infty}$. By the properties of $S_1$, $S_2$ and $S_3$ just described, it is not difficult to see that there is exactly one point $(\mu, \nu)$ of intersection of $S_1$ and $S_3$, such that

(25) $$S_1(x) < S_3(x) \text{ for } x \in (\psi(\varsigma_1), \mu)$$

and

(26) $$S_1(x) > S_3(x) \text{ for } x \in (\mu, \psi(\varsigma_1)).$$

To find the dual sets of $S$, we again make use of the sweeping function $\hbar(\lambda|\alpha)$ defined by (19) for any $\alpha \in \mathbf{R}$. We first note, in view of (21), that $\hbar(0^+|a) = \hbar(+\infty|a) = -\infty$.

Pick an arbitrary point $(a, b)$ in the plane such that $\psi(\varsigma_1) < a < \psi(\varsigma_2)$. Then by the monotonic properties of $\psi(\lambda)$ mentioned above, the equation $\psi(\lambda) = a$ has exactly three positive solutions $\lambda_1$, $\lambda_2$ and $\lambda_3$ in $(0, \varsigma_1)$, $(\varsigma_1, \varsigma_2)$ and $(\varsigma_2, +\infty)$

respectively. Furthermore, $a < \psi(\lambda)$ for $\lambda \in (0, \lambda_1) \cup (\lambda_2, \lambda_3)$ and $\psi(\lambda) < a$ for $\lambda \in (\lambda_1, \lambda_2) \cup (\lambda_3, +\infty)$. By (22), $\hbar(\lambda|a)$ is strictly decreasing on $(0, \lambda_1) \cup (\lambda_2, \lambda_3)$ and is strictly increasing on $(\lambda_1, \lambda_2) \cup (\lambda_3, +\infty)$. So $\lambda_1$ and $\lambda_3$ are local maximal points of $\hbar$, $\lambda_2$ is a local minimal point of $\hbar$ and $\hbar(\lambda_i|a) = S_i(a)$ for $i = 1, 2, 3$. See Figure 14.

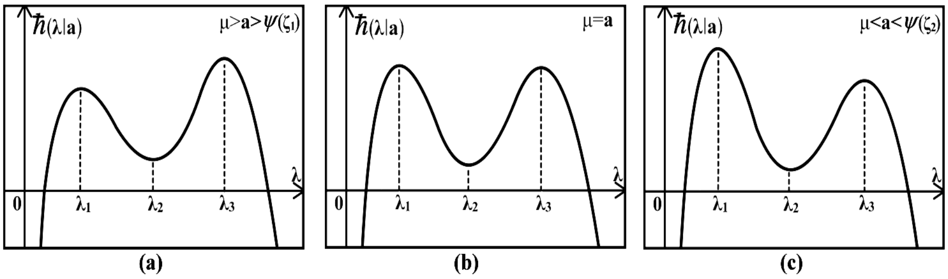



Fig. 14.

If $a \in (\psi(\varsigma_1), \mu)$, we see that $S_1(a) < S_3(a)$ by (25) (see Figure 14(a)). Therefore, if $b > S_3(a)$, then $\hbar(\lambda|a) = b$ has no positive solutions; if $b = S_3(a)$, then $\hbar(\lambda|a) = b$ has exactly one positive solution; if $S_1(a) < b < S_3(a)$ or $b < S_2(a)$, then $\hbar(\lambda|a) = b$ has exactly two positive solutions; if $b = S_1(a)$ or $b = S_2(a)$, then $\hbar(\lambda|a) = b$ has exactly three positive solutions; and if $S_2(a) < b < S_1(a)$, then $\hbar(\lambda|a) = b$ has exactly four positive solutions. See Figure 15.

A similar situation, by symmetry considerations, can be established for the case where $a \in (\mu, \psi(\varsigma_2))$ (see Figure 14(c)). As for the case where $a = \mu$, we have $S_1(a) = S_3(a)$ (see Figure 14(b)). Therefore, if $b > S_1(a)$, $\hbar(\lambda|a) = b$ has no positive solutions; if $b = S_1(a)$ or $b < S_2(a)$, then $\hbar(\lambda|a) = b$ has exactly two positive solutions; if $b = S_2(a)$, then $\hbar(\lambda|a) = b$ has exactly three positive solutions; and if $S_2(a) < b < S_1(a)$, then $\hbar(\lambda|a) = b$ has exactly four positive solutions.

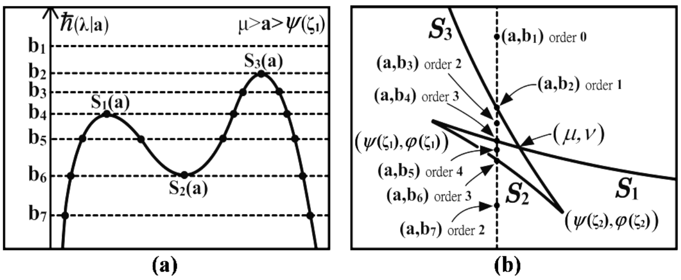



Fig. 15.

Pick another arbitrary point $(a, b)$ such that $a \geq \psi(\varsigma_2)$. Then by the monotonic properties of $\psi(\lambda)$ mentioned above, the equation $\psi(\lambda) = a$ has a positive solutions $\lambda_5 < \varsigma_1$. We have $a \leq \psi(\lambda)$ for $\lambda \in (0, \lambda_5)$ and $\psi(\lambda) < a$ for $\lambda \in (\lambda_5, +\infty)$. By (21), $\hbar(\lambda|a)$ is strictly increasing on $(0, \lambda_5)$ and is strictly decreasing on $(\lambda_5, +\infty)$. So $\lambda_5$ is a local extremal point of $\hbar$ and $\hbar(\lambda_5|a) = S_1(a)$. See Figure 16(b). If $b = b_1 > S_1(a)$, the equation $\hbar(\lambda|a) = b$ has no positive solutions; if $b = b_2 = S_1(a)$, $\hbar(\lambda|a) = b$ has exactly one positive solution; and if $b = b_3 < S_1(a)$, $\hbar(\lambda|a) = b$ has exactly two positive solutions. See Figure 17.

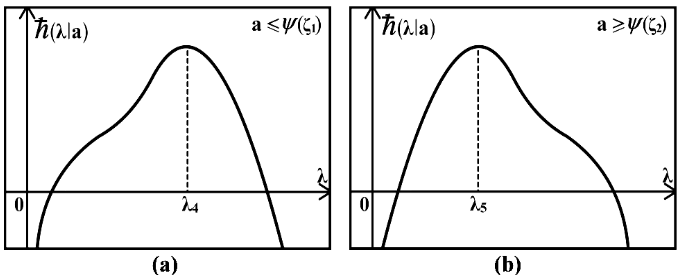



Fig. 16.

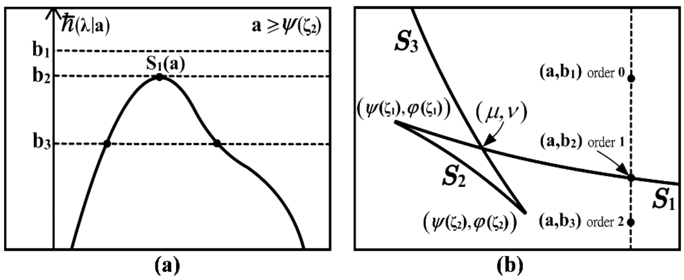



Fig. 17.

Finally if we pick an arbitrary point $(a, b)$ such that $a \leq \psi(\varsigma_1)$. Then by symmetry considerations (see Figure 16(a)), we see that if $b > S_1(a)$ or $b = S_1(a)$ or $b < S_1(a)$, then $\hbar(\lambda|a) = b$ has no positive solutions, or exactly one positive solution or exactly two positive solutions respectively.

According to our previous discussions, we may now classify all the points in the $a, b$-plane and conclude that (see Figure 18) if $(c, d) \in \Gamma_2$, then

$$\Omega_0(c, d) = \vee(S_1) \oplus \vee(S_3),$$

$$\Omega_1(c, d) = \{(a, b) \in \mathbf{R}^2 : a < \mu \text{ and } b = S_3(a)\} \cup \{(a, b) \in \mathbf{R}^2 : a > \mu \text{ and } b = S_1(a)\},$$

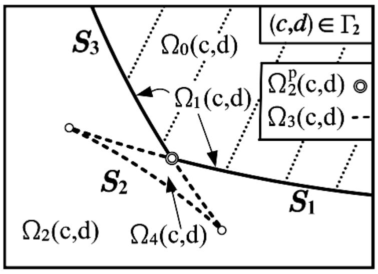



Fig. 18.

$$\Omega_3(c,d) = \{(a,b) \in \mathbf{R}^2 : \psi(\varsigma_1) < a < \mu \text{ and } b = S_1(a)\} \cup \{(a,b) \in \mathbf{R}^2 : b = S_2(a)\}$$

$$\cup\{(a,b) \in \mathbf{R}^2 : \mu < a < \psi(\varsigma_2) \text{ and } b = S_3(a)\}$$

$$\Omega_4(c,d) = \wedge(S_1\chi_{(\psi(\varsigma_1),\mu,]}) \oplus \wedge(S_3\chi_{[\mu,\psi(\varsigma_2))}) \oplus \vee(S_2),$$

and

$$\Omega_2(c,d) = \mathbf{R}^2\backslash\{\Omega_0(c,d) \cup \Omega_1(c,d) \cup \Omega_3(c,d) \cup \Omega_4(c,d)\}.$$

Note that the set

$$\Omega_2^p(c,d) = \{(a,b) \in \mathbf{R}^2 : b = S_1(a) \text{ and } b = S_3(a)\}$$

is just the point set $\{(\mu,\nu)\}$ which is a part of $\Omega_2(c,d)$.

By means of the monotonic properties of $\psi(\lambda)$, the condition (20) holds for every positive $\lambda$ except the points $\varsigma_1$ and $\varsigma_2$. Therefore we may make the following additional conclusion:

(P4) Suppose $(c,d) \in \Gamma_2$. Then $T(\lambda|a,b,c,d)$ has a positive root which is also a local extremal point if, and only if, $(a,b) \in \Omega_1(c,d) \cup \Omega_2^p(c,d) \cup \Omega_3(c,d)$. Furthermore, if $(a,b) \in \Omega_2^p(c,d)$, then both positive roots $r_1$ and $r_2$ of $T$ are local extremal points; and if $(a,b) \in \Omega_3(c,d)$, then letting $r_1$, $r_2$ and $r_3$ be the three (distinct) positive roots of $T$ such that $r_1 < r_2 < r_3$, we see that when $b = S_i(a)$ for some $i \in \{1,2,3\}$, the root $r_i$ is a local extremal point of $T$ and the other two are not.

Indeed, if $(a,b) \in \Omega_2^p(c,d)$, then $T(\lambda|a,b,c,d)$ has exactly two positive roots $r_1$ and $r_2$ such that $r_1 < r_2$ and $(a,b) = (\psi(r_i),\varphi(r_i))$, $i = 1,2$. By Lemma 10(iii), $r_1$ and $r_2$ are local extremal points. If $(a,b) \in \Omega_3(c,d)$, when $b = S_1(a)$, then $(a,b) = (\psi(r_1),\varphi(r_1))$ and $(a,b) \neq (\psi(r_j),\varphi(r_j))$ for $j = 2,3$. Thus, $r_1$ is a local extremal point, and $r_2$ and $r_3$ are not by Lemma 10(iii). The cases when $b = S_2(a)$ and when $b = S_3(a)$ are similarly proved.

We may infer from the previous discussions some of the usual properties of the quartic polynomial $T$ (such as the maximum number of positive roots) as well as some new ones. In particular, the following result will be needed in the derivation of our main results about our sextic polynomial $Q$.

**Theorem 3.** *Given a fixed pair* $(c, d)$. *Let* $T(\lambda|a, b, c, d)$ *be defined by (9). Let* $\Gamma_0$, $\Gamma_1'$, $\Gamma_1''$ *and* $\Gamma_2$ *be defined by* (*10*), (*11*), (*12*) *and* (*13*), *respectively.*

(*1*) *Suppose one of the following holds:*

(*a*) $(a, b) \in \Omega_0(c, d)$ *where* $(c, d) \in \Gamma_1' \cup \Gamma_0 \cup \Gamma_2$;

(*b*) $(a, b) \in \Omega_1(c, d)$ *where* $(c, d) \in \Gamma_1' \cup \Gamma_0 \cup \Gamma_2 \backslash \{(0, 0)\}$;

(*c*) $(a, b) \in \Omega_1^s(0, 0)$;

(*d*) $(a, b) \in \Omega_2^p(c, d)$ *where* $(c, d) \in \Gamma_2$.

*Then* $T(\lambda|a, b, c, d) \geq 0$ *for all* $\lambda > 0$.

(*2*) *Suppose one of the following holds:*

(*a*) $(a, b) \in \Omega_1(c, d)$ *where* $(c, d) \in \Gamma_1''$;

(*b*) $(a, b) \in \Omega_1^{sc}(0, 0)$;

(*c*) $(a, b) \in \Omega_2(c, d)$ *where* $(c, d) \in \Gamma_1''$.

*Then there is a positive roots* $r$ *of* $T(\lambda|a, b, c, d)$ *such that* $T(\lambda|a, b, c, d) \leq 0$ *on* $(0, r)$ *and* $T(\lambda|a, b, c, d) \geq 0$ *on* $[r, +\infty)$.

(*3*) *Suppose one of the following holds:*

(*a*) $(a, b) \in \Omega_2(c, d)$ *where* $(c, d) \in \Gamma_1' \cup \Gamma_0$;

(*b*) $(a, b) \in \Omega_2(c, d) \backslash \Omega_2^p(c, d)$ *where* $(c, d) \in \Gamma_2$;

(*c*) $(a, b) \in \Omega_3(c, d)$ *where* $(c, d) \in \Gamma_2$.

*Then there are two positive roots* $r_1$ *and* $r_2$ *of* $T(\lambda|a, b, c, d)$ *with* $r_1 < r_2$ *such that* $T(\lambda|a, b, c, d) \geq 0$ *on* $\mathbf{R}^+ \backslash (r_1, r_2)$ *and* $T(\lambda|a, b, c, d) \leq 0$ *on* $(r_2, r_3)$.

(*4*) *Suppose* $(a, b) \in \Omega_3(c, d)$ *where* $(c, d) \in \Gamma_1''$. *Then there are exactly three positive roots* $r_1$ ,$r_2$ *and* $r_3$ *of* $T(\lambda|a, b, c, d)$ *such that* $T(\lambda|a, b, c, d) > 0$ *on* $(r_1, r_2) \cup (r_3, +\infty)$ *and* $T(\lambda|a, b, c, d) < 0$ *on* $(0, r_1) \cup (r_2, r_3)$.

(*5*) *Suppose* $(a, b) \in \Omega_4(c, d)$ *where* $(c, d) \in \Gamma_2$. *Then there are exactly four positive roots* $r_1$, $r_2$, $r_3$ *and* $r_4$ *of* $T(\lambda|a, b, c, d)$ *such that* $T(\lambda|a, b, c, d) > 0$ *on* $(0, r_1) \cup (r_2, r_3) \cup (r_4, +\infty)$ *and* $T(\lambda|a, b, c, d) < 0$ *on* $(r_1, r_2) \cup (r_3, r_4)$.

*Proof.* First note that $T(+\infty|a, b, c, d) = +\infty$, and $T(0^+|a, b, c, d) = d/15$.

In case (1)-(a), $(a, b)$ is a dual point of order 0 and hence $T(\lambda|a, b, c, d)$ does not have any positive roots. Since $d \geq 0$, we see further that $T(\lambda|a, b, c, d) > 0$

on $\mathbf{R}^+$. In cases (1)-(b) and (1)-(c), $(a, b)$ is a dual point of order 1 and hence $T(\lambda|a, b, c, d)$ has a unique positive root $r$ and $r$ is a local extremal point by (P1), (P2) and (P4). Since $d \geq 0$, we see further that $T(\lambda|a, b, c, d) \geq 0$ on $\mathbf{R}^+$. In case (1)-(d), $(a, b)$ is a dual point of order 2 and hence $T(\lambda|a, b, c, d)$ has exactly two positive roots $r_1$ and $r_2$, and they are local extremal points by (P4). Since $d > 0$, we see further that $T(\lambda|a, b, c, d) \geq 0$ on $\mathbf{R}^+$. The proof of statement (1) is complete.

In cases (2)-(a) and (2)-(b), $T(\lambda|a, b, c, d)$ has a unique positive root $r$ and $r$ is not an extremal point by (P1), (P2), (P3) and (P4). Since $d \leq 0$, we see further that $T(\lambda|a, b, c, d) < 0$ on $(0, r)$ and $T(\lambda|a, b, c, d) > 0$ on $(r, +\infty)$. In case (2)-(c), we have $d \leq 0$ and $T(\lambda|a, b, c, d)$ has exactly two positive roots $r_1$ and $r_2$ with $r_1 < r_2$. By (P4) and (P5), either $r_1$ or $r_2$ is a local extremal point but not both. Assume $r_1$ is a local extremal point. Then $T(\lambda|a, b, c, d) < 0$ on $(0, r_2)\backslash\{r_1\}$ and $T(\lambda|a, b, c, d) > 0$ on $(r_2, +\infty)$. Take $r = r_2$. So $T(\lambda|a, b, c, d) \leq 0$ on $(0, r)$ and $T(\lambda|a, b, c, d) \geq 0$ on $[r, +\infty)$. In a similar manner, we may prove the case where $r_2$ is a local extremal point. The proof of statement (2) is complete.

In cases (3)-(a) and (3)-(b), we have $d \geq 0$ and $T(\lambda|a, b, c, d)$ has exactly two positive roots $r_1$ and $r_2$ and neither are extremal points by (P1), (P2), (P3) and (P4). So $T(\lambda|a, b, c, d) \geq 0$ on $\mathbf{R}^+\backslash(r_1, r_2)$ and $T(\lambda|a, b, c, d) < 0$ on $(r_1, r_2)$. In case (3)-(c), we have $d > 0$ and $T(\lambda|a, b, c, d)$ has exactly three positive roots $\lambda_1$, $\lambda_2$ and $\lambda_3$ with $\lambda_1 < \lambda_2 < \lambda_3$. By (P4), exactly one of $\lambda_1$, $\lambda_2$ and $\lambda_3$ is a local extremal point. Assume $\lambda_1$ is a local extremal point. Then $T(\lambda|a, b, c, d) > 0$ on $\{(0, \lambda_2)\backslash\{\lambda_1\}\} \cup (\lambda_3, +\infty)$ and $T(\lambda|a, b, c, d) < 0$ on $(\lambda_2, \lambda_3)$. Take $r_1 = \lambda_2$ and $r_2 = \lambda_3$. So $T(\lambda|a, b, c, d) \geq 0$ on $\mathbf{R}^+\backslash(r_1, r_2)$ and $T(\lambda|a, b, c, d) \leq 0$ on $(r_1, r_2)$. The other two cases are similarly proved. The proof of statement (3) is complete.

In case (4), we have $d \leq 0$ and $T(\lambda|a, b, c, d)$ has exactly three positive roots $r_1, r_2$ and $r_3$, and none are not extremal points by (P3). So $T(\lambda|a, b, c, d) > 0$ on $(r_1, r_2) \cup (r_3, +\infty)$ and $T(\lambda|a, b, c, d) < 0$ on $(0, r_1) \cup (r_2, r_3)$. The proof of statement (4) is complete.

In case (5), we have $d > 0$ and $T(\lambda|a, b, c, d)$ has exactly four positive roots $r_1$, $r_2$, $r_3$ and $r_4$, and none are extremal points by (P4). So $T(\lambda|a, b, c, d) > 0$ on $(0, r_1) \cup (r_2, r_3) \cup (r_4, +\infty)$ and $T(\lambda|a, b, c, d) < 0$ on $(r_1, r_2) \cup (r_3, r_4)$. The proof of (5) is complete.

We remark that in the above result, all possible cases of the pairs $(c, d)$ and $(a, b)$ have been discussed. Indeed, either $(c, d) \in \Gamma_0 \cup \Gamma_1'\backslash\{(0, 0)\}$, or, $(c, d) = (0, 0)$, or, $(c, d) \in \Gamma_1''$, or $(c, d) \in \Gamma_2$.

For $(c, d) \in \Gamma_0 \cup \Gamma_1'\backslash\{(0, 0)\}$, the sets $\Omega_0(c, d)$, $\Omega_1(c, d)$ and $\Omega_2(c, d)$ are considered in 3(1)(a), 3(1)(b) and 3(3)(a) respectively.

For $(c, d) = (0, 0)$, the sets $\Omega_0(0, 0)$, $\Omega_1^s(0, 0)$, $\Omega_1^{sc}(0, 0)$ and $\Omega_2(0, 0)$ are considered in 3(1)(a), 3(1)(c), 3(2)(b) and 3(3)(a) respectively.

For $(c, d) \in \Gamma_1''$, the sets $\Omega_1(c, d)$, $\Omega_2(c, d)$ and $\Omega_3(c, d)$ are considered in

3(2)(a), 3(2)(c), and 3(4) respectively.

For $(c, d) \in \Gamma_2$, the sets $\Omega_0(c, d), \Omega_1(c, d), \Omega_2(c, d)\backslash\Omega_2^p(c, d), \Omega_2^p(c, d), \Omega_3(c, d)$, and $\Omega_4(c, d)$ are considered in 3(1)(a), 3(1)(b), 3(3)(b), 3(1)(d), 3(3)(c), and 3(5) respectively.

## 4. Characteristic Regions of Sextic Polynomials

Consider the function $Q(\lambda|a, b, c, d, x, y)$ defined by (1). For each $\lambda > 0$, let $L_\lambda$ be the straight line in the plane defined by

$$L_\lambda : \lambda x + y = -(\lambda^6 + a\lambda^5 + b\lambda^4 + c\lambda^3 + d\lambda^2). \tag{27}$$

Note that $L_\lambda$ defined by (27) is of the form (3) and $f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda) = 1 \neq 0$. From (4), we let $G$ be the curve defined by the parametric functions

$$\begin{aligned} x(\lambda) &= -(6\lambda^5 + 5a\lambda^4 + 4b\lambda^3 + 3c\lambda^2 + 2d\lambda) \\ &\text{and } y(\lambda) = 5\lambda^6 + 4a\lambda^5 + 3b\lambda^4 + 2c\lambda^3 + d\lambda^2 \end{aligned} \tag{28}$$

for $\lambda > 0$. Then

$$x'(\lambda) = -30T(\lambda),\ y'(\lambda) = 30\lambda T(\lambda), \tag{29}$$

where $T(\lambda) = T(\lambda|a, b, c, d)$ is given by (9). Let $\Sigma = \{\lambda > 0 : T(\lambda|a, b, c, d) = 0\}$. According to our previous discussions about $T$, the positive roots of $T$ are finite in number and isolated, hence $\Sigma$ is a finite set. Furthermore, $x'(\lambda) = 0$ if, and only if, $\lambda \in \Sigma$. We see that

$$\frac{y'(\lambda)}{x'(\lambda)} = -\lambda < 0 \text{ for } \lambda \in \mathbf{R}^+\backslash\Sigma \tag{30}$$

and

$$\lim_{\lambda\to d^-} \frac{y'(\lambda)}{x'(\lambda)} = \lim_{\lambda\to d^+} \frac{y'(\lambda)}{x'(\lambda)} = -d < 0 \text{ for } d \in \Sigma. \tag{31}$$

By Theorem 1, $G$ is the envelope of the family $\{L_\lambda : \lambda > 0\}$ where $L_\lambda$ is defined by (27). We have

$$\lim_{\lambda\to 0^+} (x(\lambda), y(\lambda)) = (0, 0),\quad \lim_{\lambda\to +\infty} (x(\lambda), y(\lambda)) = (-\infty, +\infty),$$

and

$$\frac{\frac{d}{d\lambda}\left(\frac{y'(\lambda)}{x'(\lambda)}\right)}{x'(\lambda)} = \frac{1}{30T(\lambda)} \text{ for } \lambda \in \mathbf{R}^+\backslash\Sigma. \tag{32}$$

In the following results, we will find the exact region containing the coefficients $a, b, c, d, \alpha, \beta$ of $Q$ such that it has no positive roots. To this end, we first consider $(c, d)$ in different combinations of $\Gamma_0, \Gamma_1', \Gamma_2''$ and $\Gamma_2$ which are defined by (10), (11), (12) and (13) respectively. Then given a fixed pair $(c, d)$ in such a combination, we consider $(a, b)$ in different $\Omega_i(c, d)$, $i = 0, 1, 2, 3, 4$, where $\Omega_i(c, d)$ is the dual set of order of $i$ of the curve $S$ described by the parametric functions in (16). Finally, for these fixed parameters $a, b, c, d$, we only need to find the $\mathbf{C}\backslash(0, +\infty)$-characteristic region of (1) containing $(\alpha, \beta)$. In view of Theorem 2, the desired region is nothing but the dual set of order 0 of the envelope $G$ described by (28).

**Theorem 4.** (See Figure 19). *Assume $a, b, c, d, \alpha, \beta \in \mathbf{R}$. Let the parametric curve $G$ be defined by (28). Suppose the hypothesis of Theorem 3(1) holds. Then $(\alpha, \beta)$ is a point of the $\mathbf{C}\backslash(0, +\infty)$-characteristic region of (1) if, and only if, $(\alpha, \beta) \in \vee(G) \oplus \overline{\nabla}(\Theta_0)$.*

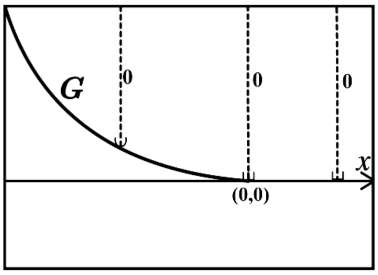



Fig. 19.

*Proof.* By Theorem 3(1), $T(\lambda|a, b, c, d) \geq 0$ for $\lambda > 0$. Since $\Sigma$ is finite and $x'(\lambda) < 0$ for $\lambda \in \mathbf{R}^+\backslash\Sigma$ by (29), $G$ is the graph of a smooth function $y = G(x)$ over $(-\infty, 0)$ by Lemma 8. By (30)-(32) and Lemma 2, we may then see that $G$ is strictly decreasing, strictly convex and $G \sim H_{-\infty}$. Then $L_{G|0} = \Theta_0$ by $G(0^-) = G'(0^-) = 0$. By Lemma 5, the dual set of order 0 of $G$ is $\vee(G) \oplus \overline{\nabla}(\Theta_0)$. The proof is complete.

**Theorem 5.** (See Figure 20). *Assume $a, b, c, d, \alpha, \beta \in \mathbf{R}$. Let the parametric curve $G$ be defined by (28). Suppose the hypothesis of Theorem 3(2) holds. Then $(\alpha, \beta)$ is a point of the $\mathbf{C}\backslash(0, +\infty)$-characteristic region of (1) if, and only if, $(\alpha, \beta) \in \vee(G_2) \oplus \overline{\nabla}(\Theta_0)$, where $G_2$ is the part of the parametric curve $G$ restricted to the interval $[r, +\infty)$ and $r$ is a positive root of $T(\lambda|a, b, c, d)$ which is not an extremal point.*

*Proof.* By Theorem 3(2), there is a positive root $r$ of $T(\lambda|a, b, c, d)$ such that $T(\lambda|a, b, c, d) \leq 0$ on $(0, r)$ and $T(\lambda|a, b, c, d) \geq 0$ on $[r, +\infty)$. Since $\Sigma$ is finite, $T(r+\delta|a, b, c, d)T(r-\delta|a, b, c, d) < 0$ for all sufficiently small positive $\delta$ and hence $r$ cannot be a local extremal point of $T(\lambda|a, b, c, d)$. The curve $G$ is composed of two pieces $G_1$ and $G_2$ restricted to the interval $(0, r)$ and to the interval $[r, +\infty)$, respectively. By (29), $x'(\lambda) > 0$ on $(0, r)\backslash\Sigma$ and $x'(\lambda) < 0$ on $[r, +\infty)\backslash\Sigma$. Then the curve $G_1$ is the graph of the smooth function $y = G_1(x)$ over $(0, x(r))$ and the curve $G_2$ is the graph of the smooth function $y = G_2(x)$ over $(-\infty, x(r)]$ by Lemma 8. By (30)-(32) and Lemma 2, we may see that $G_1$ is strictly decreasing and strictly concave such that $G_1'(0^+) = 0$ and $L_{G_1|0} = \Theta_0$; and $G_2$ is strictly decreasing and strictly convex such that $G_2 \sim H_{-\infty}$. Furthermore

$$G_1^{(v)}(x(r)^-) = G_2^{(v)}(x(r)^-),\ v = 0, 1.$$

By Theorem A.3 in [1], the intersection of the dual sets of order 0 of $G_1$ and $G_2$ is $\vee(G_2) \oplus \overline{\vee}(\Theta_0)$. So the dual set of order 0 of $G$ is also $\vee(G_2) \oplus \overline{\vee}(\Theta_0)$. The proof is complete.

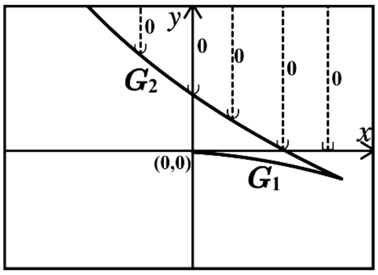



Fig. 20.

**Theorem 6.** (See Figure 21). *Assume $a, b, c, d, \alpha, \beta \in \mathbf{R}$. Let the parametric curve $G$ be defined by (28). Suppose the hypothesis Theorem 3(3) holds. Then $(\alpha, \beta)$ is a point of the $\mathbf{C}\backslash(0, +\infty)$-characteristic region of (1) if, and only if, $(\alpha, \beta) \in \vee(G_1) \oplus \vee(G_3) \oplus \overline{\vee}(\Theta_0)$, where $G_1$ is the part of the parametric curve $G$ restricted to the interval $(0, r_1]$ and $G_3$ is the part of the parametric curve $G$ restricted to the interval $[r_2, +\infty)$, and $r_1 < r_2$ are the positive roots of $T(\lambda|a, b, c, d)$ which are not extremal points.*

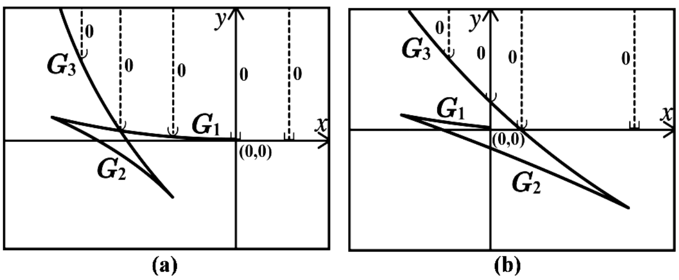



Fig. 21.

*Proof.* By Theorem 3(3), there are two positive roots $r_1$ and $r_2$ of $T(\lambda|a,b,c,d)$ with $r_1 < r_2$ such that $T(\lambda|a,b,c,d) \geq 0$ on $\mathbf{R}^+\backslash(r_1,r_2)$ and $T(\lambda|a,b,c,d) \leq 0$ on $(r_1,r_2)$. Since $\Sigma$ is finite, $r_1$ and $r_2$ are not local extremal points. The curve $G$ is composed of three pieces $G_1$, $G_2$ and $G_3$ restricted respectively to $(0,r_1]$, $(r_1,r_2)$ and $[r_2,+\infty)$. By (29), $x'(\lambda) > 0$ on $(r_1,r_2)\backslash\Sigma$ and $x'(\lambda) < 0$ on $(0,r_1]\backslash\Sigma \cup [r_2,+\infty)\backslash\Sigma$. By (28)-(32), Lemma 2 and Lemma 8, the curve $G_1$ is the graph of the function $y = G_1(x)$ which is a strictly decreasing, strictly convex, and smooth function over $[x(r_1),0)$ such that $G_1'(0^-) = 0$ and $L_{G_1|0} = \Theta_0$; the curve $G_2$ is the graph of the function $y = G_2(x)$ which is a strictly decreasing, strictly concave, and smooth function over $(x(r_1),x(r_2))$; and the curve $G_3$ is the graph of the function $y = G_3(x)$ which is a strictly decreasing, strictly convex, and smooth function over $(-\infty, x(r_2)]$ such that $G_3 \sim H_{-\infty}$. Furthermore,

$$G_1^{(v)}(x(r_1)^+) = G_2^{(v)}(x(r_1)^+) \text{ and } G_2^{(v)}(x(r_2)^-) = G_3^{(v)}(x(r_2)^-),\ v = 0,1.$$

By Theorem A.13 in [1] (which describes essentially the same distribution maps in Figure 21 and hence need not be repeated here), the intersection of the dual sets of order 0 of $G_1$, $G_2$ and $G_3$ is $\vee(G_1) \oplus \vee(G_3) \oplus \overline{\nabla}(\Theta_0)$. So the dual set of order 0 of $G$ is also $\vee(G_1) \oplus \vee(G_3) \oplus \overline{\nabla}(\Theta_0)$. The proof is complete.

**Theorem 7.** (See Figure 22). *Assume $a,b,c,d,\alpha,\beta \in \mathbf{R}$ and $(a,b) \in \Omega_3(c,d)$ where $(c,d) \in \Gamma_1''$. Let the parametric curve $G$ be defined by (28). Then $(\alpha,\beta)$ is a point of the $\mathbf{C}\backslash(0,+\infty)$-characteristic region of (1) if, and only if, $(\alpha,\beta) \in \vee(G_2) \oplus \vee(G_4) \oplus \overline{\nabla}(\Theta_0)$, where $G_2$ is the part of the parametric curve $G$ restricted to the interval $[r_1,r_2]$ and $G_4$ is the part of the parametric curve $G$ restricted to the interval $[r_3,+\infty)$, and $r_1, r_2$ and $r_3$ are the positive roots of $T(\lambda|a,b,c,d)$.*

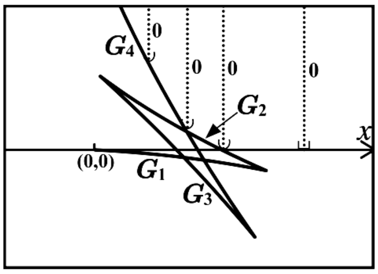



Fig. 22.

*Proof.* By Theorem 3(4), $T(\lambda|a,b,c,d)$ has exactly three positive roots $r_1$, $r_2$ and $r_3$ with $r_1 < r_2 < r_3$ such that $T(\lambda|a,b,c,d) > 0$ on $(r_1,r_2) \cup (r_3,+\infty)$ and $T(\lambda|a,b,c,d) < 0$ on $(0,r_1) \cup (r_2,r_3)$; Then the curve $G$ is composed of four pieces $G_1$, $G_2$, $G_3$ and $G_4$ restricted to the intervals $(0,r_1)$, $[r_1,r_2]$, $(r_2,r_3)$ and $[r_3,+\infty)$

respectively. By (28)-(32) Lemma 2 and Lemma 8, the curve $G_1$ is the graph of the function $y = G_1(x)$ which is a strictly decreasing, strictly concave, and smooth function over $(0, x(r_1))$ such that $G_1'(0^+) = 0$ and $L_{G_1|0} = \Theta_0$; the curve $G_2$ is the graph of the function $y = G_2(x)$ which is a strictly decreasing, strictly convex, and smooth function over $[x(r_2), x(r_1)]$; the curve $G_3$ is the graph of the function $y = G_3(x)$ which is a strictly decreasing, strictly concave, and smooth function over $(x(r_2), x(r_3))$; and the curve $G_4$ is the graph of the function $y = G_4(x)$ which is a strictly decreasing, strictly convex, and smooth function over $(-\infty, x(r_3)]$ such that $G_4 \sim H_{-\infty}$. Furthermore,

$$G_1^{(v)}(x(r_1)^-) = G_2^{(v)}(x(r_1)^-),\ G_2^{(v)}(x(r_2)^+) = G_3^{(v)}(x(r_2)^+),$$

and

$$G_3^{(v)}(x(r_3)^-) = G_4^{(v)}(x(r_3)^-),\ v = 0, 1.$$

By Lemma 7, the intersection of the dual sets of order 0 of $G_1, G_2, G_3$ and $G_4$ is then the desired characteristic region which, in view of the distribution map Figure 22, is also easy to see and is $\vee(G_2) \oplus \vee(G_4) \oplus \overline{\nabla}(\Theta_0)$. So the dual set of order 0 of $G$ is $\vee(G_2) \oplus \vee(G_4) \oplus \overline{\nabla}(\Theta_0)$. The proof is complete.

**Theorem 8.** (See Figure 23). *Assume $a, b, c, d, \alpha, \beta \in \mathbf{R}$ and $(a, b) \in \Omega_4(c, d)$ where $(c, d) \in \Gamma_2$. Let the parametric curve $G$ be defined by (28). Then $(\alpha, \beta)$ is a point of the $\mathbf{C}\backslash(0, +\infty)$-characteristic region of (1) if, and only if, $(\alpha, \beta) \in \vee(G_1) \oplus \vee(G_3) \oplus \vee(G_5) \oplus \overline{\nabla}(\Theta_0)$, where $G_1$ is the part of the parametric curve $G$ restricted to the interval $(0, r_1]$, $G_3$ is the part of the parametric curve $G$ restricted to the interval $[r_2, r_3]$, and $G_5$ is the part of the parametric curve $G$ restricted to the interval $[r_4, +\infty)$, and $r_1$, $r_2$, $r_3$ and $r_4$ are the positive roots of $T(\lambda|a, b, c, d)$.*

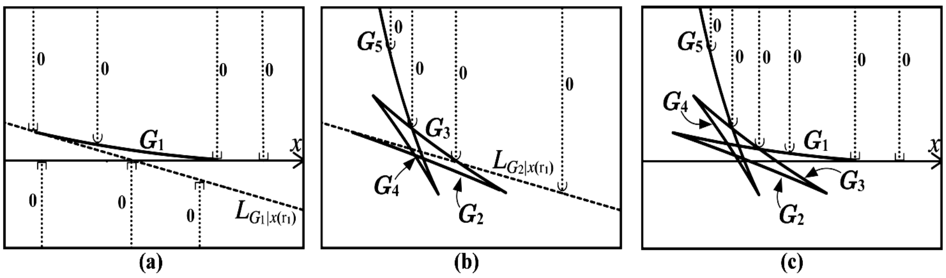



Fig. 23.

*Proof.* By Theorem 3(5), $T(\lambda|a, b, c, d)$ has exactly four positive roots $r_1$, $r_2$, $r_3$ and $r_4$ such that $T(\lambda|a, b, c, d) > 0$ on $(0, r_1) \cup (r_2, r_3) \cup (r_4, +\infty)$ and $T(\lambda|a, b, c, d) < 0$ on $(r_1, r_2) \cup (r_3, r_4)$. Then the curve $G$ is composed of five pieces $G_1$, $G_2$, $G_3$, $G_4$ and $G_5$ restricted to the intervals $(0, r_1)$, $[r_1, r_2)$, $[r_2, r_3]$,

$(r_3, r_4)$ and $[r_4, +\infty)$, respectively. By (28)-(32), Lemma 2 and Lemma 8, then the curve $G_1$ is the graph of the function $y = G_1(x)$ which is a strictly decreasing, strictly convex, and smooth function over $(x(r_1), 0)$ such that $G_1'(0^-) = 0$ and $L_{G_1|0} = \Theta_0$; the curve $G_2$ is the graph of the function $y = G_2(x)$ which is a strictly decreasing, strictly concave, and smooth function over $[x(r_1), x(r_2))$; the curve $G_3$ is the graph of the function $y = G_3(x)$ which is a strictly decreasing, strictly convex, and smooth function over $[x(r_3), x(r_2)]$; the curve $G_4$ is the graph of the function $y = G_4(x)$ which is a strictly decreasing, strictly concave, and smooth function over $(x(r_3).x(r_4))$; and the curve $G_5$ is the graph of the function $y = G_5(x)$ which is a strictly decreasing, strictly convex, and smooth function over $(-\infty, x(r_4)]$ such that $G_4 \sim H_{-\infty}$. Furthermore,

$$G_1^{(v)}(x(r_1)^+) = G_2^{(v)}(x(r_1)^+),\ G_2^{(v)}(x(r_2)^-) = G_3^{(v)}(x(r_2)^-),$$

$$G_3^{(v)}(x(r_3)^+) = G_4^{(v)}(x(r_3)^+) \text{ and } G_4^{(v)}(x(r_4)^-) = G_5^{(v)}(x(r_4)^-),\ v = 0, 1.$$

By Lemma 4, the dual set of order 0 of $G_1$ is the union of $\vee(G_1) \oplus \overline{\nabla}(L_{G_1|x(r_1)}) \oplus \overline{\nabla}(\Theta_0)$ and $\underline{\triangle}(L_{G_1|x(r_1)}) \oplus \underline{\triangle}(\Theta_0)$. By Lemma 7, the intersection of the dual sets of order 0 of $G_2, G_3, G_4$ and $G_5$ is $\vee(G_3) \oplus \vee(G_5) \oplus \overline{\nabla}(L_{G_2|x(r_1)})$. Hence, the intersection of the dual sets of order 0 of $G_1, G_2, G_3$, $G_4$ and $G_5$ is then the desired characteristic region which, in view of the distribution map Figure 23, is also easy to see and is equal to $\vee(G_1) \oplus \vee(G_3) \oplus \vee(G_5) \oplus \overline{\nabla}(\Theta_0)$. So the dual set of order 0 of $G$ is $\vee(G_1) \oplus \vee(G_3) \oplus \vee(G_5) \oplus \overline{\nabla}(\Theta_0)$. The proof is complete.

As may be noted, our main results depend in part on the roots of the quartic polynomial $T$. However, the quartic is the highest order polynomial equation that can be solved by radicals in the general case (i.e., one where the coefficients can take any value). Indeed, there are now several commercial packages that can yield symbolic roots of quartic polynomials. Hence our results, although they may depend on the roots of quartic polynomials, are good enough to show absence of positive roots of sextic polynomials. We will demonstrate our results in the following section.

## 5. Examples

We first consider the absence of 3-periodic solutions of a logistic equation mentioned in the Introduction.

**Example 1.** The logistic recurrence equation

$$x_{n+1} = x_n(x_n - 1), n = 0, 1, 2, ..., \tag{33}$$

has no real periodic solutions with least period 3.

*Proof.* Let $g : R \to R$ be defined by $g(x) = x(x-1)$, and let $g^{[2]} = g \circ g$, $g^{[3]} = g \circ g \circ g$, etc. Given $x_0 = \lambda \in R$, the unique (real) solution $\{x_n\}_{n=0}^{\infty}$ of (33) is determined by $x_1 = g(\lambda)$, $x_2 = g^{[2]}(\lambda)$, etc., which is 3-periodic if, and only if, $g(\lambda) \neq \lambda$, $g^{[2]}(\lambda) \neq \lambda$ and $g^{[3]}(\lambda) = \lambda$. Since $g(\lambda) = \lambda$ if, and only if $\lambda = 0$ or 2; and since

$$g^{[2]}(\lambda) = \lambda(\lambda-1)(\lambda(\lambda-1)-1) = \lambda$$

if, and only if, $\lambda = 0$ or 2, we see that $\{x_n\}$ is a real 3-periodic solution of (33) if, and only if $\lambda$ is a real solution of $g^{[3]}(\lambda) = \lambda$ which is different from 0 and 2. In other words, (33) does not have any real 3-periodic solutions if, and only if, $g^{[3]}(\lambda) = \lambda$ does not have any real solutions other than 0 or 2. Since we may easily check that the number 0 and 2 are simple roots of $g^{[3]}(x) - x$, thus

$$g^{[3]}(x) - x = x(x-1)Q(x),$$

where $Q$ is a polynomial with no roots equal to 0 nor 2. Hence $g^{[3]}(\lambda) = \lambda$ does not have any real solutions other than 0 and 2 if, and only if $Q(\lambda)$ does not have any real roots.

We now only need to show that the sextic polynomial

$$Q(\lambda) := \frac{g(g(g(\lambda))) - \lambda}{\lambda(\lambda-2)}$$

has no real roots. By direct verification, $Q(\lambda) = \lambda^6 - 2\lambda^5 + 2\lambda^3 - \lambda^2 + 1$. Clearly, $Q(0) \neq 0$. Let $Q_1(\lambda) = Q(-\lambda) = \lambda^6 + 2\lambda^5 - 2\lambda^3 - \lambda^2 + 1$. We see that $Q(\lambda)$ has no real roots if, and only if, $Q(\lambda)$ and $Q_1(\lambda)$ have no positive roots.

First, we claim that $Q(\lambda)$ has no positive roots. Take $a = -2$, $b = 0$, $c = 2$, $d = -1$, $\alpha = 0$ and $\beta = 1$ in (1). Let the curve $S$ be defined by (15) and the curve $G$ be defined by (28) respectively. It is easy to see that $(c, d) \in \Gamma_1''$ (see Figure 24(a)). So $P(\lambda|2, -1) = 5\lambda^4 + 2\lambda - 1$ has the unique positive root $\varsigma$ by Lemma 9. By previous discussions for the case $(c, d) \in \Gamma_1''$, the curve $S$ is composed of two pieces $S_1$ and $S_2$ (see Figure 24(b)), corresponding respectively to the case where $\lambda \in (0, \varsigma]$ and to the case where $\lambda \in (\varsigma, +\infty)$. Furthermore, the curve $S_1$ is the graph of the strictly concave function $y = S_1(x)$ over $(-\infty, \psi(\varsigma)]$ and the curve $S_2$ is the graph of the strictly convex function $y = S_2(x)$ over $(-\infty, \psi(\varsigma))$. We may see that $0.42 < \varsigma < 0.84$ since $P(0.42|2, -1)P(0.84|2, -1) < 0$. So the points $(\psi(0.42), \varphi(0.42))$ and $\psi(0.84), \varphi(0.84))$ lie in the curve $S_1$ and $S_2$ respectively. Now we consider two tangent lines $L_{S_1|0.42}$ and $L_{S_2|0.84}$ defined by

$$L_{S_1|0.42}(x) = -0.7(x - \psi(0.42)) + \varphi(0.42)$$

and

$$L_{S_2|0.84}(x) = -1.4(x - \psi(0.84)) + \varphi(0.84)$$

for $x \in \mathbf{R}$. Then $L_{S_1|0.42}(-2) < 0 < L_{S_2|0.84}(-2)$ (see Figure 24(b)). Since $S_1$ is strictly concave and $S_2$ is strictly convex, we see that the curve $S_1$ lies below the line $L_{S_1|0.42}$ and the curve $S_2$ lies above the line $L_{S_2|0.84}$. Then $(-2, 0) \in \vee(S_1) \oplus \wedge(S_2)$ (see Figure 24(b)). Thus, $(a, b) \in \Omega_3(c, d)$ where $(c, d) \in \Gamma_1''$. By Lemma 10, $T(\lambda| -2, 0, 2, -1)$ has exactly three positive roots $r_1$, $r_2$ and $r_3$ such that $r_1 < r_2 < r_3$. By (numerical) computation, $r_3 = 1$ and $r_2 \approx 0.6795$. Furthermore, $x(r_3) = y(r_3) = 0$ and $y(r_2) \approx 0.1265$. Let $G_2$ be the part of the parametric curve $G$ restricted to the interval $[r_1, r_2]$ and $G_4$ the part of the parametric curve $G$ restricted to the interval $[r_3, +\infty)$. Then $G_2$ is the graph of the function $y = G_2(x)$ which is strictly decreasing over $[x(r_2), x(r_1)]$. We have $1 > 0.1265 \approx y(r_2) = G_2(x(r_2))$ and $G_2(x(r_2)) \geq G_2(x)$ for $x \in [x(r_2), x(r_1)]$. So $(0, 1) \in \vee(G_2) \oplus \overline{\nabla}(\Theta_0)$ (see Figure 24(c)). Since $G_4$ is the graph of the function $y = G_4(x)$ which is strictly decreasing over $(-\infty, 0]$ such that $G_4(0) = 0$, we see that $(0, 1) \in \vee(G_2) \oplus \vee(G_4) \oplus \overline{\nabla}(\Theta_0)$. By Theorem 7, $Q(\lambda)$ has no positive roots.

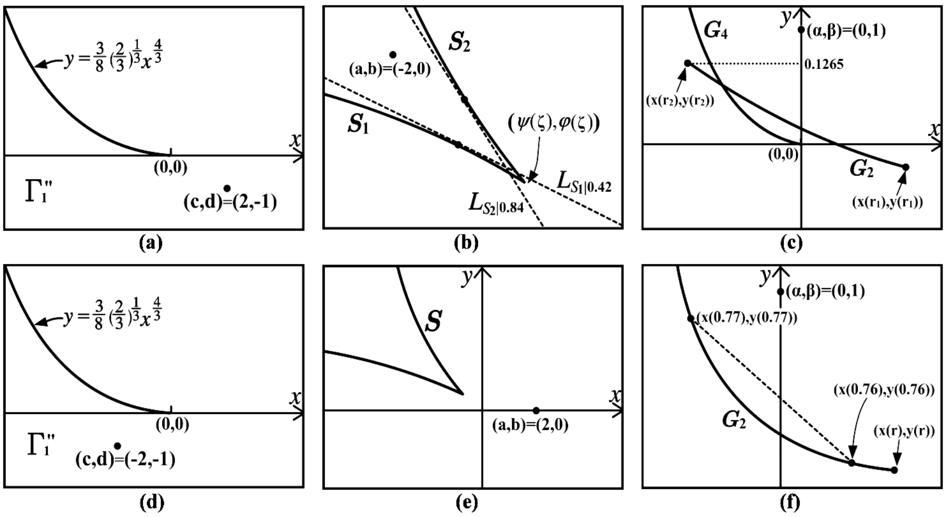



Fig. 24.

Second, we claim that $Q_1(\lambda)$ has no positive roots. Take $a = 2$, $b = 0$, $c = -2$, $d = -1$, $\alpha = 0$, $\beta = 1$. Let the curve $S$ be defined by (15) and the curve $G$ be defined by (28). It is easy to observe that $(c, d) \in \Gamma_1''$ (see Figure 24(d)). In this case

$$\psi(\lambda) = -3\lambda - \frac{3}{5\lambda^2} - \frac{1}{5\lambda^3} < 0 \text{ for } \lambda > 0.$$

So the curve $S$ lies in the left-half plane (see Figure 24(e)). Thus, $(a, b) \in \Omega_1(c, d)$ where $(c, d) \in \Gamma_1''$. By Lemma 10, $T(\lambda|2, 0, -2, -1)$ has a unique positive root $r$. Let $G_2$ be the part of the parametric curve $G$ restricted to the interval $[r, +\infty)$.

Then $G_2$ is the graph of the function $y = G_2(x)$ which is strictly decreasing and strictly convex over $(-\infty, x(r)]$. We have

$$T(0.76|2,0,-2,-1) = (0.76)^4 + \frac{4}{3}(0.76)^3 - \frac{2}{5}(0.76) - \frac{1}{15} \approx 0.54826,$$

$$(x(0.76), y(0.76)) = (0.1281, 0.6584),$$

and

$$(x(0.77), y(0.77)) = (-0.042, 0.7885).$$

Then $0.76 > r$ and the point $(0,1)$ lies strictly above the line $L$ which passes through the points $(x(0.76), y(0.76))$ and $(x(0.77), y(0.77))$. Since $G_2$ is strictly convex, $G_2(x) < L(x)$ for $x \in (x(0.77), x(0.76))$. So $(0,1) \in \vee(G_2) \oplus \overline{\nabla}(\Theta_0)$ (see Figure **??**(f)). By Theorem 5, $Q_1(\lambda)$ has no positive roots. The proof is complete.

**Example 2.** Assume $\alpha, \beta \in \mathbf{R}$, $a \geq 0$ and $b \geq 0$. Let the curve $G$ be defined by

$$x(\lambda) = -6\lambda^5 - 5a\lambda^4 - 4b\lambda^3 \text{ and } y(\lambda) = 5\lambda^6 + 4a\lambda^5 + 3b\lambda^4$$

for $\lambda > 0$. Then the sextic polynomial

$$Q(\lambda|a, b, \alpha, \beta) = \lambda^6 + a\lambda^5 + b\lambda^4 + \alpha\lambda + \beta,\ \lambda \in \mathbf{R},$$

has no positive roots if, and only if, $\alpha \geq 0$ and $\beta \geq 0$, or, $\alpha < 0$ and $(\alpha, \beta)$ lies strictly above $G$. Furthermore, if $a = b = 0$, then $Q(\lambda|0,0,\alpha,\beta)$ has no positive roots if, and only if, $\alpha \geq 0$ and $\beta \geq 0$, or, $\alpha < 0$ and $\beta > 5(\frac{\alpha}{6})^{6/5}$.

*Proof.* It is easy to observe that $(a, b) \in \Omega_0(0,0)$. By Theorem 4, $Q(\lambda|a, b, \alpha, \beta)$ has no positive roots if, and only if, $(\alpha, \beta) \in \vee(G) \oplus \overline{\nabla}(\Theta_0)$. If $a = b = 0$, then

$$y(\lambda) = 5\left(\frac{x(\lambda)}{6}\right)^{\frac{6}{5}} \text{ and } x(\lambda) < 0 \text{ for } \lambda > 0.$$

Thus, $(\alpha, \beta) \in \vee(G) \oplus \overline{\nabla}(\Theta_0)$ if, and only if, $\alpha \geq 0$ and $\beta \geq 0$, or $\alpha < 0$ and $\beta > 5(\frac{\alpha}{6})^{6/5}$. The proof is complete.

**Example 3.** The real sextic polynomial

$$Q(\lambda|\kappa) = \lambda^6 + \kappa\lambda^5 + \kappa\lambda^4 + \kappa\lambda^3 + \kappa\lambda^2 + \kappa\lambda + \kappa,\ \lambda \in \mathbf{R},$$

with one real parameter $\kappa$ has no positive roots if, and only if, $\kappa \geq 0$.

*Proof.* Suppose $\kappa < 0$. We have $Q(0|\kappa) = \kappa < 0$ and $Q(+\infty|\kappa) = +\infty$. Then $Q(\lambda|\kappa)$ has at least one positive root. Assume $\kappa \geq 0$. If $\kappa = 0$. Clearly,

$Q(\lambda|0) = \lambda^6$ has no positive roots. If $\kappa > 0$, then it is easy to observe that $(\kappa, \kappa) \in \Gamma_0$ (See Figure 25(a)). Take

$$a = b = c = d = \kappa$$

in (1). Let the curve $S$ be defined by (15) and the curve $G$ by (28). We see that the curve $S$ is the graph of a strictly decreasing function $y = S(x)$ over $\mathbf{R}$. We observe that when

$$0 = \varphi(\overline{\lambda}) = \frac{1}{2\overline{\lambda}^2}(5\overline{\lambda}^4 - 2\kappa\overline{\lambda} - \kappa)$$

for some $\overline{\lambda} > 0$, there follows

$$\psi(\overline{\lambda}) = \frac{-\kappa}{10\overline{\lambda}^3}\left(9\overline{\lambda} + 4\right) < 0.$$

Then the $x$-coordinate of the point of intersection of the $x$-axis with the curve $S$ is negative, and the $y$-coordinate of the point of intersection of the $y$-axis with the curve $S$ is negative (see Figure 25(b)). Then $(\kappa, \kappa) \in \vee(S)$. Thus $(\kappa, \kappa) \in \Omega_0(\kappa, \kappa)$ where $(\kappa, \kappa) \in \Gamma_0$. Since $G$ is the graph of the smooth function $y = G(x)$ which is strictly decreasing and strictly convex over $(-\infty, 0)$, $(\kappa, \kappa) \in \vee(G) \oplus \vee(\Theta_0)$ (see Figure 25(c)). By Theorem 4, $Q(\lambda|\kappa)$ has no positive roots. The proof is complete.

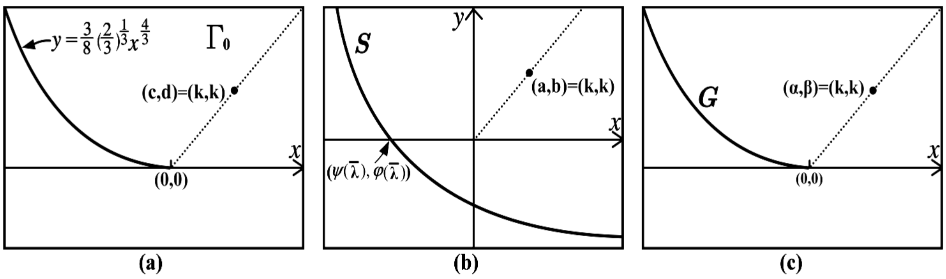



Fig. 25.

**Example 4.** Let $b, d, \alpha, \beta \in \mathbf{R}$ and $\beta \geq 0$. Let $Q(\lambda|b, d, \alpha, \beta) = \lambda^6 + b\lambda^4 + d\lambda^2 + \alpha\lambda + \beta$ for $\lambda \in \mathbf{R}$ and

$$r = \sqrt{\frac{-b + \sqrt{b^2 - \frac{5}{3}d}}{5}}$$

(1) Suppose either (a) $d > 0$ and $b \geq -\sqrt{5d/3}$; or (b) $d = 0$ and $b \geq 0$. Then $Q(\lambda|b, d, \alpha, \beta)$ has no positive roots if $\alpha \geq 0$.

(2) Suppose either (a) $d \geq 0$ and $b < -\sqrt{5d/3}$; or (b) $d < 0$. Then $Q(\lambda|b, d, \alpha, \beta)$ has no positive roots if $\alpha \geq \max\{0, (-1.6)r(br^2 + d)\}$.

*Proof.* Let the curve $S$ be described by parametric functions

$$\psi(\lambda)=\frac{-1}{5\lambda^3}(15\lambda^4-d) \text{ and } \varphi(\lambda)=\frac{1}{2\lambda^2}(5\lambda^4-d)$$

for $\lambda>0$, and the curve $G$ by parametric functions

$$x(\lambda)=-2\lambda(3\lambda^4+2b\lambda^2+d) \text{ and } y(\lambda)=\lambda^2(5\lambda^4+3b\lambda^2+d)$$

for $\lambda>0$. Observe that when $d>0$, $\psi(\lambda)$ has the unique positive root $(d/15)^{1/4}$. So $\varphi((d/15)^{1/4})=-\sqrt{5d/3}<0$ is the $y$-coordinate of the point of intersection of the vertical line $x=0$ with the curve $S$.

Assume $d>0$. Then $(0,d)\in\Gamma_0$ and the curve $S$ is the graph of a strictly decreasing function $y=S(x)$ over $\mathbf{R}$. If $b=b_1\geq-\sqrt{5d/3}$, the point $(0,b_1)$ lies above the curve $S$. Thus, $(0,b_1)\in\Omega_0(0,d)\cup\Omega_1(0,d)$ where $(0,d)\in\Gamma_0$ (see Figure 26(a)). By the properties of the curve $G$ in this case, $(\alpha,\beta)\in\vee(G)\oplus\nabla(\Theta_0)$ when $\alpha\geq0$ (see Figure 26(d)). By Theorem 4, $Q(\lambda|b,d,\alpha,\beta)$ has no positive roots when $\alpha\geq0$. So the proof under the assumption (1)(a) is completed. If $b=b_2<-\sqrt{5d/3}$. we see that $(0,b_2)\in\Omega_2(0,d)$ where $(0,d)\in\Gamma_0$ (see Figure 26(a)). By Lemma 10,

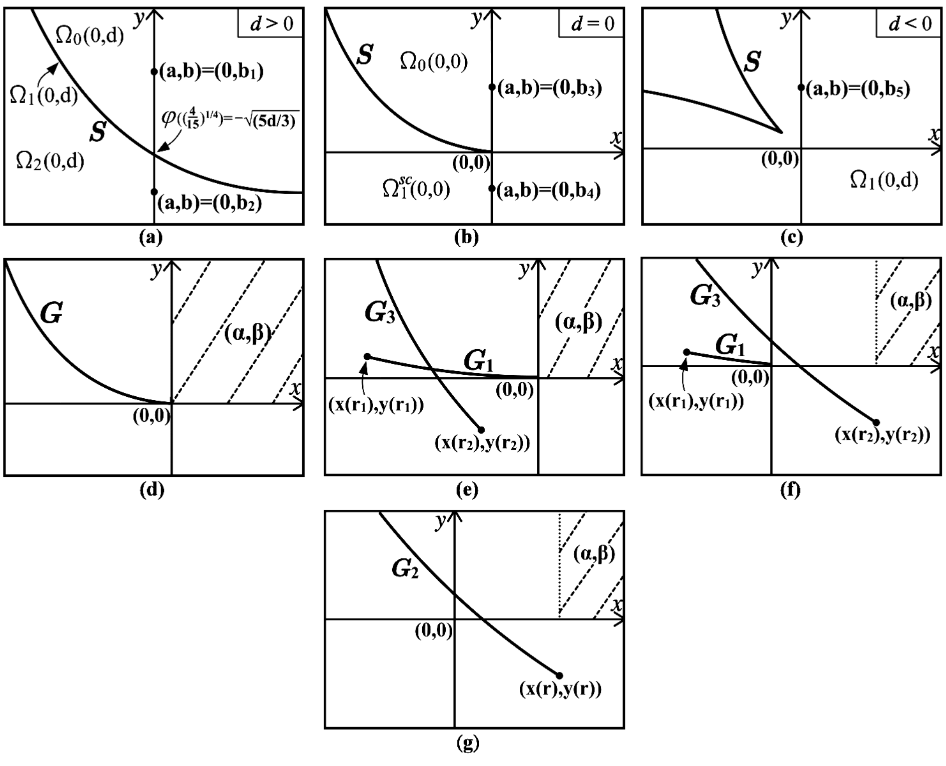



Fig. 26.

$T(\lambda|b,d) = \lambda^4 + 2b/5\lambda^2 + d/15$ has exactly two positive roots $r_1$ and $r_2$ such that $r_1 < r_2$. We may solve the positive roots of the quartic $T(\lambda|b,d)$ and check that $r_2 = r$. By Theorem 6, $Q(\lambda|b,d,\alpha,\beta)$ has no positive roots if, and only if, $(\alpha,\beta) \in \vee(G_1) \oplus \vee(G_3) \oplus \overline{\nabla}(\Theta_0)$, where $G_1$ is the part of the parametric curve $G$ restricted to the interval $(0, r_1]$ and $G_3$ is the part of the parametric curve $G$ restricted to the interval $[r_2, +\infty)$. We see that $x(r) = -8r(br^2+d)/5$. If $\alpha \geq \max\{0, x(r)\}$, then $(\alpha,\beta) \in \vee(G_1) \oplus \vee(G_3) \oplus \overline{\nabla}(\Theta_0)$ (see Figures **??**(e) and **??**(f)). It follows that $Q(\lambda|b,d,\alpha,\beta)$ has no positive roots. The proof under the assumption (2)(a) is completed.

Assume $d = 0$. Then $(0,d) = (0,0) \in \Gamma_0$ and the curve $S$ is the graph of a strictly decreasing function $y = S(x)$ over $(-\infty, 0)$. If $b = b_3 \geq 0$, then $(0, b_3) \in \Omega_0(0,0)$ (see Figure 26(b)). Similarly, by Theorem 4, $Q(\lambda|b,0,\alpha,\beta)$ has no positive roots when $\alpha \geq 0$ (see Figure 26(d)). The proof under the assumption (1)(b) is completed. If $b = b_4 < 0$, then $(0, b_4) \in \Omega_1^{sc}(0,0)$ (see Figure 26(b)). By Lemma 10, $T(\lambda|b,d) = \lambda^4 + 2b/5\lambda^2 + d/15$ has the unique positive root $r$. So $x(r) = -8br^3/5 > 0$. By Theorem 5, $Q(\lambda|b,0,\alpha,\beta)$ has no positive roots if, and only if, $(\alpha,\beta) \in \vee(G_2) \oplus \overline{\nabla}(\Theta_0)$, where $G_2$ is the part of the parametric curve $G$ restricted to the interval $[r, +\infty)$. If $\alpha \geq x(r)$, then $(\alpha,\beta) \in \vee(G_2) \oplus \overline{\nabla}(\Theta_0)$ (see Figure 26(g)). It follows that $Q(\lambda|b,d,\alpha,\beta)$ has no positive roots. The proof under the assumption (2)(a) is completed.

Assume $d < 0$. Then $(0,d) \in \Gamma_1''$. In this case, $\psi(\lambda) < 0$ and $\varphi(\lambda) > 0$ for $\lambda \in \mathbf{R}^+$. For any $b = b_5 \in \mathbf{R}$, $(0, b_5) \in \Omega_1(0,d)$ (see Figure 26(c)). By Theorem 5, $Q(\lambda|b,d,\alpha,\beta)$ has no positive roots if, and only if, $(\alpha,\beta) \in \vee(G_2) \oplus \overline{\nabla}(\Theta_0)$, where $G_2$ is the part of the parametric curve $G$ restricted to the interval $[r, +\infty)$. Similarly, if $\alpha \geq x(r)$, $Q(\lambda|b,d,\alpha,\beta)$ has no positive roots (see Figure 26(g)). The proof under the assumption (2)(b) is complete.

We remark that in example 4, we can also get more precise conditions for $\alpha$ and $\beta$ such that $Q(\lambda|b,d,\alpha,\beta)$ has no positive roots. But this conditions are cumbersome to state and do not add to further understanding.

**Example 5.** Our principal objective is to consider the absence of positive roots of the sextic polynomial $Q(\lambda|a,b,c,d,\alpha,\beta)$. We have also seen in Example 1 that by a simple symmetry transformation, the absence of real roots can also be handled. However, we like to mention that the same technique used to derive our main Theorems can also be used to handle the absence of real roots. We illustrate this by considering the simple sextic polynomial

$$Q(\lambda|d,\alpha,\beta) = \lambda^6 + d\lambda^2 + \alpha\lambda + \beta,\ \lambda \in \mathbf{R},$$

where the parameters $d, \alpha, \beta \in \mathbf{R}$ and $d > 0$. We may show that it has no real roots if, and only if,$(\alpha,\beta)$ lies strictly above the curve $G$, where $G$ is defined by the parametric functions

$$x(\lambda) = -2\lambda(3\lambda^4 + d) \text{ and } y(\lambda) = \lambda^2(5\lambda^4 + d) \text{ for } \lambda \in R.$$

Indeed, we consider the family $\{L_\lambda : \lambda \in \mathbf{R}\}$ where

$$L_\lambda : \lambda x + y = -\lambda^6 - d\lambda^2.$$

Since $x'(\lambda) = -30\lambda^4 - 2d < 0$ and $y'(\lambda) = 30\lambda^5 + 2\lambda d$ for $\lambda \in \mathbf{R}$. We see that

$$\frac{y'(\lambda)}{x'(\lambda)} = -\lambda < 0 \text{ and } \frac{\frac{d}{\lambda}\frac{y'(\lambda)}{x'(\lambda)}}{x'(\lambda)} = \frac{1}{30}(\lambda^4 + \frac{d}{15}) > 0 \text{ for } \lambda \in \mathbf{R}.$$

By Theorem 1, $G$ is the envelope of the family $\{L_\lambda : \lambda \in \mathbf{R}\}$. We have

$$(x(-\infty), y(-\infty)) = (+\infty, +\infty) \text{ and } (x(+\infty), y(+\infty)) = (-\infty, +\infty).$$

It is easy to see that the curve $G$ is the graph of a smooth function $y = G(x)$ which is strictly convex such that $G \sim H_{+\infty}$ and $G \sim H_{-\infty}$ over $\mathbf{R}$ (see Figure 27). By Lemma 3, $(\alpha, \beta)$ is a dual point of order 0 of $G$ if, and only if $(\alpha, \beta) \in \vee(G)$. By Theorem 2, $Q(\lambda|d, \alpha, \beta)$ has no real roots if, and only if $(\alpha, \beta) \in \vee(G)$, that is, $(\alpha, \beta)$ lies strictly above the graph of $G$.

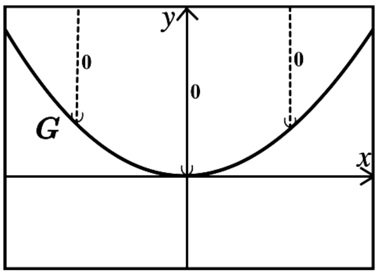



Fig. 27.

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