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THE FIRST INITIAL BOUNDARY VALUE PROBLEM FOR HYPERBOLIC SYSTEMS IN INFINITE NONSMOOTH CYLINDERS

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Abstract. This paper is concerned with the first initial boundary value problem for hyperbolic systems in infinite cylinders with the nonsmooth base. Some results on the regularity of generalized solutions of this problem are given.

1. Introduction

The theory of general boundary-value problems for partial differential equations and systems in smooth domains is nearly completely studied (see [1,3,11]). The boundary value problems for partial differential equations and systems of various classes in nonsmooth domains attract the attention of many researchers. These problems for elliptic equations and systems on domains containing conical points have been investigated in the works [7,8,10]. The first initial boundary value problem for parabolic systems in cylinders with the nonsmooth base was described in [5]. The second initial boundary value problem in cylinders with the base containing conical points has been dealt with for hyperbolic systems in [9] and for Schrodinger systems in [6], in which the authors considered the existence, uniqueness and smoothness of the generalized solution of the mentioned problems. Regularity of solutions to the first initial boundary value problem for hyperbolic systems in nonsmooth cylinders was considered in [4]. However, the results of the work [4] were obtained only in the finite cylinders.

In this paper we are concerned with the first initial boundary value problem for hyperbolic systems in infinite cylinders with the nonsmooth base. The aim of the paper is to establish some results on the regularity of generalized solutions of the problem. First, we study the regularity with respect to time variable of the generalized solutions in the infinite cylinders with nonsmooth base. After that, we can apply the results for elliptic boundary value problems to deal with the regularity

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with respect to spatial variables of the solutions in infinite cylinders with conical points.

The paper is organized in the following way. In Section 2 we introduce some notations and formulate our problem. The regularity with respect to time variable is presented in Section 3. In the last section we give results on the regularity with respect to both time and spatial variables of the generalized solutions in a neighborhood of the conical point.

2. Preliminaries

Let Ω be a bounded domain in R^n with the boundary $\partial\Omega$. Denote $\Omega_\infty = \Omega \times (0, \infty)$ and $S_\infty = \partial\Omega \times (0, \infty)$.

Let $L(x, t, D)$ be a differential operator of order $2m$:

$$L(x,t,D) \equiv \sum_{|\alpha|,|\beta|=0}^{m} D^\alpha (a_{\alpha\beta}(x,t) D^\beta) \tag{2.1}$$

where $a_{\alpha\beta} \equiv a_{\alpha\beta}(x,t)$ are $s \times s$ matrices of bounded measurable complex functions in $\overline{\Omega}_\infty$; $a_{\alpha\beta} = (-1)^{|\alpha|+|\beta|} a^*_{\beta\alpha}$, $a^*_{\beta\alpha}$ are complex conjugate transportation matrices of $a_{\alpha\beta}$. Suppose that $a_{\alpha\beta}$ are continuous with respect to $x \in \overline{\Omega}$ uniformly with respect to $t \in [0, \infty)$ if $|\alpha| = |\beta| = m$, and satisfy the inequality

$$\sum_{|\alpha|,|\beta|=m} a_{\alpha\beta}(x,t)\xi^\alpha \xi^\beta \eta\overline{\eta} \geq \nu_0 |\xi|^{2m} |\eta|^2 \tag{2.2}$$

for all $\xi \in R^n \backslash \{0\}$, $\eta \in C^s \backslash \{0\}$ and $(x,t) \in \overline{\Omega}_\infty$, where $\nu_0 = \text{const} > 0$.

We use the following notation. For each multi-index $\alpha = (\alpha_1, ...\alpha_n) \in N^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$ and $D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial^{\alpha_1}_{x_1} .....\partial^{\alpha_n}_{x_n}} = u_{x_1^{\alpha_1}...x_n^{\alpha_n}}$ is the generalized derivative up to order α with respect to $x = (x_1, ..., x_n)$; $u_{t^k} = \partial^k u / \partial t^k$ is the generalized derivative up to order k with respect to t.

In this paper we use the usual functional spaces:

$H^m(\Omega)$ is the space of all functions $u(x)$, $x \in \Omega$, with the norm

$$\|u\|_{H^m(\Omega)} = \Big(\sum_{|\alpha|=0}^{l} \int_\Omega |D^\alpha u|^2 dx \Big)^{1/2}.$$

$\overset{o}{H}{}^m(\Omega)$ is the completion of $\overset{o}{C}{}^\infty(\Omega)$ in the norm of the space $H^m(\Omega)$.

$H^m_\beta(\Omega)$ is the space of all functions $u(x)$ satisfying

$$\|u\|^2_{H^m_\beta(\Omega)} = \sum_{|\alpha|\leq l} \int_\Omega r^{2(\beta+|\alpha|-l)} |D^\alpha u|^2 dx < \infty.$$

$H^{m,k}(e^{-\gamma t}, \Omega_\infty)$ is the space of all functions $u(x,t)$, $(x,t) \in \Omega_\infty$, with the norm

$$\|u\|_{H^{m,k}(e^{-\gamma t},\Omega_\infty)} = \Big(\int\limits_{\Omega_\infty} \big(\sum_{|\alpha|=0}^{l} |D^\alpha u|^2 + \sum_{j=1}^{k} |u_{t^j}|^2\big) e^{-2\gamma t} dxdt\Big)^{1/2}.$$

$\overset{o}{H}{}^{m,k}(e^{-\gamma t}, \Omega_\infty)$ is the closure in $H^{m,k}(e^{-\gamma t}, \Omega_\infty)$ of the set consisting of all infinitely differentiable in Ω_∞ functions which belong to $H^{m,k}(e^{-\gamma t}, \Omega_\infty)$ and vanish near S_∞.

$H^{m,k}_\beta(e^{-\gamma t}, \Omega_\infty)$ is the space of all functions $u(x,t)$ satisfying

$$\|u\|^2_{H^{m,k}_\beta(e^{-\gamma t},\Omega_\infty)} = \int\limits_{\Omega_\infty} \big(\sum_{|\alpha|=0}^{l} r^{2(\beta+|\alpha|-l)} |D^\alpha u|^2 + \sum_{j=1}^{k} |u_{t^j}|^2\big) e^{-2\gamma t} dxdt < \infty.$$

$H^{m}_\beta(e^{-\gamma t}, \Omega_\infty)$ is the space of all functions $u(x,t)$ with the norm

$$\|u\|_{H^{m}_\beta(e^{-\gamma t},\Omega_\infty)} = \Big(\int\limits_{\Omega_\infty} \big(\sum_{|\alpha|+j=0}^{l} r^{2(\beta+|\alpha|-l)} |D^\alpha u_{t^j}|^2\big) e^{-2\gamma t} dxdt\Big)^{1/2}.$$

Let X be a Banach space. Denote by $L^\infty(0,\infty;X)$ the space consisting of all measurable functions $u(x,\cdot) : (0,\infty) \to X$, $t \mapsto u(x,t)$ satisfying

$$\|u\|_{L^\infty(0,\infty;X)} = ess \sup_{t>0} \|u(x,t)\|_X < \infty.$$

We consider the following problem in the infinite cylinder Ω_∞:

$$(-1)^{m-1} L(x,t,D)u - u_{tt} = f(x,t), \tag{2.3}$$

$$u|_{t=0} = 0, u_t|_{t=0} = 0, \tag{2.4}$$

$$\frac{\partial^j u}{\partial \nu_j}\Big|_{S_\infty} = 0; j = 0, ...(m-1), \tag{2.5}$$

where ν is the outer unit normal to S_∞.

The function $u(x,t)$ is called a generalized solution of problem $(2.3)-(2.5)$ in the space $H^{m,1}(e^{-\gamma t}, \Omega_\infty)$ if $u(x,t) \in \overset{o}{H}{}^{m,1}(e^{-\gamma t}, \Omega_\infty)$, $u(x,0) = 0$ and for each $T > 0$ the equality

$$\begin{aligned}(-1)^{m-1} \int\limits_{\Omega_\infty} \Big(\sum_{|\alpha|,|\beta|=0}^{m} (-1)^\alpha a_{\alpha\beta} D^\beta u \overline{D^\alpha \varphi}\Big) dxdt + \int\limits_{\Omega_\infty} u_t \overline{\varphi}_t dxdt \\ = \int\limits_{\Omega_\infty} f\ \overline{\varphi} dxdt\end{aligned} \tag{2.6}$$

holds for all test functions $\varphi \in \overset{o}{H}{}^{m,k}(e^{-\gamma t}, \Omega_\infty)$, where $\varphi(x,t) = 0$ for $t \in [T, \infty)$.

By the same argument as in [4, p. 105-106] we can prove the following lemma.

Lemma 2.1. *Assume that $u(x,t)$ is a generalized solution of problem (2.3) – (2.5) in the space $H^{m,1}(e^{-\gamma t}, \Omega_\infty)$ and $u_{tt},\ f(x,t) \in L^\infty(0,\infty; L_2(\Omega))$. Then for almost all $t \in (0, \infty)$:*

$$(2.7) \qquad (-1)^{m-1} \int\limits_\Omega \Big(\sum_{|\alpha|,|\beta|=0}^{m} (-1)^\alpha a_{\alpha\beta} D^\beta u \overline{D^\alpha \chi} \Big) dx = \int\limits_\Omega (u_{tt} + f) \overline{\chi} dx$$

holds for all functions $\chi = \chi(x) \in \overset{o}{H}{}^m(\Omega)$.

For convenience, in the rest of this paper we use the notation:

$$(2.8) \qquad B(u,u)(t) = \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} \int\limits_\Omega a_{\alpha\beta} D^\beta u \overline{D^\alpha u} dx.$$

For almost all $t \in (0, \infty)$ the function $x \longmapsto u(x,t)$ belongs to $\overset{o}{H}{}^m(\Omega)$. On the other hand, since the principal coefficients $a_{\alpha\beta}$ are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0, \infty)$ if $|\alpha| = |\beta| = m$ and the constant ν_0 is independent of t, by repeating the proof of the Garding inequality [3, p. 44] we have the following assertion.

Lemma 2.2. *Assume that coefficients of the operator L(x,D,t) satisfy condition (2.2) and $a_{\alpha\beta}$ are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0, \infty)$ if $|\alpha| = |\beta| = m$. Then there exist two constants $\mu_0 > 0$, $\lambda_0 \geq 0$ such that*

$$(2.9) \qquad (-1)^m B(u,u)(t) \geq \mu_0 \|u\|^2_{H^m(\Omega)} - \lambda_0 \|u\|^2_{L_2(\Omega)}$$

for all functions $u = u(x,t) \in \overset{o}{H}{}^{m,1}(e^{-\gamma t}, \Omega_\infty)$.

3. Regularity with Respect to Time Variable

In this section we investigate the regularity with respect to time variable of generalized solutions of problem (2.3)-(2.5). It is shown that the regularity depends on the coefficients and the right-hand side of the problem.

Denote by m^* the number of multi-indexes which have order not exceeding m and $\mu = \mathrm{const} > 0$. For a non-negative integer s we use the notation:

$$\gamma_s = \frac{m^*\mu(2s+1) + \sqrt{(m^*\mu(2s+1))^2 + 8\mu_0\lambda_0^2}}{4\mu_0}.$$

Let γ_0^* be a number such that $\gamma_0^* > \gamma_0$. We have the following assertion.

Theorem 3.1. *Assume that problem (2.3)-(2.5) has exactly one genenalized solution* $u(x,t)$ *in the space* $H^{m,1}(e^{-\gamma_0^* t},\Omega_\infty)$ *and the following conditions are fulfilled:*

(i) $\sup\limits_{(x,t)\in\overline{\Omega}_\infty}\left|\dfrac{\partial^k a_{\alpha\beta}}{\partial t^k}\right| \le \mu; 1\le|\alpha|,|\beta|\le m,\ 0\le k\le h+1,$

(ii) $f_{t^k}\in L^\infty(0,\infty;L_2(\Omega)),\ 0\le k\le h,$

(iii) $f_{t^k}(x,0)=0,\ 0\le k\le h-1.$

Then for every $\gamma>\max\{\gamma_h,\gamma_0^*\}$*, the function* $u(x,t)$ *has derivatives with respect to* t *up to order h belonging to the space* $\overset{o}{H}{}^{m,1}(e^{-\gamma t},\Omega_\infty)$*. Moreover, the funtion* $u(x,t)$ *satisfies the inequality*

$$\|u_{t^h}\|^2_{H^{m,1}(e^{-\gamma t},\Omega_\infty)} \le C\sum_{k=0}^{h}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))}, \tag{3.1}$$

where $\mathrm{C}=\mathrm{const}>0$ *is independent of* u *and* f*.*

Proof. We shall use the Galerkin's approximate method to prove the theorem. Let $\{\psi_k\}\subset \overset{o}{C}{}^\infty(\Omega)$ be an orthogonal system in $L^2(\Omega)$ such that its linear closure in $H^m(\Omega)$ coincides with $\overset{o}{H}{}^m(\Omega)$. For each natural number N, we consider the function

$$u^N(x,t)=\sum_{k=1}^{N}c_k^N(t)\psi_k(x),$$

where $c_k^N(t)$ are the solutions of the system of ordinary differential equations:

$$(-1)^m\int\limits_\Omega\Big(\sum_{|\alpha|,|\beta|=0}^{m}(-1)^{|\alpha|}a_{\alpha\beta}D^\beta u^N\overline{D^\alpha\psi_l}\Big)dx+\int\limits_\Omega u^N_{tt}\overline{\psi}_l dx=-\int\limits_\Omega f\overline{\psi}_l dx, \tag{3.2}$$

$l=1,2,\dots,N$ with the initial conditions

$$c_k^N(0)=\frac{d}{dt}c_k^N(0)=0. \tag{3.3}$$

Since (3.2) is a linear ordinary differential system with initial condition (3.3), it has the unique solution c_k^N. Moreover, for each $T>0$ we have $d^{s+2}c_k^N/dt^{s+2}\in L^2(0,T)$. Therefore, from identity (3.2) we have

$$\begin{aligned}&(-1)^m\int\limits_\Omega\frac{\partial^s}{\partial t^s}\Big(\sum_{|\alpha|,|\beta|=0}^{m}(-1)^{|\alpha|}a_{\alpha\beta}D^\beta u^N\overline{D^\alpha\psi_l}\Big)dx+\int\limits_\Omega u^N_{t^{s+2}}\overline{\psi}_l dx\\&=-\int\limits_\Omega f_{t^s}\overline{\psi}_l dx,\end{aligned} \tag{3.4}$$

$l = 1, 2, \ldots, N$. Multiplying (3.4) by $d^{s+1}c_k^N/dt^{s+1}$, taking the sum with respect to l and integrating the obtained equality with respect to t on $(0, T)$, we get

$$\begin{aligned}
&\int\limits_{\Omega_T} \Big(u^N_{t^{s+2}} \overline{u^N_{t^{s+1}}} + \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} (a_{\alpha\beta} D^\beta u^N)_{t^s} \overline{D^\alpha u^N_{t^s+1}} \Big) dxdt \\
&= -\int\limits_{\Omega_T} f_{t^s} \overline{u^N_{t^{s+1}}} dxdt,
\end{aligned}$$

where $\Omega_T = \Omega \times (0, T)$. Adding this equality to its complex conjugate we obtain

$$\begin{aligned}
&\int\limits_{\Omega_T} \frac{\partial}{\partial t}\Big(u^N_{t^{s+1}} \overline{u^N_{t^{s+1}}} \Big) dxdt \\
&+2\mathrm{Re} \int\limits_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} \frac{\partial^s}{\partial t^s} \Big(a_{\alpha\beta} D^\beta u^N \Big) \overline{D^\alpha u^N_{t^s+1}} dxdt \\
&= -2\mathrm{Re} \int\limits_{\Omega_T} f_{t^s} \overline{u^N_{t^{s+1}}} dxdt
\end{aligned} \tag{3.5}$$

By using the hypothesis (iv) and condition (3.3) we can see easily that $D^p u^N_{t^k}$ $(x, 0) = 0$ with $0 \le k \le s,\ 0 \le |p| \le m$. Therefore, for the first term of inequality (3.5) we have

$$\int\limits_{\Omega_T} \frac{\partial}{\partial t}\Big(u^N_{t^{s+1}} \overline{u^N_{t^{s+1}}} \Big) dxdt = \| u^N_{t^{s+1}}(x, T) \|^2_{L^2(\Omega)}. \tag{3.6}$$

Denoting $\begin{pmatrix} k \\ s \end{pmatrix} = \dfrac{s!}{k!(s-k)!}$ and noting that $a_{\alpha\beta} = (-1)^{|\alpha|+|\beta|} a^*_{\beta\alpha}$, we obtain for the second term of inequality (3.5) :

$$\begin{aligned}
&2\mathrm{Re} \int\limits_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} \frac{\partial^s}{\partial t^s} (a_{\alpha\beta} D^\beta u^N) \overline{D^\alpha u^N_{t^s+1}} dxdt \\
&= 2\mathrm{Re} \int\limits_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} \sum_{k=1}^{s} (-1)^{m+|\alpha|} \begin{pmatrix} k \\ s \end{pmatrix} \frac{\partial}{\partial t} \Big(\frac{\partial^k a_{\alpha\beta}}{\partial t^k} D^\beta u^N_{t^{s-k}} \overline{D^\alpha u^N_{t^s}} \Big) dxdt \\
&- 2\mathrm{Re} \int\limits_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} \sum_{k=1}^{s} (-1)^{m+|\alpha|} \begin{pmatrix} k \\ s \end{pmatrix} \frac{\partial^{k+1} a_{\alpha\beta}}{\partial t^{k+1}} D^\beta u^N_{t^{s-k}} \overline{D^\alpha u^N_{t^s}} dxdt \\
&- 2\mathrm{Re} \int\limits_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} \sum_{k=1}^{s} (-1)^{m+|\alpha|} \begin{pmatrix} k \\ s \end{pmatrix} \frac{\partial^k a_{\alpha\beta}}{\partial t^k} D^\beta u^N_{t^{s-k+1}} \overline{D^\alpha u^N_{t^s}} dxdt \\
&- \mathrm{Re} \int\limits_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} \frac{\partial a_{\alpha\beta}}{\partial t} D^\beta u^N_{t^s} \overline{D^\alpha u^N_{t^s}} dxdt
\end{aligned} \tag{3.7}$$

$$+\int_0^T (-1)^m \frac{\partial}{\partial t} B(u^N_{t^s}, u^N_{t^s})(t)dt.$$

Since $D^p u^N_{t^k}(x,0) = 0,\ 0 \le k \le s,\ 0 \le |p| \le m$, by applying the integration by parts, from (3.7) we obtain

$$\begin{aligned}
&2\mathrm{Re}\int_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} \frac{\partial^s}{\partial t^s}(a_{\alpha\beta}D^\beta u^N)\overline{D^\alpha u^N_{t^s+1}}dxdt \\
&= (-1)^m B(u^N_{t^s}, u^N_{t^s})(T) \\
&+ 2\mathrm{Re}\int_{\Omega} \sum_{|\alpha|,|\beta|=0}^{m} \sum_{k=1}^{s} (-1)^{m+|\alpha|} \binom{k}{s} \frac{\partial^k a_{\alpha\beta}}{\partial t^k} D^\beta u^N_{t^{s-k}} \overline{D^\alpha u^N_{t^s}}\Big|_{t=T} dx \\
&- 2\mathrm{Re}\int_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} \sum_{k=1}^{s} (-1)^{m+|\alpha|} \binom{k}{s} \frac{\partial^{k+1} a_{\alpha\beta}}{\partial t^{k+1}} D^\beta u^N_{t^{s-k}} \overline{D^\alpha u^N_{t^s}} dxdt \\
&- 2\mathrm{Re}\int_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} \sum_{k=1}^{s} (-1)^{m+|\alpha|} \binom{k}{s} \frac{\partial^{k} a_{\alpha\beta}}{\partial t^{k}} D^\beta u^N_{t^{s-k+1}} \overline{D^\alpha u^N_{t^s}} dxdt \\
&- \mathrm{Re}\int_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} \frac{\partial a_{\alpha\beta}}{\partial t} D^\beta u^N_{t^s} \overline{D^\alpha u^N_{t^s}} dxdt.
\end{aligned} \tag{3.8}$$

Since

$$2\mathrm{Re}\int_{\Omega_T} \lambda_0 u^N_{t^s} \overline{u^N_{t^{s+1}}} dxdt = \lambda_0 \|u^N_{t^s}(x,T)\|^2_{L^2(\Omega)},$$

from (3.5), (3.6) and (3.8) we obtain

$$\begin{aligned}
&\|u^N_{t^{s+1}}(x,T)\|^2_{L^2(\Omega)} + (-1)^m B(u^N_{t^s}, u^N_{t^s})(T) + \lambda_0 \|u^N_{t^s}(x,T)\|^2_{L^2(\Omega)} \\
&= -2\mathrm{Re}\int_{\Omega} \sum_{|\alpha|,|\beta|=0}^{m} \sum_{k=1}^{s} (-1)^{m+|\alpha|} \binom{k}{s} \frac{\partial^k a_{\alpha\beta}}{\partial t^k} D^\beta u^N_{t^{s-k}} \overline{D^\alpha u^N_{t^s}}\Big|_{t=T} dx \\
&+ 2\mathrm{Re}\int_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} \sum_{k=1}^{s} (-1)^{m+|\alpha|} \binom{k}{s} \frac{\partial^{k+1} a_{\alpha\beta}}{\partial t^{k+1}} D^\beta u^N_{t^{s-k}} \overline{D^\alpha u^N_{t^s}} dxdt \\
&+ 2\mathrm{Re}\int_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} \sum_{k=1}^{s} (-1)^{m+|\alpha|} \binom{k}{s} \frac{\partial^{k} a_{\alpha\beta}}{\partial t^{k}} D^\beta u^N_{t^{s-k+1}} \overline{D^\alpha u^N_{t^s}} dxdt \\
&+ \mathrm{Re}\int_{\Omega_T} \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{m+|\alpha|} \frac{\partial a_{\alpha\beta}}{\partial t} D^\beta u^N_{t^s} \overline{D^\alpha u^N_{t^s}} dxdt \\
&+ 2\mathrm{Re}\int_{\Omega_T} \lambda_0 u^N_{t^s} \overline{u^N_{t^{s+1}}} dxdt - 2\mathrm{Re}\int_{\Omega_T} f_{t^s} \overline{u^N_{t^{s+1}}} dxdt.
\end{aligned} \tag{3.9}$$

We have

$$
\begin{aligned}
&(\mathrm{I}) \equiv 2\mathrm{Re}\int_{\Omega} \sum_{\substack{|\alpha|=|\beta|=0\\ 1\le|\alpha|,|\beta|\le m}} \sum_{k=1}^{s}(-1)^{m+|\alpha|+1}\binom{k}{s}\frac{\partial^k a_{\alpha\beta}}{\partial t^k}D^\beta u^N_{t^{s-k}}\overline{D^\alpha u^N_{t^s}}\Big|_{t=T}dx\\
&\quad\le \varepsilon_1\|u^N_{t^s}(x,T)\|^2_{H^m(\Omega)}+C(\varepsilon_1)\sum_{k=0}^{s-1}\|u^N_(x,T)t^k\|^2_{H^m(\Omega)},\\
&(\mathrm{II}) \equiv \mathrm{Re}\int_{\Omega_T} \sum_{\substack{|\alpha|=|\beta|=0\\ 1\le|\alpha|,|\beta|\le m}} (-1)^{m+|\alpha|}\frac{\partial a_{\alpha\beta}}{\partial t}D^\beta u^N_{t^s}\overline{D^\alpha u^N_{t^s}}dxdt\\
&\quad\le m^*\mu\int_0^T\|u^N_{t^s}\|^2_{H^m(\Omega)}dt,\\
&(\mathrm{III}) \equiv 2\mathrm{Re}\int_{\Omega_T} \sum_{\substack{|\alpha|=|\beta|=0\\ 1\le|\alpha|,|\beta|\le m}} \sum_{k=1}^{s}(-1)^{m+|\alpha|}\binom{k}{s}\frac{\partial^{k+1} a_{\alpha\beta}}{\partial t^{k+1}}D^\beta u^N_{t^{s-k}}\overline{D^\alpha u^N_{t^s}}dxdt\\
&\qquad+2\mathrm{Re}\int_{\Omega_T} \sum_{\substack{|\alpha|=|\beta|=0\\ 1\le|\alpha|,|\beta|\le m}} \sum_{k=1}^{s}(-1)^{m+|\alpha|}\binom{k}{s}\frac{\partial^{k} a_{\alpha\beta}}{\partial t^{k}}D^\beta u^N_{t^{s-k+1}}\overline{D^\alpha u^N_{t^s}}dxdt\\
&\quad\le 2m^*\mu s\int_0^T\|u^N_{t^s}\|^2_{H^m(\Omega)}dt+\frac{\varepsilon_2}{2}\int_0^T\|u^N_{t^s}\|^2_{H^m(\Omega)}dt\\
&\qquad+C(\varepsilon_2)\sum_{k=0}^{s-1}\int_0^T\|u^N_{t^k}\|^2_{H^m(\Omega)}dt\\
&(\mathrm{IV}) \equiv \lambda_0\|u^N_{t^s}(x,T)\|^2_{L^2(\Omega)}-2\mathrm{Re}\int_{\Omega_T} f_{t^s}\overline{u^N_{t^{s+1}}}dxdt.\\
&\quad\le(\delta\lambda_0)^2\int_0^T\|u^N_{t^s}\|^2_{H^m(\Omega)}dt+\frac{2}{\delta^2}\int_0^T\|u^N_{t^{s+1}}\|^2_{L^2(\Omega)}dt+\delta^2\int_0^T\|f_{t^s}\|^2_{L^2(\Omega)}dt.
\end{aligned}
\tag{3.10}
$$

where $C(\varepsilon_1)>0$ depends on ε_1, and $C(\varepsilon_2)>0$ depends on ε_2. Using the Cauchy inequality and Lemma 2.2, from (3.10) we get

$$
\begin{aligned}
&\|u^N_{t^{s+1}}(x,T)\|^2_{L^2(\Omega)}+(\mu_0-\varepsilon_1)\|u^N_{t^s}(x,T)\|^2_{H^m(\Omega)}\\
&\le\frac{2}{\delta^2}\int_0^T\|u^N_{t^{s+1}}\|^2_{L^2(\Omega)}dt+(m^*\mu(2s+1)\\
&\quad+(\delta\lambda_0)^2+\varepsilon_2)\int_0^T\|u^N_{t^s}\|^2_{H^m(\Omega)}dt\\
&\quad+\delta^2\int_0^T\|f_{t^s}\|^2_{L^2(\Omega)}dt\\
&\quad+C(\varepsilon_1)\sum_{k=0}^{s-1}\|u^N_{t^k}\|^2_{H^m(\Omega)}+C(\varepsilon_2)\sum_{k=0}^{s-1}\int_0^T\|u^N_{t^k}\|^2_{H^m(\Omega)}dt\\
&\le\frac{2}{\delta^2}\int_0^T\Big(\|u^N_{t^{s+1}}\|^2_{L^2(\Omega)}+\frac{(m^*\mu(2s+1+(\delta\lambda_0)^2+\varepsilon_2)\delta^2}{2}\|u^N_{t^s}\|^2_{H^m(\Omega)}\Big)dt
\end{aligned}
\tag{3.11}
$$

$$+\,\delta^2\int_0^T\|f_{t^s}\|^2_{L^2(\Omega)}dt+C(\varepsilon_1)\sum_{k=0}^{s-1}\|u^N_{t^k}(x,T)\|^2_{H^m(\Omega)}$$
$$+\,C(\varepsilon_2)\sum_{k=0}^{s-1}\int_0^T\|u^N_{t^k}\|^2_{H^m(\Omega)}dt.$$

We find a solution δ^2 from the equation:

$$\frac{(m^*\mu(2s+1)+(\delta\lambda_0)^2+\varepsilon_2)\delta^2}{2}=\mu_0-\varepsilon_1,$$

where $0<\varepsilon_1<\mu_0$ and $\varepsilon_2>0$. We get

$$\lambda_0^2\delta^4+(m^*\mu(2s+1)+\varepsilon_2)\delta^2-2(\mu_0-\varepsilon_1)=0.$$

Denote by δ_* the positive solution of this equation, we obtain

$$\delta_*^2=\begin{cases}\dfrac{2(\mu_0-\varepsilon_1)}{m^*\mu(2s+1)+\varepsilon_2} & \text{if } \lambda_0=0,\\ \dfrac{-(m^*\mu(2s+1)+\varepsilon_2)+\sqrt{(m^*\mu(2s+1)+\varepsilon_2)^2+8(\mu_0-\varepsilon_1)\lambda_0^2}}{2\lambda_0^2} & \text{if } \lambda_0\neq 0..\end{cases}$$

Therefore, we have

$$\frac{2}{\delta_*^2}=\frac{(m^*\mu(2s+1)+(\delta\lambda_0)^2+\varepsilon_2)}{\mu_0-\varepsilon_1}$$
$$=\frac{m^*\mu(2s+1)+\varepsilon_2+\sqrt{(m^*\mu(2s+1)+\varepsilon_2)^2+8(\mu_0-\varepsilon_1)\lambda_0^2}}{2(\mu_0-\varepsilon_1)}$$

We consider the function of variables ε_1 and ε_2:

$$\gamma_s(\varepsilon_1,\varepsilon_2)=\frac{1}{\delta_*^2(\varepsilon)}=\frac{m^*\mu(2s+1)+\varepsilon_2+\sqrt{(m^*\mu(2s+1)+\varepsilon_2)^2+8(\mu_0-\varepsilon_1)\lambda_0^2}}{4(\mu_0-\varepsilon_1)}$$

with $0<\varepsilon_1<\mu_0$ and $\varepsilon_2>0$. Rewrite this equation in the form:

$$\gamma_s(\varepsilon_1,\varepsilon_2)=\frac{m^*\mu(2s+1)+\varepsilon_2}{4(\mu_0-\varepsilon_1)}+\sqrt{\frac{(m^*\mu(2s+1)+\varepsilon_2)^2}{16(\mu_0-\varepsilon_1)^2}+\frac{\lambda_0^2}{2(\mu_0-\varepsilon_1)}}\ .$$

It is easy to check that $\dfrac{\partial\gamma_s}{\partial\varepsilon_1}>0$ with $\forall\varepsilon_1\in(0,\mu_0)$ and $\dfrac{\partial\gamma_s}{\partial\varepsilon_2}>0$ with $\forall\varepsilon_2>0$. Put

$$\gamma_s=\gamma_s(\varepsilon_1=0,\varepsilon_2=0)=\frac{1}{\delta^2(0)}=\frac{m^*\mu(2s+1)+\sqrt{(m^*\mu(2s+1))^2+8\mu_0\lambda_0^2}}{4\mu_0}.$$

Take $s=h$. Since $\gamma>\gamma_h$ (see Theorem 3.1), there exist two constants $\varepsilon_1,\varepsilon_2:\ 0<\varepsilon_1<\mu_0,\ \varepsilon_2>0$ such that $\gamma=\gamma_h(\varepsilon_1,2\varepsilon_2)$. Denote $\gamma^*=\gamma_h(\varepsilon_1,\varepsilon_2)$. We have

$$\gamma = \gamma_h(\varepsilon_1, 2\varepsilon_2) > \gamma_h(\varepsilon_1, \varepsilon_2) = \gamma^*. \tag{3.12}$$

From this fact and inequality (3.11) it follows that

$$\begin{aligned}
&\|u^N_{t^{h+1}}(x,T)\|^2_{L_2(\Omega)} + (\mu_0 - \epsilon_1)\|u^N_{t^h}(x,T)\|^2_{H^m(\Omega)} \\
&\le 2\gamma^* \int_0^T \Big(\|u^N_{t^{h+1}}(x,t)\|^2_{L_2(\Omega)} + (\mu_0 - \epsilon_1)\|u^N_{t^h}(x,t)\|^2_{H^m(\Omega)}\Big)dt \\
&\quad + C_h(\varepsilon_1,\varepsilon_2)\Big(\sum_{k=0}^{h-1}\|u^N_{t^k}(x,t)\|^2_{H^m(\Omega)} \\
&\quad + \sum_{k=0}^{h-1}\int_0^T \|u^N_{t^k}(x,t)\|^2_{H^m(\Omega)}dt + \int_{\Omega_T}|f_{t^h}|^2 dxdt\Big) \\
&\le 2\gamma^* \int_0^T \Big(\|u^N_{t^{h+1}}(x,t)\|^2_{L_2(\Omega)} + (\mu_0 - \epsilon_1)\|u^N_{t^h}(x,t)\|^2_{H^m(\Omega)}\Big)dt \\
&\quad + C_h(\varepsilon_1,\varepsilon_2)\Big\{\sum_{k=0}^{h-1}\Big[\|u^N_{t^k}(x,T)\|^2_{H^m(\Omega)} \\
&\quad + \int_0^T \|u^N_{t^k}(x,t)\|^2_{H^m(\Omega)}dt\Big] + T\|f_{t^s}\|^2_{L^\infty(0,\infty;L^2(\Omega))}\Big\}.
\end{aligned} \tag{3.13}$$

where $C_h(\varepsilon_1,\varepsilon_2) > 0$ depends on ε_1 and ε_2.

Let l be a non-negative integer and $l \le h-1$. We now use the induction to show that

$$\begin{aligned}
&\sum_{k=0}^{l}\Big[\|u^N_{t^k}(x,T)\|^2_{H^m(\Omega)} + \int_0^T \|u^N_{t^k}(x,t)\|^2_{H^m(\Omega)}dt\Big] \\
&\le C^*_l(\varepsilon_1,\varepsilon_2)e^{2T\gamma_l(\varepsilon_1,\varepsilon_2)}\sum_{k=0}^{l}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))},
\end{aligned} \tag{3.14}$$

where $\varepsilon_1,\ \varepsilon_2$ are the constants as in (3.13), $C_s(\varepsilon_1,\varepsilon_2) > 0$ depends on ε_1, ε_2 and $\gamma_l(\varepsilon_1,\varepsilon_2)$ is the constant as in (3.12).

From (3.5) with $s = 0$ we have

$$\begin{aligned}
&\int_{\Omega_T}\frac{\partial}{\partial t}\Big(u^N_t\overline{u^N_t}\Big)dxdt + 2\mathrm{Re}\int_{\Omega_T}\sum_{|\alpha|,|\beta|=0}^{m}(-1)^{m+|\alpha|}a_{\alpha\beta}D^\beta u^N\overline{D^\alpha u^N_t}dxdt \\
&= -2\mathrm{Re}\int_{\Omega_T} f\overline{u^N_t}dxdt
\end{aligned} \tag{3.15}$$

In the same way as while the proof of inequality (3.11), from (3.15) we obtain

$$
\begin{aligned}
&\|u_t^N\|^2_{L^2(\Omega)} + \mu_0\|u^N\|^2_{H^m(\Omega)} \\
&\le (m^*\mu + (\delta\lambda_0)^2)\int_0^T \|u^N\|^2_{H^m(\Omega)}dt + \frac{2}{\delta^2}\int_0^T \|u_t^N\|^2_{L^2(\Omega)}dt \\
&+ \delta^2\int_0^T \|f\|^2_{L^2(\Omega))} \\
&\le \frac{2}{\delta^2}\int_0^T \Big(\|u_t^N\|^2_{L^2(\Omega)} + \frac{(m^*\mu + (\delta\lambda_0)^2)\delta^2}{2}\|u^N\|^2_{H^m(\Omega)}\Big)dt \\
&+ \delta^2\int_0^T \|f\|^2_{L^2(\Omega))}.
\end{aligned}
\tag{3.16}
$$

Choosing δ^2 such that $\dfrac{(m^*\mu + (\delta\lambda_0)^2)\delta^2}{2} = \mu_0$, we get

$$
\delta^2 = \begin{cases} \dfrac{2\mu_0}{m^*\mu} & \text{if } \lambda_0 = 0, \\ \dfrac{-m^*\mu + \sqrt{(m^*\mu)^2 + 8\mu_0\lambda_0^2}}{2\lambda_0^2} & \text{if } \lambda_0 \neq 0. \end{cases}
$$

Put

$$
J_N(t) = \|u^N(x,t)\|^2_{L^2(\Omega)} + \mu_0\|u^N(x,t)\|^2_{H^m(\Omega)}.
$$

From (3.16) we obtain

$$
\begin{aligned}
J_N(T) &\le \frac{m^*\mu + \sqrt{(m^*\mu)^2 + 8\mu_0\lambda_0^2}}{2\mu_0^2}\int_0^T J_N(t)dt + \delta^2\int_0^T \|f\|^2_{L^2(\Omega))} \\
&\le \frac{m^*\mu + \sqrt{(m^*\mu + 8\mu_0\lambda_0^2}}{2\mu_0^2}\int_0^T J_N(t)dt + CT\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}, \ C = \text{const.}
\end{aligned}
$$

From this estimate and from the Gronwall- Bellman inequality, we obtain

$$
\begin{aligned}
&\|u_t^N(x,T)\|^2_{L^2(\Omega)} + \mu_0\|u^N(x,T)\|^2_{H^m(\Omega)} \\
&\le C_0 \exp\Big\{\Big(\frac{m^*\mu + \sqrt{(m^*\mu + 8\mu_0\lambda_0^2}}{2\mu_0^2}\Big)T\Big\}\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))},
\end{aligned}
$$

where $C_0 = \text{const} > 0$, i.e.,

$$
\|u_t^N(x,T)\|^2_{L^2(\Omega)} + \mu_0\|u^N(x,T)\|^2_{H^m(\Omega)} \le C_0 e^{2T\gamma_0}\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}, \tag{3.17}
$$

Since

$$2\gamma_0(\epsilon_1,\epsilon_2)=\frac{m^*\mu+\epsilon_2+\sqrt{(m^*\mu+\epsilon_2)^2+8(\mu_0-\varepsilon_1)\lambda_0^2}}{2(\mu_0-\varepsilon_1)^2}>2\gamma_0,$$

we have

$$\|u_t^N(x,T)\|^2_{L^2(\Omega)}+\mu_0\|u^N(x,T)\|^2_{H^m(\Omega)}\le C_0e^{2T\gamma_0(\epsilon_1,\epsilon_2)}\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))} \tag{3.18}$$

From this inequality and by the arbitrariness of T, we obtain

$$\int_0^T\|u^N(x,t)\|^2_{H^m(\Omega)}dt\le\frac{C_0}{\mu_0}e^{2T\gamma_0(\epsilon_1,\epsilon_2)}\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}. \tag{3.19}$$

From (3.18), (3.19) it follows that

$$\|u^N(x,T)\|^2_{H^m(\Omega)}+\int_0^T\|u^N(x,t)\|^2_{H^m(\Omega)}dt\le C_0^*e^{2T\gamma_0(\epsilon_1,\epsilon_2)}\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

Therefore, inequality (3.14) holds for $l=0$.

Assume that (3.14) holds for all $j\le l-1$. From inequality (3.13) with $s=j+1$ we have

$$\begin{aligned}&\|u^N_{t^{j+2}}(x,T)\|^2_{L_2(\Omega)}+(\mu_0-\epsilon_1)\|u^N_{t^{j+1}}(x,T)\|^2_{H^m(\Omega)}\\
&\le 2\gamma_{j+1}(\varepsilon_1,\varepsilon_2)\int_0^T\Big(\|u^N_{t^{j+2}}(x,t)\|^2_{L_2(\Omega)}+(\mu_0-\epsilon_1)\|u^N_{t^{j+1}}(x,t)\|^2_{H^m(\Omega)}\Big)dt\\
&\quad+C_{j+1}(\varepsilon_1,\varepsilon_2)\Big(\sum_{k=0}^{j}\|u^N_{t^k}(x,T)\|^2_{H^m(\Omega)}\\
&\quad+\sum_{k=0}^{j}\int_0^T\|u^N_{t^k}(x,t)\|^2_{H^m(\Omega)}dt+T\|f_{t^{j+1}}\|^2_{L^\infty(0,\infty;L^2(\Omega))}\Big),\end{aligned}$$

where $C_{j+1}(\varepsilon_1,\varepsilon_2)>0$ depends on $\varepsilon_1,\ \varepsilon_2$. From this inequality and by using the induction on j we obtain

$$\begin{aligned}&\|u^N_{t^{j+2}}(x,T)\|^2_{L_2(\Omega)}+(\mu_0-\epsilon_1)\|u^N_{t^{j+1}}(x,T)\|^2_{H^m(\Omega)}\\
&\le 2\gamma_{j+1}(\varepsilon_1,\varepsilon_2)\int_0^T\Big(\|u^N_{t^{j+2}}(x,t)\|^2_{L_2(\Omega)}+(\mu_0-\epsilon_1)\|u^N_{t^{j+1}}(x,t)\|^2_{H^m(\Omega)}\Big)dt\\
&\quad+C_{j+1}(\varepsilon_1,\varepsilon_2)\Big(C_j^*(\varepsilon_1,\varepsilon_2)e^{2T\gamma_j(\varepsilon_1,\varepsilon_2)}\sum_{k=0}^{j}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))}\\
&\quad+T\|f_{t^{j+1}}\|^2_{L^\infty(0,\infty;L^2(\Omega))}\Big),\end{aligned}$$

By the Gronwall- Bellman inequality we get

$$\begin{aligned}&\|u^N_{t^{j+2}}(x,T)\|^2_{L_2(\Omega)}+(\mu_0-\epsilon_1)\|u^N_{t^{j+1}}(x,T)\|^2_{H^m(\Omega)}\\&\le C_{j,j+1}(\varepsilon_1,\varepsilon_2)e^{2T\gamma_{j+1}(\varepsilon_1,\varepsilon_2)}\sum_{k=0}^{j+1}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))},\end{aligned}$$

where $C_{j,j+1}(\varepsilon_1,\varepsilon_2)>0$ depends on $\varepsilon_1,\ \varepsilon_2$. Hence

$$\|u^N_{t^{j+1}}(x,T)\|^2_{H^m(\Omega)}\le\frac{C_{j,j+1}(\varepsilon_1,\varepsilon_2)}{\mu_0-\epsilon_1}e^{2T\gamma_{j+1}(\varepsilon_1,\varepsilon_2)}\sum_{k=0}^{j+1}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

In the same way as in the proof of inequality (3.19) we have

$$\begin{aligned}&\int_0^T\|u^N_{t^{j+1}}(x,t)\|^2_{H^m(\Omega)}dt\\&\le\frac{C_{j,j+1}(\varepsilon_1,\varepsilon_2)}{2(\mu_0-\epsilon_1)\gamma_{j+1}(\varepsilon_1,\varepsilon_2)}e^{2T\gamma_{j+1}(\varepsilon_1,\varepsilon_2)}\sum_{k=0}^{j+1}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))}.\end{aligned}$$

From these inequalities and by the induction hypothesis, we get

$$\begin{aligned}&\sum_{k=0}^{j+1}\Big[\|u^N_{t^k}(x,T)\|^2_{H^m(\Omega)}+\int_0^T\|u^N_{t^k}(x,t)\|^2_{H^m(\Omega)}dt\Big]\\&\le C^*_{j+1}(\varepsilon_1,\varepsilon_2)e^{2T\gamma_{j+1}(\varepsilon_1,\varepsilon_2)}\sum_{k=0}^{j+1}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))}.\end{aligned}$$

Therefore, (3.14) holds for all $j\le l$ and the proof of (3.14) is completed.

Now we return to inequality (3.13). By using (3.14), from (3.13) we have

$$\begin{aligned}&\|u^N_{t^{h+1}}(x,T)\|^2_{L_2(\Omega)}+(\mu_0-\epsilon_1)\|u^N_{t^h}(x,T)\|^2_{H^m(\Omega)}\\&\le 2\gamma^*\int_0^T\Big(\|u^N_{t^{h+1}}(x,t)\|^2_{L_2(\Omega)}+(\mu_0-\epsilon_1)\|u^N_{t^h}(x,t)\|^2_{H^m(\Omega)}\Big)dt\\&\quad+C_h(\varepsilon_1,\varepsilon_2)\Big(C^*_{h-1}(\varepsilon_1,\varepsilon_2)e^{2T\gamma_{h-1}(\varepsilon_1,\varepsilon_2)}\sum_{k=0}^{h-1}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))}\\&\quad+T\|f_{t^h}\|^2_{L^\infty(0,\infty;L^2(\Omega))}\Big).\end{aligned}\tag{3.20}$$

Put

$$J^N_h(T)=\|u^N_{t^{h+1}}(x,T)\|^2_{L_2(\Omega)}+(\mu_0-\epsilon_1)\|u^N_{t^h}(x,T)\|^2_{H^m(\Omega)},$$

$$\phi(T) = C_h(\varepsilon_1, \varepsilon_2)\Big(C^*_{h-1}(\varepsilon_1, \varepsilon_2)e^{2T\gamma_{h-1}(\varepsilon_1,\varepsilon_2)}\sum_{k=0}^{h-1}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))}$$
$$+T\|f_{t^h}\|^2_{L^\infty(0,\infty;L^2(\Omega))}\Big)$$

From inequality (3.20) we have

$$J^N_h(T) \le 2\gamma^*\int_0^T J^N_h(t)dt + \phi(T).$$

Applying the Gronwall- Bellman inequality, from inequality (3.19) we obtain

$$J^N_h(T) \le \int_0^T e^{2\gamma^*(T-t)}\phi'(t)dt. \tag{3.21}$$

Since $\gamma_{h-1}(\varepsilon_1, \varepsilon_2) < \gamma_h(\varepsilon_1, \varepsilon_2) = \gamma^*$, it follows from (3.21) that

$$J^N_h(T) \le C_{h-1,h}(\varepsilon_1, \varepsilon_2)e^{2\gamma^* T}\sum_{k=0}^{h}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

This implies that

$$\begin{aligned}&\|u^N_{t^{h+1}}(x,T)\|^2_{L_2(\Omega)} + \|u^N_{t^h}(x,T)\|^2_{H^m(\Omega)}\\ &\le C^*_{h-1,h}(\varepsilon_1, \varepsilon_2)e^{2\gamma^* T}\sum_{k=0}^{h}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))}.\end{aligned} \tag{3.22}$$

From (3.12) we have $\gamma > \gamma^*$. Therefore, by multiplying $e^{-2\gamma T}$ to the both sides of inequality (3.22) and integrating it with respect to T from 0 to ∞, we obtain

$$\|u^N_{t^h}\|^2_{H^{m,1}(e^{-\gamma t},\Omega_\infty)} \le C\sum_{k=0}^{h}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))}, \tag{3.23}$$

where $C = \text{const} > 0$ is independent of N and f.

We now return to inequality (3.17). Since $\gamma_0^* > \gamma_0$, by multiplying the both sides of inequality (3.17) on $e^{-2T\gamma_0^*}$ and integrating with respect to $T \in (0, \infty)$, we have

$$\|u^N\|^2_{H^{m,1}(e^{-\gamma_0^* t},\Omega_\infty)} \le C\|f\|^2_{L^\infty(0,\infty;L^2(\Omega))},$$

where $C = \text{const}$. Therefore, there exist a subsequence which converges weakly to a function v in $H^{m,1}(e^{-\gamma_0^* t}, \Omega_\infty)$. It is easy to check that v is a generalized solution of problem (2.3)-(2.5) in the space $H^{m,1}(e^{-\gamma_0^* t}, \Omega_\infty)$. Since this problem has exactly one generalized solution in the space $H^{m,1}(e^{-\gamma_0^* t}, \Omega_\infty)$, we have $v \equiv u$.

From (3.23) it follows that $\left\{u_{t^h}^N\right\}$ is bounded in $H^{m,1}(e^{-\gamma t},\Omega_\infty)$. We can choose a subsequence which converges weakly to a function u_h in $H^{m,1}(e^{-\gamma t},\Omega_\infty)$. By passing in (3.23) to the limit for a weakly convergent subsequence, we obtain

$$\|u_{t^h}\|^2_{H^{m,1}(e^{-\gamma t},\Omega_\infty)} \le C\sum_{k=0}^{h}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},$$

where $C=\mathrm{const}>0$ is independent of u and f. The proof of Theorem 3.1 is completed. ■

4. Regularity with Respect to Both of Time and Spatial Variables

Let Ω be a bounded domain in R^n with the boundary $\partial\Omega$. We suppose that $\partial\Omega$ is a infinitely differentiable surface everywhere except the coordinate origin, in a neighborhood of which the domain Ω coincides with the cone $K=\{x : x/|x|\in G\}$, where G is a smooth domain on the unit sphere S^{n-1}.

Suppose that $w=(w_1,...,w_{n-1})$ is a local coordinate system on the unit sphere S^{n-1} and $L_0(0,t,D)$ is the principal part of the operator $L(x,t,D)$ at the coordinate origin. Then we can write $L_0(0,t,D)$ in the form

$$L_0(0,t,D)=r^{-2m}Q(w,t,rD_r,D_w)$$

where $Q(w,t,rD_r,D_w)$ is the linear operator with smooth coefficients, $D_r=i\partial/\partial_r$ and $D_w=\partial/\partial w_1.....\partial w_{n-1}$. Consider the spectral problem:

$$Q(\omega,t,\lambda,D_w)v(w)=0, w\in G, \tag{4.1}$$

$$D_w^\alpha v(w)=0, w\in\partial G, |\alpha|=0,1,...,m-1. \tag{4.2}$$

It is well known (see [2]; p. 39) that for every $t\in[0,\infty)$ its spectrum is discrete.

We consider the Dirichlet problem for an elliptic system with the parameter t:

$$(-1)^{m-1}L_0(0,t,D)=F(x,t), x\in\Omega. \tag{4.3}$$

The function $u(x,t)$ is called a generalized solution of the Dirichlet problem for system (4.3) in the space $H^m(\Omega)$ if $u(x,t)\in \overset{\circ}{H}{}^m(\Omega)$ for almost all $t\in[0,\infty)$ and the identity

$$(-1)^{m-1}\int_\Omega\sum_{|\alpha|,|\beta|=1}^{m}(-1)^{|\alpha|}a_{\alpha\beta}(0,t)D^\beta u(x,t)\overline{D^\alpha\varphi(x)}dx=-\int_\Omega F(x,t)\overline{\varphi(x)}dx$$

holds for all test functions $\varphi(x)\in \overset{0}{H}{}^m(\Omega)$, $t\in[0,\infty)$.

From Lemma 2.1 of Section 2 of this paper and by using the similar arguments as in the proof of Lemma 3.2 in [4] we obtain the following result.

Lemma 4.1. *Suppose $F(x,t) \in H^{l,0}_\beta(e^{-\gamma t}, \Omega_\infty)$ for almost all $t \in [0,\infty)$ and $u(x,t)$ is a generalized solution of Dirichlet problem for system (4.3) in the space $H^m(\Omega)$ such that $u(x,t) \equiv 0$ outside U_0. Then $u(x,t) \in H^{2m+l,0}_\beta(e^{-\gamma t}, \Omega_\infty)$ and*

$$\|u\|^2_{H^{2m+l,0}_\beta(e^{-\gamma t},\Omega_\infty)} \le C\Big(\|u\|^2_{H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t},\Omega_\infty)} + \|F\|^2_{H^{l,0}_\beta(e^{-\gamma t},\Omega_\infty)}\Big),$$

where $C = \mathrm{const} > 0$ is independent of u and F.

We surround the origin by a neighborhood U_0 of a diameter sufficiently small so that the intersection of Ω and U_0 coincides with the cone K. From Theorem 3.1 with $h = 1$ and by using the similar arguments as in the proof of Lemma 3.1 in [4], we have the following assertion.

Lemma 4.2. *Assume that problem (2.3)-(2.5) has exactly one generalized solution $u(x,t)$ in the space $H^{m,1}(e^{-\gamma_0^* t}, \Omega_\infty)$. and the following conditions are fulfilled:*

(i) $\sup\limits_{(x,t)\in\overline{\Omega}_\infty} \Big|\dfrac{\partial^k a_{\alpha\beta}}{\partial t^k}\Big| \le \mu; 1 \le |\alpha|, |\beta| \le m,\ 0 \le k \le 2,$

(ii) $f_{t^k} \in L^\infty(0,\infty; L_2(\Omega)),\ 0 \le k \le 1,$

(iii) $f(x,0) = 0.$

In addition, assume that $u(x,t) \equiv 0$ outside U_0. Then for every $\gamma > \max\{\gamma_1, \gamma_0^*\}$ the generalized solution $u(x,t)$ belongs to $H^{2m,2}_m(e^{-\gamma t}, \Omega_\infty)$ and

$$\|u\|^2_{H^{2m,2}_m(e^{-\gamma t},\Omega_\infty)} \le C(\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))} + \|f_t\|^2_{L^\infty(0,\infty;L_2(\Omega))}, \tag{4.1}$$

where $C = \mathrm{const} > 0$ does not depend on u and f.

Theorem 4.3. *Assume that problem (2.3)-(2.5) has exactly one genenalized solution $u(x,t)$ in the space $H^{m,1}(e^{-\gamma_0^* t}, \Omega_\infty)$, and the following conditions are fulfilled:*

(i) $\sup\limits_{(x,t)\in\overline{\Omega}_\infty} \Big|\dfrac{\partial^k a_{\alpha\beta}}{\partial t^k}\Big| \le \mu; 1 \le |\alpha|, |\beta| \le m,\ 0 \le k \le 2m+l+1,$

(ii) $f_{t^k} \in L^\infty(0,\infty; L_2(\Omega)),\ 0 \le k \le l+2m,$

(iii) $f_{t^k}(x,0) = 0,\ 0 \le k \le l+2m-1.$

In addition, suppose that the strip

$$m - \frac{n}{2} \le \mathrm{Im}\lambda \le 2m + l - \frac{n}{2}$$

does not contain the points of the spectrum of problem $(2.7)-(2.8)$ for every $t \in [0,\infty)$. Then for every $\gamma > \max\{\gamma_{2m+l}, \gamma_0^*\}$ the generalized solution $u(x,t)$ belongs to the space $H^{2m+l}_0(e^{-\gamma t}, \Omega_\infty)$ and the following inequality holds

$$\|u\|^2_{H_0^{2m+l}(e^{-\gamma t},\Omega_\infty)} \le C\sum_{k=0}^{2m+l}\|f_{t^k}\|^2_{L^\infty(0,\infty;L^2(\Omega))},$$

where $C = \text{const} > 0$ *is independent of* u *and* f.

Proof.

Case 1. We prove that the Theorem is true for a generalized solution $u(x,t)$ of problem (2.3)-(2.5) in the space $H^{m,1}(e^{-\gamma t},\Omega_\infty)$ satisfying $u(x,t)\equiv 0$ outside U_0. First, we consider the case $l = 0$ and rewrite system (2.3) in the form

$$(-1)^{m-1}L_0(0,t,D)u = F(x,t), \tag{4.5}$$

where $F(x,t) = u_{tt} + f + (-1)^{m-1}[L_0(0,t,D) - L(x,t,D)]u$. From Lemma 4.2 and Theorem 3.1 it follows that $F(x,t)\in H^{0,0}_{m-1}(e^{-2\gamma t},\Omega_\infty)$. Therefore, $F(x,t)\in H^0_{m-1}(\Omega)$ for almost all $t\in[0,\infty)$. On the other hand, for every $t\in[0,\infty)$ the strip $m-(n/2)\le Im\lambda\le m+1-(n/2)$ does not contain any points of the spectrum of problem (4.2)-(4.3). So from the results of the work [8] it follows that for almost all $t\in[0,\infty)$ the function $u(x,t)$ belongs to the space $H^{2m}_{m-1}(\Omega)$ and

$$\|u\|^2_{H^{2m}_{m-1}(\Omega)} \le C\Big(\|u_{tt}\|^2_{L_2(\Omega)} + \|f\|^2_{L^2(\Omega)}\Big),$$

where $C = \text{const} > 0$ is independent of u and f. Using similar arguments we can show that

$$\|u\|^2_{H_0^{2m}(\Omega)} \le C\Big(\|u_{tt}\|^2_{L_2(\Omega)} + \|f\|^2_{L^2(\Omega)}\Big),$$

where $C = \text{const} > 0$ is independent of u and f. Multiplying by $e^{-4\gamma t}$ the both sides of this inequality and integrating with respect to t from 0 to ∞, we obtain

$$\|u\|^2_{H_0^{2m,0}(e^{-2\gamma t},\Omega_\infty)} \le C\Big(\|u_{tt}\|^2_{L^2(\Omega_\infty)} + \|f\|^2_{L^\infty(0,\infty;L_2(\Omega))}\Big),$$

where $C = \text{const} > 0$ is independent of u and f. This inequality and Theorem 3.1 imply

$$\|u\|^2_{H_0^{2m,0}(e^{-2\gamma t},\Omega_\infty)} \le C\Big(\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))} + \|f_t\|^2_{L^\infty(0,\infty;L_2(\Omega))},\tag{4.6}$$

where $C = \text{const} > 0$ is independent of u and f.

We claim that the following inequality is valid:

$$\|u_{t^s}\|^2_{H_0^{2m,0}(e^{-2\gamma t},\Omega_\infty)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},\tag{4.7}$$

where $C = \text{const} > 0$ is independent of u and f.

Indeed, differentiating system (2.3) with respect to t and putting $v = u_{t^s}$, we have

$$(4.8) \qquad (-1)^{m-1}L(x,t,D)v = v_{tt} + f_{t^s} + (-1)^m \sum_{k=1}^{s} \binom{s}{k} L_{t^k} u_{t^{s-k}},$$

where

$$L_{t^k} = \sum_{|\alpha|,|\beta|=1}^{m} D^\alpha \Big(\frac{\partial^k a_{\alpha\beta}}{\partial t^k} D^\beta \Big) + \sum_{|\alpha|=1}^{m} \frac{\partial^k a_\alpha}{\partial t^k} D^\alpha + \frac{\partial^k a}{\partial t^k}.$$

Put

$$F_1 = v_{tt} - f_{t^s} - (-1)^m \sum_{k=1}^{s} \binom{s}{k} L_{t^k} u_{t^{s-k}} + \big(-1\big)^{m-1} \big(L_0(0,t,D) - L(x,t,D)\big)v.$$

Therefore, we have the system:

$$(4.9) \qquad (-1)^{m-1}L_0(0,t,D)v = F_1(x,t).$$

Using the induction hypothesis and the similar arguments as in the proof of (4.6), we obtain the inequality

$$(4.10) \quad \|u_{t^s}\|^2_{H_0^{2m,0}(e^{-2\gamma t},\Omega_\infty)} = \|v\|^2_{H_0^{2m,0}(e^{-2\gamma t},\Omega_\infty)} \le C \sum_{k=0}^{2m} \|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},$$

where $C = \text{const} > 0$. It follows that inequality (4.7) is true and so the claim is proved.

Since

$$\|u\|^2_{H_0^{2m}(e^{-2\gamma t},\Omega_\infty)} \le \sum_{k=0}^{2m-1} \|u_{t^k}\|^2_{H_0^{2m,0}(e^{-2\gamma t},\Omega_\infty)} + \|u_{t^{2m}}\|^2_{H_0^{0,0}(e^{-2\gamma t},\Omega_\infty)},$$

from inequality (4.10) and Theorem 3.1, we have

$$\|u\|^2_{H_0^{2m}(e^{-2\gamma t},\Omega_\infty)} \le C \sum_{k=0}^{2m} \|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},$$

where $C = \text{const} > 0$ is independent of u and f. Hence the Theorem is proved for $l = 0$.

Suppose that the conclusion of the Theorem is true for all $s \le l-1$, that is

$$(4.11) \qquad \|u\|^2_{H_0^{2m+s}(e^{-\gamma t},\Omega_\infty)} \le C \sum_{k=0}^{2m+s} \|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))}, \ s \le l-1,$$

where $C = \text{const} > 0$ is independent of u and f. We need to show that the conclusion of the Theorem holds for all $s \le l$.

First, we prove the following inequality:

$$(4.12) \qquad \|u_{t^s}\|^2_{H_0^{2m+l-s}(e^{-\gamma t},\Omega_\infty)} \le C \sum_{k=0}^{2m+l} \|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))}$$

for $s = l, l-1, \ldots.., 0$, where $C = \mathrm{const} > 0$ is independent of u and f. Since $f_{t^k} \in L^\infty(0,\infty; L_2(\Omega))$ for $k \le l+2m$, $f_{t^k}(x,0) = 0$ for $k \le l+2m-1$, from Theorem 3.1 it follows that $u_{t^{l+2}} \in H_0^{0,0}(e^{-2\gamma t}, \Omega_\infty)$. Using this fact and by the similar arguments as in the proof of inequality (4.7) we obtain inequality (4.12) for $s = l$. Let us assume that inequality (4.12) is true for $s = l, l-1, .., j+1$. Set $v = u_{t^j}$. From identity (4.8) it follows that

$$(-1)^{m-1} Lv = F, \tag{4.13}$$

where

$$F = F(x,t) = v_{tt} + f_{t^j} + (-1)^m \sum_{k=1}^{j} \binom{j}{k} L_{t^k} u_{t^{j-k}},$$

$$L_{t^k} = \sum_{|\alpha|,|\beta|=1}^{m} D^\alpha \Big(\frac{\partial^k a_{\alpha\beta}}{\partial t^k} D^\beta\Big) + \sum_{|\alpha|=1}^{m} \frac{\partial^k a_\alpha}{\partial t^k} D^\alpha + \frac{\partial^k a}{\partial t^k}.$$

By virtue of the induction hypothesis with respect to l, we have

$$\sum_{k=1}^{j} \binom{j}{k} L_{t^k} u_{t^{j-k}} \in H_0^{l-j}(e^{-\gamma t}, \Omega_\infty).$$

On the other hand, in view of the induction assumption with respect to s,

$$v_{tt} \in H_0^{l-j}(e^{-\gamma t}, \Omega_\infty).$$

Therefore, $F(x,t) \in H_0^{l-j}(e^{-2\gamma t}, \Omega_\infty)$. From this fact and the relation

$$H_0^{l-j}(e^{-2\gamma t}, \Omega_\infty) \subset H_{-1}^{l-j-1,0}(e^{-2\gamma t}, \Omega_\infty)$$

it follows that $F(x,t) \in H_{-1}^{l-j-1,0}(e^{-2\gamma t}, \Omega_\infty)$.

By repeating arguments that are analogous to those which were used in the proof of this theorem with $l = 0$, we obtain $v \in H_{-1}^{2m+l-j-1,0}(e^{-2\gamma t}, \Omega_\infty)$. The application of Lemma 4.1 yields $u_{t^j} = v \in H_0^{2m+l-j,0}(e^{-2\gamma t}, \Omega_\infty)$ and

$$\|v\|^2_{H_0^{2m+l-j,0}(e^{-2\gamma t}, \Omega_\infty)} \le C \sum_{k=0}^{2m+l} \|f_{t^k}\|^2_{L^\infty(0,\infty; L_2(\Omega))}, \tag{4.14}$$

where $C = \mathrm{const} > 0$ is independent of u and f.

Since vspace-0.1cm

$$\begin{aligned} &\|u_{t^j}\|^2_{H_0^{2m+l-j}(e^{-\gamma t}, \Omega_\infty)} \\ &\le \|u_{t^{j+1}}\|^2_{H_0^{2m+l-j-1}(e^{-\gamma t}, \Omega_\infty)} + \|u_{t^j}\|^2_{H_0^{2m+l-j,0}(e^{-\gamma t}, \Omega_\infty)}, \end{aligned} \tag{4.15}$$

by the induction hypothesis with respect to s, it follows that

$$\|u_{t^j}\|^2_{H_0^{2m+l-j}(e^{-\gamma t},\Omega_\infty)} \le C\sum_{k=0}^{2m+l}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},$$

where $C = \text{const} > 0$ is independent of u and f. Hence we obtain the conclusion of the theorem for $j = 0$.

Case 2. We now prove the theorem for the general case. Take a function $u_0 = \varphi_0 u$, where $\varphi_0 \in \overset{\circ}{C}{}^\infty(U_0)$ and $\varphi_0 \equiv 1$ in a neighborhood of the coordinate origin. The function φ_0 satisfies the system

$$(-1)^{m-1}L(x,t,D)u_0 - (u_0)_{tt} = \varphi_0 f + L_1(x,t,D)u,$$

where L_1 is a linear differential operator having order less than 2m. The coefficients of this operator depend on the choice of the function φ_0 and are equal to 0 outside U_0. Using this fact and by the similar arguments as in the proof of the case 1 we have

$$\|\varphi_0 u\|^2_{H_0^{2m+l}(e^{-\gamma t},\Omega_\infty)} \le C\sum_{k=0}^{2m+l}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))}, \tag{4.16}$$

where $C = \text{const} > 0$ is independent of u and f.

The function $\varphi_1 u = (1-\varphi_0)u$ is equal to 0 in a neighborhood of the coordinate origin. We now apply this function to the theorem on the smoothness of a solution of the elliptic problem in a smooth domain to conclude that $\varphi_1 u \in H_0^{2m+l}(e^{-\gamma t},\Omega_\infty)$ and

$$\|\varphi_1 u\|^2_{H_0^{2m+l}(e^{-\gamma t},\Omega_\infty)} \le C\sum_{k=0}^{2m+l}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))}, \tag{4.17}$$

where $C = \text{const} > 0$ is independent of u and f. Since $u = \varphi_0 u + \varphi_1 u$, it follows from the inequalities (4.16) and (4.17) that

$$\|u\|^2_{H_0^{2m+l}(e^{-\gamma t},\Omega_\infty)} \le C\sum_{k=0}^{2m+l}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},$$

where $C = \text{const} > 0$ is independent of u and f. The proof of the theorem is completed. ∎

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