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POMPEIU PROBLEM FOR THE HEISENBERG BALL

Wayne M. Eby

Abstract. The standard Pompeiu results for complex balls in the setting of the Heisenberg group $\mathbf{H}^n$ are also shown to carry over to Heisenberg balls. This extension is important because it allows for integration over sets of the same dimension as the ambient space $\mathbf{H}^n$. Several different concepts of the Heisenberg ball are considered, using differing definitions for the metric on $\mathbf{H}^n$. For purposes of this work, none of these is more natural than the others. The results for each of the spaces L^2, L^p for $1 \leq p < \infty$, and for L^∞, are directly comparable to the Pompeiu results for complex balls in $\mathbf{H}^n$, as in [2, 3, 4]. The natural expression for the Pompeiu problem in $\mathbf{H}^n$ is integration over complex balls, and the extension to the Heisenberg ball builds upon the methods for this case. The extra dimension primarily leads to extra complexity in the integrals. At the level of L^∞, where the results require balls of two radii satisfying appropriate conditions, the additional dimension adds to the complexity in the functions defining the conditions for the radii. The different concepts of the Heisenberg ball lead to different forms for these arithmetic conditions defining the radii. The differences between these balls can also be seen when they are rotated with $\mathbf{H}^n$, an issue to be further considered in a later work. The standard Pompeiu results have been extended to all of the cases considered here.

1. INTRODUCTION

Let us briefly review some of the mathematical problems that fall under the heading of the Pompeiu problem. In a general sense the Pomepiu problem refers to the conditions under which an integral transformation of the form

$$\mathcal{P}(f)(\mathbf{x}) = \int_S f(\mathbf{y}+\mathbf{x})d\mu_S(\mathbf{y}) \qquad \text{for all } \mathbf{x} \in \mathbf{R}^n$$

uniquely characterizes the function f. Here $S \subset \mathbf{R}^n$ is some bounded subset of the ambient space with area measure μ_S, and $f \in C(\mathbf{R}^n)$, although sometimes

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alternative function spaces are considered. Please see the articles [18, 19, 20] for more in depth discussion of these ideas and some of their extensions.

The Pompeiu problem has been studied in the setting of the Heisenberg group in the papers [1, 2, 3]. See also [9, 10, 12] for more recent work involving sets which are not spherically symmetric and sets which are of higher codimension. In each of these cases, the integrals are taken over some subset of $\mathbf{C}^n \times \{0\}$ rather than a subset of the larger space $\mathbf{H}^n$. This setting is actually the natural one to consider for the Pompeiu problem in the Heisenberg setting, especially when approaching the Pompeiu problem as an application of methods of harmonic analysis. The paper [17] of Strichartz describes methods of harmonic analysis on the Heisenberg group based on joint eigenfuntions of the Heisenberg sub-Laplacian $\square$ and the extra direction iT; see Section 2. In some sense, the work on the Pomepiu problem in [1, 2, 3] has been an extension of the work of Strichartz both in harmonic analysis and in his definition of the Radon transform for $\mathbf{H}^n$ as

$$\mathcal{R}f(\mathbf{g}) = \int L_{\mathbf{g}} f(\mathbf{z}, 0) dm(\mathbf{z}).$$

This integral transformation is defined in terms of integration over translations of the complex plane $\mathbf{C}^n \times \{0\}$. It is then natural to extend to integration over bounded subsets of the complex plane and their translations. This approach to the Pompeiu problem on the Heisenberg group, initiated in papers [1, 2, 3], has been very successful in extending results to $\mathbf{H}^n$ in a context where the Heisenberg group is considered as quantization of Euclidean space and the Laguerres are the quantization of Bessel functions. There appears to be a sense of natural pairing between this definition of the Pompeiu problem on $\mathbf{H}^n$ and the harmonic analysis used there. However, complications arise and the analysis is not as straightforward when integrating over bounded subsets $S \subset \mathbf{H}^n$, such as a Heisenberg ball, which are not contained in $\mathbf{C}^n \times \{0\}$.

Previous work on the Pompeiu problem on the Heisenberg group has primarily dealt with the cases of spherically symmetric sets, the ball of radius r, B_r, as well as its boundary S_r. When realizing these within $\mathbf{C}^n \times \{0\} \subset \mathbf{H}^n$, the integrals are taken over the complex ball $B_r \times \{0\}$, a set of codimension one in $\mathbf{H}^n$, and its boundary $S_r \times \{0\}$, of codimension two. However we would like to work with the Pompeiu problem on $\mathbf{H}^n$ in cases where the set S has the same dimension as the ambient space. In particular, we are concerned in this paper with some of the various definitions of a Heisenberg ball.

In the course of this paper we apply methods of harmonic analysis to extend the Pompeiu results from the complex ball $B_r \times \{0\}$ to each one of the above forms of the Heisenberg ball B_r^H. In each case this is accomplished by breaking $\mathbf{H}^n$ into sheets $\mathbf{C}^n \times \{t_0\}$ and applying the results of [1, 2, 3, 4]. Although the results do extend, the most natural form for the problem remains integration over

$B_r \times \{0\} \subset \mathbf{H}^n$, as considered in previous papers.

2. Heisenberg Group

We first define the Heisenberg group $\mathbf{H}^n$ using the coordinates $\{[\mathbf{z}, t] : \mathbf{z} \in \mathbf{C}^n, t \in \mathbf{R}\}$ with operation given by the following group law

$$[\mathbf{z}, t] \cdot [\mathbf{w}, s] = [\mathbf{z} + \mathbf{w}, t + s + 2\mathrm{Im} \sum_{j=1}^{n} z_j \bar{w}_j].$$

$\mathbf{H}^n$ can be realized as a manifold in the space $\mathbf{C}^{n+1}$ as the boundary of the Siegel upper half space

$$\Omega_{n+1} = \{(\mathbf{z}, z_{n+1} \in C^{n+1} : \mathrm{Im}\, z_{n+1} > |\mathbf{z}|^2\}.$$

The dilation of $\mathbf{H}^n$ is given by

$$\delta \circ [\mathbf{z}, t] = [\delta \mathbf{z}, \delta^2 t].$$

The concept of dilation will be important when defining a Heisenberg ball B_r^{H}, as it is important to have the concept of the Heisenberg ball preserved under dilations, i.e.

$$\begin{aligned} r \circ B_1^{\mathrm{H}} &= \{r \circ [\mathbf{z}, t] : [\mathbf{z}, t] \in B_1^{\mathrm{H}}\} \\ &= \{[r\mathbf{z}, r^2 t] : \|[\mathbf{z}, t]\| \leq 1\}, \end{aligned}$$

which must equal

$$B_r^{\mathrm{H}} = \{[\mathbf{z}, t] : \|[\mathbf{z}, t]\| \leq r\}$$

The concept of the Heisenberg ball is directly linked to the definiton of a distance on the Heisenberg group. However, there is not a uniquely defined distance for the Heisenberg group, but rather several concepts of distance. We consider several possible balls, including the Koranyi ball based on thh metric $\|[\mathbf{z}, t]\| = \sqrt[4]{r^4 + t^2}$.

The left-invariant vector fields spanning the tangent space are given by

$$Z_j = \frac{\partial}{\partial z_j} + i \bar{z}_j \frac{\partial}{\partial t}$$

and

$$\bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - i z_j \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$, together with the additional direction $T = \frac{\partial}{\partial t}$. The bracket relations are given by

$$[\bar{Z}_j, Z_k] = 2i\delta_{j,k} T,$$

and the group is a step-2 nilpotent Lie group, as the complex vector fields $\{Z_1, \ldots, Z_n, \bar{Z}_1, \ldots, \bar{Z}_n\}$ generate the tangent space. The Heisenberg group $\mathbf{H}^n$ is also a CR manifold since the complexified tangent bundle $\mathbf{C}T^*(M)$ has the decomposition

$$\mathbf{C}T^*(M) = T^{(1,0)} \oplus \overline{T^{(1,0)}} \oplus \mathbf{C}\mathcal{N},$$

where $T^{(1,0)}$ is spanned by $\{Z_1, \ldots, Z_n\}$, $\overline{T^{(1,0)}} = T^{(0,1)}$ is spanned by $\{\bar{Z}_1, \ldots, \bar{Z}_n\}$, and $\mathbf{C}\mathcal{N}$ is the complexified vector bundle of the real line bundle $\mathcal{N}$ generated by the vector field T.

From these complex vector fields we can form the Heisenberg sub-Laplacian

$$\Box = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_j Z_j),$$

which plays a fundamental role in our analysis and in the concept of harmonic analysis as defined by Strichartz. Similarly to Strichartz, we look at the following bounded $U(n)$-spherical functions, which are the joint eigenfunctions of $\Box$, representing radialization in the complex directions, and of T, representing the extra real direction. For more information, see [6]. The joint spectrum of these operators is given by the Heisenberg fan

$$\mathcal{H} = \left(\cup_{\lambda\in\mathbf{Z}_+}\mathcal{H}_k\right) \cup \mathcal{H}_\rho,$$

where

$$\mathcal{H}_k = \{(\lambda, (4k+2)|\lambda|) : \lambda \in \mathbf{R}\}$$

and

$$\mathcal{H}_\rho = \{(0,\rho) : \rho^2 \in \mathbf{R}_+\}.$$

This space can be parameterized by the points $\{(\lambda, k) \in \mathbf{R}^* \times \mathbf{Z}_+\} \cup \{(0,\rho) \in \mathbf{R}_+\}$, with each point corresponding to a separate $U(n)$-spherical function. For the points $(\lambda, \nu) \in \mathbf{R}^* \times \mathbf{Z}_+$, the Laguerre part of the spectrum, we have the bounded $U(n)$-spherical functions

$$\psi^\lambda_{\nu,\nu}(\mathbf{z}, t) = \binom{\nu+n-1}{\nu}^{-1} e^{2\pi i\lambda t} e^{-2\pi|\lambda||\mathbf{z}|^2} L^{(n-1)}_\nu(4\pi|\lambda||\mathbf{z}|^2), \quad (\lambda,\nu) \in \mathbf{R}^* \times \mathbf{Z}_+.$$

The remaining points in the spectrum are $(0,\rho) \in \mathbf{R}_+$, the Bessel part of the spectrum associated with the bounded $U(n)$-spherical functions

$$\mathcal{J}^\rho_{n-1}(\mathbf{z}, t) = (n-1)!2^{n-1}\frac{J_{n-1}(\rho|\mathbf{z}|)}{(\rho|\mathbf{z}|)^{n-1}} \quad \rho \in \mathbf{R}_+.$$

A Gelfand transform, or spherical function transform is then defined for functions $f \in L^1_*(\mathbf{H}^n)$ by $\tilde{f}(p)$, which for $p = (\lambda, \nu) \in \mathbf{R}^* \times \mathbf{Z}_+$ takes the form

$$\tilde{f}(\lambda, \nu) = \int_{\mathbf{H}^n} f(\mathbf{g})\overline{\psi^\lambda_{\nu,\nu}(\mathbf{g})}d\mathbf{g}.$$

Similarly, for $p = (0,\rho) \in \mathbf{R}_+$, this takes the form

$$\tilde{f}(0, \rho) = \int_{\mathbf{H}^n} f(\mathbf{g})\overline{\mathcal{J}^\rho_{n-1}(\mathbf{g})}d\mathbf{g}.$$

The following Tauberian theorem will be used when applying the $U(n)$ spherical transform.

Theorem 2.1. *Let J be a closed ideal in $L^1_*(\mathbf{H}^n)$ and suppose that*

(1) *For any $(\lambda,\nu)\in\mathbf{R}^*\times\mathbf{Z}_+$ there exists some $f\in J$ such that*

$$\tilde{f}(\lambda,\nu)\neq 0$$

(2) *For any $\rho\in\mathbf{R}_+$ there exists some $f\in J$ such that*

$$\tilde{f}(0,\rho)\neq 0$$

Then $J = L^1_*(\mathbf{H}_n) = \{f\in L^1(\mathbf{H}_n) : f(U\mathbf{z},t) = f(\mathbf{z},t)$ for every $U\in U(n)\}$.

When working with the function space $L^2(\mathbf{H}^n)$, easier methods are available, in particular the partial Fourier transform in the real variable t. For $f\in L^1(\mathbf{H}^n)$ denote by

$$\tilde{f}^\tau(\mathbf{z}) = f(\mathbf{z},\cdot)^\wedge(\tau) = \int_{\mathbf{R}} f(\mathbf{z},t)e^{-2\pi i\tau t}dt.$$

The interaction of a convolution of functions under the partial Fourier transform leads to the concept of twisted convolution. Given any $\lambda\in\mathbf{R}^*$ and two functions $f,g\in L^2(\mathbf{C}^n)$ define

$$(f*^\lambda g)(\mathbf{z}) = \int_{\mathbf{C}^n} e^{-4\pi i\lambda \mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}}} f(\mathbf{z}-\mathbf{w})g(\mathbf{w})d\mathbf{w},$$

the λ-twisted convolution of f and g. There is the following relation with the partial Fourier transform and the twisted convolution:

$$\widetilde{(f*g)}^\lambda = \tilde{f}^\lambda *^\lambda \tilde{g}^\lambda,$$

which is verified as follows

$$\begin{aligned}
\widetilde{(f*g)}^\lambda &= \int_{\mathbf{H}^n\times\mathbf{R}} e^{-2\pi i\tau t} f(\mathbf{z}-\mathbf{w},t-s-2\mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}})g(\mathbf{w},t)d\mathbf{w}dsdt\\
&= \int_{\mathbf{H}^n\times\mathbf{R}} e^{-2\pi i\tau(t-s)}e^{-2\pi i\tau s}e^{-4\pi i\lambda\mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}}} f(\mathbf{z}-\mathbf{w},t-s)g(\mathbf{w},s)d\mathbf{w}dsdt\\
&= \int_{\mathbf{C}^n} e^{-4\pi i\lambda\mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}}}\tilde{f}(\mathbf{z}-\mathbf{w},\tau)\tilde{g}(\mathbf{w},\tau)d\mathbf{w}\\
&= \tilde{f}^\lambda *^\lambda \tilde{g}^\lambda.
\end{aligned}$$

The partial Fourier transform $\tilde{f}^\lambda$ of a function f that is radial in $\mathbf{z}\in\mathbf{C}^n$ is expressible as a Laguerre series using the orthonormal basis $\{\mathcal{W}^\lambda_{kk}(\mathbf{z})\}_{k\in\mathbf{Z}_+}$ for $L^2_0(\mathbf{C}^n)$, where

$$\mathcal{W}^\lambda_{k,k}(\mathbf{z}) = ce^{-2\pi|\lambda||\mathbf{z}|^2}L_k^{(n-1)}(4\pi|\lambda||\mathbf{z}|^2).$$

The properties of the $\mathcal{W}^\lambda_{kk}$ under $*^\lambda$ twisted convolution will be utilized. See [2] and [6]. An important function determined by integration of these $\mathcal{W}^\lambda_{k,k}(\mathbf{z})$ is the following

$$\Psi_k^{(n-1)}(x) = \int_0^x e^{-t/2} L_k^{(n-1)}(t) t^{n-1} dt.$$

When applying our analysis to functions $f \in L^p(\mathbf{H}^n)$ for $1 < p < \infty$, we will require some additional tools. The main tool is the Abel summability of the decomposition [13]:

$$f = \lim_{r \to 1^-} \sum_{|\mathbf{k}|=0}^{\infty} r^{|\mathbf{k}|} f * \mathcal{W}_{\mathbf{k}}^{(0)}$$

in L^p- norm, for all $f \in L^p(\mathbf{H}^n)$. This summation is also written in the form

$$\begin{aligned} f(\mathbf{z}, t) &= \lim_{r \to 1^-} \sum_{|\mathbf{k}| \geq 0} r^{|\mathbf{k}|} \left[P_{\mathbf{k},+}(f)(\mathbf{z}, t) + P_{\mathbf{k},-}(f)(\mathbf{z}, t) \right] \\ &= \lim_{r \to 1^-} \sum_{|\mathbf{k}| \geq 0} r^{|\mathbf{k}|} \left[\int_0^\infty (f * \phi_{\mathbf{k},+}^\lambda)(\mathbf{z}, t) d\lambda + \int_0^\infty (f * \phi_{\mathbf{k},-}^\lambda)(\mathbf{z}, t) d\lambda \right] \end{aligned}$$

Here the projections $P_{\mathbf{k},+}$ are to be considered as projections along rays of the Heisenberg fan, which is bounded in L^p, [13].

To apply this summation formula, we establish mean-periodicity relations with a measure representing the set S for the integral conditions of the Pompeiu problem. In [4] the relation

$$\left(\psi_k^\lambda \right) * \sigma_r(\mathbf{z}, t) = \psi_k^{-\lambda}(r, 0) \cdot \psi_k^\lambda(\mathbf{z}, t)$$

is established, where σ_r is area measure on a sphere in $\mathbf{C}^n \times \{0\} \subset \mathbf{H}^n$. Section 4 will extend a similar relation from $\mathbf{C}^n \times \{0\}$ to certain sets $S \subset \mathbf{H}^n$ of codimension 0, namely the three notions of the Heisenberg ball given earlier. This mean-periodicity result can be used together with the Abel summation, as demonstrated in [4] and [5]. This now concludes this section on the Heisenberg group $\mathbf{H}^n$ and the tools for analysis in $\mathbf{H}^n$.

3. L^2 Results

We show that the L^2 results for the Pompeiu problem on the Heisenberg group extend from the complex ball in $\mathbf{C}^n \times \{0\}$ to a Heisenberg ball. And we illustrate that this extension works for each of the notions of Heisenberg ball described above. In particular, we consider the following sets of integral conditions. First, let $B_r^{H_1}$ be defined by $\{[\mathbf{z}, t] \in \mathbf{H}^n : \|\mathbf{z}\|^4 + t^2 \leq r^4\}$. The first set of integral conditions is then given by

$$\int_{B_r^{H_1}} L_{\mathbf{g}} f(\mathbf{z}, t) d\hat{\mu}_r(\mathbf{z}, t) \qquad \text{for all } \mathbf{g} \in \mathbf{H}^n. \tag{1}$$

Here $\hat{\mu_r}$ is volume measure onthe Heisenberg ball $B_r^{H_1}$, and μ_r is volume measure on the complex ball $\{\mathbf{Z} \in \mathbf{C}^n : \|\mathbf{z}\| \leq r\}$ in $\mathbf{C}^n \times \{0\}$. Such notation is also used in the remainder of the paper. Similarly for $B_r^{H_2}$ defined by $\{[\mathbf{z},t] \in \mathbf{H}^n : max(\|\mathbf{z}\|, |t|) \leq r\}$ and $B_r^{H_3}$ defined by $\{[\mathbf{z},t] \in \mathbf{H}^n : \|\mathbf{z}\|^2 + |t| \leq r^2\}$, we have the integral conditions

$$\int_{B_r^{H_2}} L_{\mathbf{g}} f(\mathbf{z},t) d\hat{\mu}_r(\mathbf{z},t) \qquad \text{for all } \mathbf{g} \in \mathbf{H}^n, \tag{2}$$

and

$$\int_{B_r^{H_3}} L_{\mathbf{g}} f(\mathbf{z},t) d\hat{\mu}_r(\mathbf{z},t) \qquad \text{for all } \mathbf{g} \in \mathbf{H}^n. \tag{3}$$

In addition we consider a direct extension of the ball based on the Euclidean type ball, where $B_r^{H_4}$ is defined by $\{[\mathbf{z},t] \in \mathbf{H}^n : \|\mathbf{z}\|^2 + t^2 \leq r^2\}$, with the associated integral conditions

$$\int_{B_r^{H_4}} L_{\mathbf{g}} f(\mathbf{z},t) d\hat{\mu}_r(\mathbf{z},t) \qquad \text{for all } \mathbf{g} \in \mathbf{H}^n. \tag{4}$$

However, this form of the ball does not have self-scaling under dilations fo the Heisenberg group. As a result the L^∞ results, requiring multiple radii, given in Section 5, will not extend to this case, although the L^2 and L^p results of Sections 3 and 4 do work.

In each of these cases the following theorem addresses the Pompeiu problem for the function space L^2.

Theorem 3.2. *Let $f \in L^2(\mathbf{H}^n) \cap C(\mathbf{H}^n)$, and let $r > 0$. Suppose f satisfies integral conditions (1), (2), (3), or (4). That is, for $B_r^H = B_r^{H_1}, B_r^{H_2}, B_r^{H_3},$ or $B_r^{H_4}$ as defined above and μ_r volume measure on this set, suppose*

$$\int_{B_r^H} L_{\mathbf{g}} f(\mathbf{z},t) d\hat{\mu}_r(\mathbf{z},t) = 0 \qquad \textit{for all } \mathbf{g} \in \mathbf{H}^n.$$

Then $f \equiv 0$.

Proof. The proofs for each of the cases (1), (2), (3) and (4) are based on the same idea.

We begin with the case of integral conditions (2) since the steps are clearest for this product form. In this case, the measure T_{2S} is represented by the integral

$$\begin{aligned} \langle \phi, T_{2S} \rangle &= \int_{\max(|\mathbf{z}|,|t|)<r} \phi(\mathbf{z},t) d\hat{\mu}_{2,r}(\mathbf{z},t) \\ &= \int_{-r}^{r} \int_{|\mathbf{z}|<r} \phi(\mathbf{z},t) d\mu_r(\mathbf{z}) dt, \end{aligned}$$

which breaks down into integration over complex balls of radius r, when decomposed into level sets. The integral conditions (2) are then written as the convolution equation $f * T_{2S} \equiv 0$, also expressible as

$$\begin{aligned} f * T_{2S}(\mathbf{z}, t) &\equiv 0 \\ \int_{\max(|\mathbf{w}|,|s|)<r} f(\mathbf{z}-\mathbf{w}, t-s-2\mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}})d\hat{\mu}_{2,r}(\mathbf{w}, s) &= 0 \\ \int_{-r}^{r}\int_{|\mathbf{w}|<r} f(\mathbf{z}-\mathbf{w}, t-s-2\mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}})d\mu_r(\mathbf{w})ds &= 0 \\ \int_{-r}^{r}\left(f *_{\mathrm{tw}} T_r\right)(\mathbf{z}, t)(s)ds &= 0, \end{aligned}$$

where, for each level set $\mathbf{C}^n \times \{s\}$ we have twisted convolution with respect to the same distribution T_r. As a consequence, in this case the result will follow directly from the Pompeiu problem for T_r, as in [2, 8]. By applying the partial Fourier transform to the convolution equation we have the following

$$\begin{aligned} &(f * T_{2S})^{\frown\lambda} \\ &= \int_{\mathbf{R}} e^{-2\pi i\lambda t}\int_{-r}^{r}\int_{|\mathbf{w}|<r} f(\mathbf{z}-\mathbf{w}, t-s-2\mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}})d\mu_r(\mathbf{w})dsdt \\ &= \int_{-r}^{r}\int_{|\mathbf{w}|<r}\int_{\mathbf{R}} e^{-2\pi i\lambda t} f(\mathbf{z}-\mathbf{w}, t-s-2\mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}})dtd\mu_r(\mathbf{w})ds \\ &= \int_{-r}^{r}\int_{|\mathbf{w}|<r} e^{-2\pi i\lambda s}e^{-4\pi i\lambda \mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}}}\left(\int_{\mathbf{R}} e^{-2\pi i\lambda t} f(\mathbf{z}-\mathbf{w}, t)dt\right)d\mu_r(\mathbf{w})ds \\ &= \int_{-r}^{r} e^{-2\pi i\lambda s}\int_{|\mathbf{w}|<r} e^{-4\pi i\lambda \mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}}}\hat{f}^{\lambda}(\mathbf{z}-\mathbf{w})d\mu_r(\mathbf{w})ds \\ &= \int_{-r}^{r} e^{-2\pi i\lambda s}\left(\hat{f}^{\lambda} *^{\lambda} T_r\right)(\mathbf{z})ds. \end{aligned}$$

Note this integral then splits apart as a product. The reduction of the convolution, as in [2], will then allow us to reach the desired conclusion. Thus, the integral conditions (2) lead to the following

$$\begin{aligned} 0 &= (f * T_{2S})^{\frown\lambda} \\ &= \int_{-r}^{r} e^{-2\pi i\lambda s}\hat{f}^{\lambda} *^{\lambda} T_r(\mathbf{z})ds \\ 0 &= \frac{\sin(2\pi\lambda r)}{\pi\lambda}\left(\hat{f}^{\lambda} *^{\lambda} T_r\right)(\mathbf{z}). \end{aligned}$$

The proof of [2] that $f \equiv 0$ is then based on decomposition of $\hat{f}^{\lambda}$ as a Laguerre series

$$\hat{f}^{\lambda} = \sum_{k\in\mathbf{Z}_+} c_k(\lambda)\mathcal{W}_{kk}^{\lambda}(\mathbf{z})$$

and demonstration that each $c_k(\lambda) = 0$ for almost every λ. The function $\frac{\sin(2\pi\lambda r)}{\pi\lambda}$ is zero on the set $\{\lambda \neq \frac{n}{2r} : n \in \mathbf{Z}, n \neq 0\}$. For other values of λ, the conclusions of $\hat{f} *^{\lambda} T_r = 0$ from [2] remain unchanged. Thus we maintain the conclusion that $c_k(\lambda) = 0$ for almost every λ, for all k. It follows that $f \equiv 0$, as we intended to prove. Note that in this case

$$\int_{B_r^{H_2}} \psi_k^{\lambda}(\mathbf{z}, t) d\hat{\mu}_r(\mathbf{z}, t) = \frac{\sin(2\pi\lambda r)}{\pi\lambda} \Psi(k, \lambda, r^2). \tag{5}$$

The next case to consider is integral conditions (4), the extension of the Euclidean type metric. Here the convolution equation $f * T_{4S} = 0$ simplifies as

$$\begin{aligned} f * T_{4S}(\mathbf{z}, t) &\equiv 0 \\ \int_{|\mathbf{w}|^2 + s^2 < r^2} f(\mathbf{z} - \mathbf{w}, t - s - 2\mathrm{Im}\, \mathbf{z} \cdot \bar{\mathbf{w}}) d\hat{\mu}_{3,r}(\mathbf{w}, s) &= 0 \\ \int_{-r}^{r} \int_{|\mathbf{w}| < \sqrt{r^2 - s^2}} f(\mathbf{z} - \mathbf{w}, t - s - 2\mathrm{Im}\, \mathbf{z} \cdot \bar{\mathbf{w}}) d\mu_{\sqrt{r^2 - s^2}}(\mathbf{w}) ds &= 0 \\ \int_{-r}^{r} \left(f *_{\mathrm{tw}} T_{\sqrt{r^2 - s^2}} \right) (\mathbf{z}, t)(s) ds &= 0 \end{aligned}$$

where, for the level sets $\mathbf{C}^n \times \{s\}$, the twisted convolution with the with $T_{\sqrt{r^2 - s^2}}$ has a radius varying with s. Applying the partial Fourier transform we may write

$$\begin{aligned} & (f * T_{4S})^{\frown\lambda} \\ &= \int_{\mathbf{R}} e^{-2\pi i\lambda t} \int_{-r}^{r} \int_{|\mathbf{w}| < \sqrt{r^2 - s^2}} f(\mathbf{z} - \mathbf{w}, t - s - 2\mathrm{Im}\, \mathbf{z} \cdot \bar{\mathbf{w}}) d\mu_{\sqrt{r^2 - s^2}}(\mathbf{w}) ds dt \\ &= \int_{-r}^{r} \int_{|\mathbf{w}| < \sqrt{r^2 - s^2}} \int_{\mathbf{R}} e^{-2\pi i\lambda t} f(\mathbf{z} - \mathbf{w}, t - s - 2\mathrm{Im}\, \mathbf{z} \cdot \bar{\mathbf{w}}) dt d\mu_{\sqrt{r^2 - s^2}}(\mathbf{w}) ds \\ &= \int_{-r}^{r} \int_{|\mathbf{w}| < \sqrt{r^2 - s^2}} e^{-2\pi i\lambda s} e^{-4\pi i\lambda \mathbf{z} \cdot \bar{\mathbf{w}}} \left(\int_{\mathbf{R}} e^{-2\pi i\lambda t} f(\mathbf{z} - \mathbf{w}, t) dt \right) d\mu_{\sqrt{r^2 - s^2}}(\mathbf{w}) ds \\ &= \int_{-r}^{r} e^{-2\pi i\lambda s} \int_{|\mathbf{w}| < \sqrt{r^2 - s^2}} e^{-4\pi i\lambda \mathbf{z} \cdot \bar{\mathbf{w}}} \hat{f}^{\lambda}(\mathbf{z} - \mathbf{w}) d\mu_{\sqrt{r^2 - s^2}}(\mathbf{w}) ds \\ &= \int_{-r}^{r} e^{-2\pi i\lambda s} \left(\hat{f}^{\lambda} *^{\lambda} T_{\sqrt{r^2 - s^2}} \right) (\mathbf{z}) ds \end{aligned}$$

Here the convolution depends on the level set $\mathbf{C}^n \times \{s\}$, and the evaluation will involve more careful attention to the λ-convolution $\hat{f}^{\lambda} *^{\lambda} T_{\sqrt{r^2 - s^2}}$. Express $\hat{f}^{\lambda}$ as the Laguerre series

$$\hat{f}^{\lambda} = \sum_{k \in \mathbf{Z}_+} c_k(\lambda) \mathcal{W}_{kk}^{\lambda}(z).$$

The integral conditions (4) are expressed as $f * T_{4S} \equiv 0$, and after taking a partial Fourier transform, we then write

$$\begin{aligned} 0 &= (f * T_{4S})(\mathbf{z}) \\ &= \int_{-r}^{r} e^{-2\pi i\lambda s} \left(\hat{f}^{\lambda} *^{\lambda} T_{\sqrt{r^2-s^2}}\right) ds \\ &= \sum_{k\in\mathbf{Z}_+} c_k(\lambda) \int_{-r}^{r} e^{-2\pi i\lambda s} \left(\mathcal{W}_{kk}^{\lambda} *^{\lambda} T_{\sqrt{r^2-s^2}}\right)(\mathbf{z}) ds. \end{aligned}$$

The λ-convolution $\mathcal{W}_{kk}^{\lambda} *^{\lambda} T_{\sqrt{r^2-s^2}}$ is of the form evaluated in [2, 8], and we have that

$$\left(\mathcal{W}_{kk}^{\lambda} *^{\lambda} T_{\sqrt{r^2-s^2}}\right)(\mathbf{z}) = \Psi(k,\lambda,r^2-s^2)\mathcal{W}_{kk}^{\lambda}(\mathbf{z})$$

where

$$\begin{aligned} \Psi(k,\lambda,r^2-s^2) &= c\int_0^{r^2-s^2} e^{-2\pi|\lambda|t} L_k^{(n-1)}(4\pi|\lambda|t)t^{n-1}dt \\ &= \frac{c}{(4\pi|\lambda|)^n}\int_0^{4\pi|\lambda|(r^2-s^2)} e^{-x/2}x^{n-1}L_k^{(n-1)}(x)dx. \end{aligned}$$

Observe that Ψ is an exponential polynomial in $|\lambda|$ and therefore is real analytic in $|\lambda|$. It then follows that

$$0 = \sum_{k\in\mathbf{Z}_+} c_k(\lambda)\left(\int_{-r}^{r} e^{-2\pi i\lambda s}\Psi(k,\lambda,r^2-s^2)ds\right)\mathcal{W}_{kk}^{\lambda}(\mathbf{z}).$$

For each $k \in \mathbf{Z}_+$, we must then have

$$c_k(\lambda)\left(\int_{-r}^{r} e^{-2\pi i\lambda s}\Psi(k,\lambda,r^2-s^2)ds\right) = 0 \qquad \text{for all } \lambda\in\mathbf{R}^*.$$

Next observe that $\int_{-r}^{r} e^{-2\pi i\lambda s}\Psi(k,\lambda,r^2-s^2)ds$ is real analytic in λ. We can see, as done in [12], that $\Psi(k,\lambda,r^2-s^2)$ is an exponential polynomial in $|\lambda|$, giving the real analyticity we need. This real analyticity extends also to $\int_{-r}^{r} e^{-2\pi i\lambda s}\Psi(k,\lambda,r^2-s^2)ds$. See Proposition 3.2 where the real-analyticity is proven explicitly using series. It then follows, as above, that $c_k(\lambda) = 0$ for almost every $\lambda\in\mathbf{R}^*$ and all $k\in\mathbf{Z}_+$. Thus $f\equiv 0$, and the proof for conditions (3) is also complete. Note that in this case

$$\int_{B_r^{H_4}} \psi_k^{\lambda}(\mathbf{z},t)d\hat{\mu}_r(\mathbf{z},t) = \int_{-r}^{r} e^{-2\pi i\lambda s}\Psi(k,\lambda,r^2-s^2)ds. \tag{6}$$

The proof for the integral conditions (1) is very similar. Here the convolution $f * T_{1S}$ will instead reduce as

$$(f * T_{1S})^{\frown\lambda} = \int_{-r}^{r} e^{-2\pi i\lambda s}\left(\hat{f}^{\lambda} *^{\lambda} T_{\sqrt[4]{r^4-s^2}}\right) ds.$$

Following the same steps in the previous case, we find that for each $k \in \mathbf{Z}_+$

$$c_k(\lambda) \int_{-r}^{r} e^{-2\pi i\lambda s} \Psi(k, \lambda, \sqrt{r^4 - s^2}) ds = 0 \qquad \text{for all } \lambda \in \mathbf{R}^*$$

with the goal of demonstrating that each $c_k(\lambda) = 0$, as above. We then only need to show

$$\int_{-r}^{r} e^{-2\pi i\lambda s} \Psi(k, \lambda, \sqrt{r^4 - s^2}) ds$$

is real analytic in λ. The rest of the arguement is the same as in the above case. Proposition 3.2 again contains the details of the real-analyticity of this function in the variable λ. Note that in this case

$$\int_{B_r^{H_1}} \psi_k^\lambda(\mathbf{z}, t) d\hat{\mu}_r(\mathbf{z}, t) = \int_{-r}^{r} e^{-2\pi i\lambda s} \Psi(k, \lambda, \sqrt{r^4 - s^2}) ds. \tag{7}$$

The proof for the integral conditions (3) is very similar. Here the convolution $f * T_{3S}$ will instead reduce as

$$(f * T_{3S})^{\frown\lambda} = \int_{-r}^{r} e^{-2\pi i\lambda s} \left(\hat{f}^\lambda *^\lambda T_{r^2 - |s|} \right) ds.$$

Following the same steps in the previous case, we find that for each $k \in \mathbf{Z}_+$

$$c_k(\lambda) \int_{-r}^{r} e^{-2\pi i\lambda s} \Psi(k, \lambda, r^2 - |s|) ds \qquad \text{for all } \lambda \in \mathbf{R}^*$$

with the goal of demonstrating that each $c_k(\lambda) = 0$, as above. We then only need to show

$$\int_{-r}^{r} e^{-2\pi i\lambda s} \Psi(k, \lambda, r^2 - |s|) ds$$

is real analytic in λ. The rest of the arguement is the same as in the above case. See also Proposition 3.2 for the details of the real-analyticity. Note that in this case

$$\int_{B_r^{H_3}} \psi_k^\lambda(\mathbf{z}, t) d\hat{\mu}_r(\mathbf{z}, t) = \int_{-r}^{r} e^{-2\pi i\lambda s} \Psi(k, \lambda, \sqrt{r^2 - |s|}) ds. \tag{8}$$

This completes the proof of Theorem 3.1.

Proposition 3.3. *Each of the functions defined in the integrals* (6), (7), *and* (8) *is real-analytic in the variable* λ.

Proof. The method is based on use of the real-analyticity of the integrand to investigate the power series and prove real-analyticity. We begin with the function

$$h_3(k, \lambda, r) = \int_{-r}^{r} e^{-2\pi i\lambda s} \Psi(k, \lambda, r^2 - |s|) ds$$

associated to the integral conditions (8) for $B_r^{H_3}$. In this case the form of the integral simplifies sufficiently so that power series are not required. By expanding the integral

$$\begin{aligned}&\int_{-r}^{r} e^{-2\pi i\lambda s}\Psi(k,\lambda,r^2-|s|)ds\\ &=\frac{1}{2(4\pi|\lambda)^n}\int_{-r}^{r} e^{-2\pi i\lambda s}\left(\int_0^{4\pi|\lambda|(r^2-|s|)} e^{-x/2}L_k^{(n-1)}(x)x^{n-1}\right)ds\end{aligned}$$

and expressing the Laguerre polynomial in the integral as a finite sum of terms $L_k^{(n-1)}(x)=\sum_{j=0}^{k} c_j^{k,n-1}x^j$, we obtain a finite sum of integrals

$$\sum_{j=0}^{k} c_j^{k,n-1}\int_{-r}^{r} e^{-2\pi i\lambda s}\left(\int_0^{4\pi|\lambda|(r^2-|s|)} e^{-x/2}x^{n+j-1}dx\right)ds.$$

Clearly it is enough to show each of these is real-analytic. As shown in [8], the inner integral evaluates to a finite sum of terms of the form

$$\sum_{\gamma,\text{finite}} c_\gamma\left(4\pi|\lambda|(r^2-|s|)\right)^\gamma e^{-2\pi|\lambda|(r^2-|s|)}.$$

Since the integral for h_3 now has the form

$$\sum_{\gamma,\text{finite}} c_\gamma\int_{-r}^{r} e^{-2\pi i\lambda s}\left(4\pi|\lambda|\right)^\gamma\left(r^2-|s|\right)^\gamma e^{-2\pi|\lambda|(r^2-|s|)}ds$$

it is possible to represent this as a finite sum of terms of the form

$$c' r^{2\ell}\int_{-r}^{r} e^{-2\pi i\lambda s}e^{-2\pi|\lambda|(r^2-|s|)}|s|^k ds$$

and show that each of these is real-analytic. In this case we do not need power series, but rather can write the integral in the form

$$\begin{aligned}\int_{-r}^{r} e^{-2\pi i|\lambda|s}e^{-2\pi|\lambda|(r^2-|s|)}|s|^k ds &= \int_{-r}^{r} e^{-2\pi|\lambda|(r^2+i\operatorname{sgn}(\lambda)-\operatorname{sgn}(s))}|s|^k ds\\ &= e^{-2\pi|\lambda|r^2}\int_{-r}^{r} e^{-2\pi|\lambda|\omega s}|s|^k ds\end{aligned}$$

where $\omega=-\operatorname{sgn}(s)+i\operatorname{sgn}(\lambda)$. We use the same result of [8] to state this is a finite sum of terms of the form $(2\pi|\lambda|\omega r)$. Since each of these terms is real-analytic, this completes this first case of conditions (8).

Next we move to integral conditions (6) associated to the ball $B_r^{H_4}$, for which the function $h_4(k, \lambda, r)$ must be shown to be real-analytic in λ. As in the previous case, it is again possible to expand the integral

$$\begin{aligned}&\int_{-r}^{r} e^{-2\pi i\lambda s}\Psi(k,\lambda,r^2-s^2)ds\\&=\frac{1}{2(4\pi|\lambda|)^n}\int_{-r}^{r} e^{-2\pi i\lambda s}\left(\int_0^{4\pi|\lambda|(r^2-s^2)} e^{-x/2}L_k^{(n-1)}(x)x^{n-1}\right)ds\end{aligned}$$

and rewrite it as a finite sum of terms of the form

$$\sum_{j=0}^{k} c_j^{k,n-1}\int_{-r}^{r} e^{-2\pi i\lambda s}\left(\int_0^{4\pi|\lambda|(r^2-s^2)} e^{-x/2}x^{n+j-1}dx\right)ds.$$

As before, this also reduces to showing real-analyticity in λ for each of the integrals of the form

$$c'r^{2\ell}\int_{-r}^{r} e^{-2\pi i\lambda s}e^{-2\pi|\lambda|(r^2-s^2)}s^{2k}ds.$$

After grouping, completing the square, and changing variables, this integral is simplified as follows

$$\begin{aligned}&\int_{-r}^{r} e^{-2\pi i\lambda s}e^{-2\pi|\lambda|(r^2-s^2)}s^{2k}ds\\&=e^{-2\pi\lambda r^2}\int_{-r}^{r} e^{-2\pi|\lambda|(i\mathrm{sgn}(\lambda)s-s^2)}s^{2k}ds\\&=e^{-2\pi|\lambda|r^2}e^{\pi|\lambda|/2}\int_{-r}^{r} e^{-2\pi|\lambda|(i\mathrm{sgn}(\lambda)s-s^2+1/4)}s^{2k}ds\\&=e^{-\pi|\lambda|(2r^2-1/2)}\int_{-r}^{r} e^{-2\pi|\lambda|(s-i\mathrm{sgn}(\lambda)/2)^2}s^{2k}ds\\&=e^{-\pi|\lambda|(2r^2-1/2)}\int_{-r}^{r} e^{-2\pi|\lambda|s^2}\left(s+\frac{i\mathrm{sgn}(\lambda)}{2}\right)^{2k}ds\\&=e^{-\pi|\lambda|(2r^2-1/2)}\sum_{j=0}^{2k}\binom{2k}{j}\left(\frac{i\mathrm{sgn}(\lambda)}{2}\right)^j\int_{-r}^{r} e^{-2\pi|\lambda|s^2}s^{2k-j}ds\end{aligned}$$

Each of the terms in this finite sum is real-analytic in λ, because the integral $\int_{-r}^{r} e^{-2\pi|\lambda|s^2}s^{2k-j}ds$ is real-analytic in λ, as we have observed in previously from the result of [8].

Finally, for integral conditions (7) associated to the ball $B_r^{H_1}$, the function to be shown real-analytic is $h_1(k, \lambda, r)$ given by the integral

$$\int_{-r^2}^{r^2} e^{-2\pi i\lambda s}\Psi(k,\lambda,\sqrt{r^4-s^2})ds$$
$$= \frac{1}{2(4\pi|\lambda|)^n}\int_{-r}^{r} e^{-2\pi i\lambda s}\left(\int_0^{4\pi|\lambda|(\sqrt{r^4-s^2})} e^{-x/2}L_k^{(n-1)}(x)x^{n-1}\right)ds.$$

Similarly to the previous cases, this integral is reduced to a finite sum of terms of the form

$$\int_{-r^2}^{r^2} e^{-2\pi i\lambda s}\,(4\pi|\lambda|)^{\gamma}\left(\sqrt{r^4-s^2}\right)^{\gamma} e^{-2\pi|\lambda|\sqrt{r^4-s^2}}ds$$

each of which must be shown to be real-analytic in λ. Expanding using power series, we find

$$\int_{-r^2}^{r^2} e^{-2\pi i\lambda s}e^{-2\pi|\lambda|\sqrt{r^4-s^2}}\left(\sqrt{r^4-s^2}\right)^{\gamma}ds$$
$$= \int_{-r^2}^{r^2} e^{-2\pi|\lambda|(i\mathrm{sgn}(\lambda)s+\sqrt{r^4-s^2})}\left(\sqrt{r^4-s^2}\right)^{\gamma}ds$$
$$= \sum_{j=0}^{\infty}\left[\int_{-r^2}^{r^2}\left(i\mathrm{sgn}(\lambda)s+\sqrt{r^4-s^2}\right)^{j}\left(\sqrt{r^4-s^2}\right)^{\gamma}ds\right]|\lambda|^{j}$$

where the inner integral can be written as

$$\sum_{k=0}^{j}\binom{j}{k}(i\mathrm{sgn}(\lambda))^{k}\int_{-r^2}^{r^2} s^{k}\left(\sqrt{r^4-s^2}\right)^{\gamma+k-j}ds.$$

Using the symmetry, the sum is reduced to a sum of only the even terms, for which the sum is non-zero. This sum of integrals is then further reduced with trigonometric substitution and substitution, to the form

$$\sum_{k\le j,\ k\text{ even}}\binom{j}{k}(-1)^{k/2}2r^{2(\gamma+j+1)}\int_0^1 u^k\left(\sqrt{1-u^2}\right)^{\gamma+j+k}du.$$

In order to display real-analyticity of the series

$$S(\lambda) = \sum_{j=0}^{\infty}\frac{(-2\pi)^j}{j!}\left[\sum_{k\le j,\ k\text{ even}}\binom{j}{k}(-1)^{k/2}2r^{2(\gamma+j+1)}\right.$$
$$\left.\int_0^1 u^k\left(\sqrt{1-u^2}\right)^{\gamma+j+k}du\right]|\lambda|^{j},$$

we break this series into four separate series, each of which is alternating. Then the bound of the remainder term for each such series is shown to go to 0 in order to demonstrate real-analyticity of each. The division of the terms into the four series is based on the value of j in $\binom{j}{k}$ and also corresponding to the exponent in $|\lambda|^j$. This is considered together with the sign of the term, as determined by $(-1)^{k/2}$. For each j, the positive and negative terms are split between two series. Four series are needed because a new positive term is added every fourth row, i.e. from j to $j+4$. Thus $S(\lambda) = s_1(\lambda) + s_2(\lambda) + s_3(\lambda) + s_4(\lambda)$. The series s_1 contains positive terms from rows $j \equiv 0(\mathrm{mod}4)$ and negative terms from rows $j \equiv 2(\mathrm{mod}4)$, while s_2 contains positive terms from rows $j \equiv 1(\mathrm{mod}4)$ and negative terms from rows $j \equiv 3(\mathrm{mod}4)$. Similarly s_3 contains positive terms from rows $j \equiv 2(\mathrm{mod}4)$ and negative terms from rows $j \equiv 0(\mathrm{mod}4)$, while s_2 contains positive terms from rows $j \equiv 3(\mathrm{mod}4)$ and negative terms from rows $j \equiv 1(\mathrm{mod}4)$.

We then show that each of s_1, s_2, s_3, and s_4 is real-analytic in λ by showing the remainder terms $R_{i,n}$ in the sums $s_i = \sum_{j=0}^{n-1} c_j|\lambda|^{2j} + R_{i,n}(\lambda)$, for $i = 1, \ldots, 4$, converge to 0. Using that

$$\sum_{k \leq j,\ k \text{ even}} \binom{j}{k} = 2^{j-1}$$

and, for k even,

$$\int_{-r^2}^{r^2} s^k \left(\sqrt{r^4 - s^2}\right)^{\gamma+j-k} ds = 2r^{2(\gamma+j+1)} \int_0^1 u^k \left(\sqrt{1-u^2}\right)^{\gamma+j-k} du \leq 2r^{s(\gamma+j+1)},$$

we see that $|c_j| \leq \frac{1}{(2j)!} r^{\gamma+1}(4r)^j$. It then follows that

$$|R_{i,n}(\lambda)| \leq \frac{1}{(2n)!} r^{\gamma+1}(4r)^n |\lambda^2|^n.$$

For any fixed λ,

$$\lim_{n\to\infty} |R_{i,n}(\lambda)| = \lim_{n\to\infty} \frac{r^{\gamma+1}(4r)^n|\lambda^2|^n}{(2n)!} = 0,$$

thus $s_i(\lambda)$ is real-analytic.

Since each $s_i(\lambda)$, for $i = 1, \ldots, 4$, is real-analytic in λ, it follows that each of the terms of the form

$$\int_{-r^2}^{r^2} e^{-2\pi i \lambda s} (4\pi|\lambda|)^\gamma \left(\sqrt{r^4 - s^2}\right)^\gamma e^{-2\pi|\lambda|\sqrt{r^4-s^2}} ds$$

is real-analytic, implying the same result for the function $h_1(k, \lambda, r)$. This completes the proof of Proposition 3.2.

In closing, note that the results of Theorem 3.1 show that the Pompeiu property extends to these four versions of a Heisenberg ball, for the function space L^2. This is as expected based on comparison to the Euclidean case, where each individual set S will have the Pompeiu property at the level of the function space L^2. By the usual means this result extends to L^p for $1 \leq p \leq 2$.

4. L^p RESULTS

In this section we extend the results from L^p for $1 \leq p \leq 2$ to L^p for $1 \leq p < \infty$. The method used is that developed in [4], where the one radius results for the standard version of the Pomepiu problem in $\mathbf{H}^n$ were extended from L^p for $1 \leq p \leq 2$ to L^p for $1 \leq p < \infty$. This method relies on the L^p summability of $f \in L^p(\mathbf{H}^n)$ for $1 < p < \infty$ in terms of its spectral projections, as proven in [13]. It also requires a property known as mean-periodicity of the exponential Laguerre functions $\{\psi^\lambda_{\mathbf{k}}\}$ with respect to the measures on the balls which are being considered, T_{iS}, for $i = 1, 2, 3, 4$, where these represent the measures on the balls $B_r^{H_1}, B_r^{H_2}, B_r^{H_3}$, and $B_r^{H_4}$, respectively, as defined in Section 3. The property of mean-periodicity expresses a specific relation under convolution, namely there exist functions f, g, and h, not dependent on $[\mathbf{z}, t] \in \mathbf{H}^n$, such that

$$\left(\psi^\lambda_{\mathbf{k}} * T_{1S}\right)(\mathbf{z}, t) = h_1(\mathbf{k}, \lambda, r) \cdot \psi^\lambda_{\mathbf{k}}(\mathbf{z}, t),$$

$$\left(\psi^\lambda_{\mathbf{k}} * T_{2S}\right)(\mathbf{z}, t) = h_2(\mathbf{k}, \lambda, r) \cdot \psi^\lambda_{\mathbf{k}}(\mathbf{z}, t),$$

$$\left(\psi^\lambda_{\mathbf{k}} * T_{3S}\right)(\mathbf{z}, t) = h_3(\mathbf{k}, \lambda, r) \cdot \psi^\lambda_{\mathbf{k}}(\mathbf{z}, t),$$

and

$$\left(\psi^\lambda_{\mathbf{k}} * T_{4S}\right)(\mathbf{z}, t) = h_4(\mathbf{k}, \lambda, r) \cdot \psi^\lambda_{\mathbf{k}}(\mathbf{z}, t).$$

We first prove this result in the following Lemma.

Lemma 4.1. *Let $(\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$. Each exponential Laguerre function is mean-periodic with respect to the measures T_{1S}, T_{2S}, T_{3S}, and T_{4S}. In particular, we may write*

$$\left(\psi^\lambda_{\mathbf{k}} * T_{1S}\right)(\mathbf{z}, t) = h_1(\mathbf{k}, \lambda, r) \cdot \psi^\lambda_{\mathbf{k}}(\mathbf{z}, t),$$

$$\left(\psi^\lambda_{\mathbf{k}} * T_{2S}\right)(\mathbf{z}, t) = h_2(\mathbf{k}, \lambda, r) \cdot \psi^\lambda_{\mathbf{k}}(\mathbf{z}, t),$$

$$\left(\psi^\lambda_{\mathbf{k}} * T_{3S}\right)(\mathbf{z}, t) = h_3(\mathbf{k}, \lambda, r) \cdot \psi^\lambda_{\mathbf{k}}(\mathbf{z}, t),$$

and

$$\left(\psi^\lambda_{\mathbf{k}} * T_{4S}\right)(\mathbf{z}, t) = h_4(\mathbf{k}, \lambda, r) \cdot \psi^\lambda_{\mathbf{k}}(\mathbf{z}, t),$$

where $h_1(\lambda, \mathbf{k}, r)$, $h_2(\lambda, \mathbf{k}, r)$, $h_3(\lambda, \mathbf{k}, r)$, and $h_4(\lambda, \mathbf{k}, r)$ are computed in the proof below.

Proof. We use the mean periodicity of $\psi^\lambda_{\mathbf{k}}$ with respect to T_α, where T_α is the measure assoicated to the ball $B_r = \{\mathbf{z} : \|\mathbf{z}\| \leq r\}$ in $\mathbf{C}^n \times \{0\}$. We have

$$\begin{aligned}
(\psi^\lambda_{\mathbf{k}} * T_\alpha)(\mathbf{z}, t) &= e^{-2\pi i \lambda t} \int_{|\mathbf{z}|<\alpha} e^{-4\pi i \lambda \mathrm{Im}\, \mathbf{z}\cdot\bar{\mathbf{w}}} \mathcal{W}^\lambda_{kk}(\mathbf{z}) d\mu_\alpha(\mathbf{z}) \\
&= \Psi(\mathbf{k}, \lambda, \alpha^2) \psi^\lambda_{\mathbf{k}}(\mathbf{z}, t),
\end{aligned}$$

where $\Psi(\mathbf{k},\lambda,\alpha^2)=\frac{c}{(4\pi|\lambda|)^n}\int_0^{4\pi|\lambda|\alpha^2}e^{-x/2}x^{n-1}L_k^{(n-1)}(x)dx$, as given in the previous section; see for instance [6, 5]. Thus we can write

$$\begin{aligned}
\psi_{\mathbf{k}}^{\lambda}*T_{2S}(\mathbf{z},t) &= \int_{\|\mathbf{w}\|<r,|s|<r}\psi_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w},t-s-2\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle)d\hat{\mu}_r(\mathbf{w},s)\\
&= \int_{\|\mathbf{w}\|<r,|s|<r} ce^{2\pi i\lambda(t-s-2\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle)}\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w})d\hat{\mu}_r(\mathbf{w},s)\\
&= e^{2\pi i\lambda t}\int_{|s|<r}e^{-2\pi i\lambda s}\left(e^{-4\pi i\lambda\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle}\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w})d\mu_r(\mathbf{w})\right)ds\\
&= e^{2\pi i\lambda t}\left(\sin(2\pi\lambda r)\right)\left(\Psi(\mathbf{k},\lambda,r^2)\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z})\right)\\
&= \Psi(\mathbf{k},\lambda,r^2)\left(\sin(2\pi\lambda r)\right)\psi_{\mathbf{k}}^{\lambda}(\mathbf{z},t).
\end{aligned}$$

Thus $\psi_{\mathbf{k}}^{\lambda}*T_{2S}(\mathbf{z},t)=h_2(k,\lambda,r)\cdot\psi_k^{\lambda}(\mathbf{z},t)$, where $h_2(k,\lambda,r)=\sin(2\pi\lambda r)\Psi(k,\lambda,r^2)$. The other cases are computed similarly. In the case of T_{4S}, we have

$$\begin{aligned}
&\psi_{\mathbf{k}}^{\lambda}*T_{4S}(\mathbf{z},t)\\
&= \int_{\|\mathbf{w}\|^2+s^2<r^2}\psi_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w},t-s-2\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle)d\hat{\mu}_r(\mathbf{w},s)\\
&= \int_{-r}^{r}\left(\int_{\|\mathbf{w}\|\le\sqrt{r^2-s^2}}e^{2\pi i\lambda(t-s-2\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle)}\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w})d\mu_{\sqrt{r2-s^2}}(\mathbf{w})\right)ds\\
&= e^{2\pi i\lambda t}\int_{-r}^{r}e^{-2\pi i\lambda s}\left(\int_{\|\mathbf{w}\|\le\sqrt{r^2-s^2}}e^{-4\pi i\lambda\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle}\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w})d\mu_{\sqrt{r^2-s^2}}(\mathbf{w})\right)ds\\
&= e^{2\pi i\lambda t}\left(\int_{-r}^{r}e^{-2\pi i\lambda s}\Psi(\mathbf{k},\lambda,r^2-s^2)ds\right)\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z})\\
&= \left(h_4(k,\lambda,r)\right)\cdot\psi_{\mathbf{k}}^{\lambda}(\mathbf{z},t),
\end{aligned}$$

where $h_4(k,\lambda,r)=\int_{-r}^{r}e^{-2\pi i\lambda s}\Psi(k,\lambda,r^2-s^2)ds$. Next consider the case of T_{1S}.

$$\begin{aligned}
&\psi_{\mathbf{k}}^{\lambda}*T_{1S}(\mathbf{z},t)\\
&= \int_{\|\mathbf{w}\|^4+s^2<r^4}\psi_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w},t-s-2\langle\mathbf{z},\mathbf{w}\rangle)d\hat{\mu}_r(\mathbf{w},s)\\
&= \int_{-r}^{r}\left(\int_{\|\mathbf{w}\|\le\sqrt[4]{r^4-s^2}}e^{2\pi i\lambda(t-s-2\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle)}\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w})d\mu_{\sqrt[4]{r^4-s^2}}(\mathbf{w})\right)ds\\
&= e^{2\pi i\lambda t}\int_{-r}^{r}e^{-2\pi i\lambda s}\left(\int_{\|\mathbf{w}\|\le\sqrt[4]{r^4-s^2}}e^{-4\pi i\lambda\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle}\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w})d\mu_{\sqrt[4]{r^4-s^2}}(\mathbf{w})\right)ds\\
&= e^{2\pi i\lambda t}\left(\int_{-r}^{r}e^{-2\pi i\lambda s}\Psi(\mathbf{k},\lambda,\sqrt{r^4-s^2})ds\right)\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z})\\
&= \left(h_1(k,\lambda,r)\right)\cdot\psi_{\mathbf{k}}^{\lambda}(\mathbf{z},t),
\end{aligned}$$

where $h_1(k,\lambda,r)=\int_{-r}^{r} e^{-2\pi i\lambda s}\Psi(k,\lambda,\sqrt{r^4-s^2})ds$. Finally, for the case of T_{3S},

$$\begin{aligned}
&\psi_{\mathbf{k}}^{\lambda} * T_{3S}(\mathbf{z},t)\\
&= \int_{\|\mathbf{w}\|^2+|s|<r^2} \psi_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w},t-s-2\langle\mathbf{z},\mathbf{w}\rangle)d\hat{\mu}_r(\mathbf{w},s)\\
&= \int_{-r}^{r}\left(\int_{\|\mathbf{w}\|\le\sqrt{r^2-|s|}} e^{2\pi i\lambda(t-s-2\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle)}\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w})d\mu_{\sqrt{r^2-|s|}}(\mathbf{w})\right)ds\\
&= e^{2\pi i\lambda t}\int_{-r}^{r} e^{-2\pi i\lambda s}\left(\int_{\|\mathbf{w}\|\le\sqrt{r^2-|s|}} e^{-4\pi i\lambda\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle}\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}-\mathbf{w})d\mu_{\sqrt{r^2-|s|}}(\mathbf{w})\right)ds\\
&= e^{2\pi i\lambda t}\left(\int_{-r}^{r} e^{-2\pi i\lambda s}\gamma(\mathbf{k},\lambda,\sqrt{r^2-|s|})ds\right)\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z})\\
&= (h_3(k,\lambda,r))\cdot\psi_{\mathbf{k}}^{\lambda}(\mathbf{z},t),
\end{aligned}$$

where $h_3(k,\lambda,r)=\int_{-r}^{r} e^{-2\pi i\lambda s}\Psi(k,\lambda,r^2-|s|)ds$. This completes the proof of Lemma 4.1.

We observe that functions h_1, h_2, h_3, and h_4 as defined in the lemma are the same functions that arose in the proof of Theorem 3.1. At that point, each was shown to be real-analytic in $|\lambda|$. This real-analyticity is an important part of the proof of the next result.

We are now able to extend Theorem 3.1 to the function spaces L^p for $1\le p<\infty$.

Theorem 4.2. *Let $f\in L^p(\mathbf{H}^n)\cap C(\mathbf{H}^n)$, for $1\le p<\infty$ and let $r>0$. Suppose f satisfies integral conditions (1), (2), (3), or (4). That is, for $B_r^H = B_r^{H_1}, B_r^{H_2}, B_r^{H_3}$ or $B_r^{H_4}$ as defined above and μ_r volume measure on this set, assume*

$$\int_{B_r^H} L_{\mathbf{g}} f(\mathbf{z},t)d\mu_r(\mathbf{z},t)=0 \qquad \textit{for all } \mathbf{g}\in\mathbf{H}^n.$$

Then $f\equiv 0$.

Proof. Since $f\in L^p(\mathbf{H}^n)$ for $1<p<\infty$, we use the summation

$$f(\mathbf{z},t)=\lim_{r\to 1^-}\sum_{\mathbf{k}\ge\mathbf{0}} r^{|\mathbf{k}|}\left(P_{\mathbf{k},+}(f)(\mathbf{z},t)+P_{\mathbf{k},-}(f)(\mathbf{z},t)\right).$$

The integral conditions (1), (2), (3), and (4) are expressed by the convolution equations $f * T_{iS}$, for $i=1,2,3$. Using the Abel means, the convolution equations become

$$\begin{aligned}
f * T_{iS} &= \lim_{r\to 1^-}\sum_{k\ge 0} r^k\left(\int_0^\infty (f*\phi_{k,+}^{\lambda} * T_{iS})(\mathbf{z},t)d\lambda+\int_0^\infty (f*\phi_{k,-}^{\lambda} * T_{iS})(\mathbf{z},t)\right)\\
&= \lim_{r\to 1^-}\sum_{k\ge 0} r^k\int_{-\infty}^{\infty} f*\phi_k^{\lambda} * T_{iS}ds.
\end{aligned}$$

Using the relationship $(\psi_k^\lambda * T_{iS})(\mathbf{z},t) = h_i(k,\lambda,r)\psi_k^\lambda(\mathbf{z},t)$ for $i = 1,2,3,4$, we may write

$$\begin{aligned} f * T_{iS} &= \lim_{r\to 1^-}\sum_{k\geq 0} r^k \left(\int_0^\infty (f*\phi_{k,+}^\lambda * T_{iS})(\mathbf{z},t)d\lambda + \int_0^\infty (f*\phi_{k,-}^\lambda * T_{iS})(\mathbf{z},t)\right) \\ &= \lim_{r\to 1^-}\sum_{k\geq 0} r^k \int_{-\infty}^\infty h_i(k,\lambda,r)(f*\phi_k^\lambda)(\mathbf{z},t)d\lambda, \end{aligned}$$

which equals 0 by the integral conditions. Applying the projection operators P_k to the equation

$$\lim_{r\to 1^-}\sum_{k\geq 0} r^k \int_{-\infty}^\infty h_i(k,\lambda,r)(f*\phi_k^\lambda)(\mathbf{z},t)d\lambda = 0,$$

we obtain that

$$\int_{-\infty}^\infty h_i(k,\lambda,r)\left(f*\psi_k^\lambda\right)(\mathbf{z},t)|\lambda|^n d\lambda = 0$$

for each $k \in (\mathbf{Z}_+)$. Now choose a sequence $\{f_j\}$ converging to f in L^p-norm, such that each $f_j \in \mathcal{S}(\mathbf{H}^n)$. We may then write

$$\lim_{j\to\infty}\int_{-\infty}^\infty h(k,\lambda,r)\left(f_j*\psi_k^\lambda\right)(\mathbf{z},t)|\lambda|^n d\lambda = 0,$$

and thus

$$\lim_{j\to\infty}\int_{-\infty}^\infty h(k,\lambda,r)(\widetilde{P_k f_j})^\lambda(\mathbf{z})e^{i\lambda t}d\lambda = 0.$$

Since the above sequence converges to 0 in the L^p-norm, the sequence of partial Fourier transforms converges to 0 in the sense of distributions.

$$\lim_{j\to\infty} h(k,\lambda,r)(\widetilde{P_k f_j})^\lambda(\mathbf{z}) = h(k,\lambda,r)(\widetilde{P_k f})^\lambda(\mathbf{z}) = 0$$

Observe also that h_1, h_2, h_3, and h_4 correspond to the functions (7), (5), (8), and (6), respectively, from Section 3. There, in Proposition 3.2 and in the proof of Theorem 3.1, it is demonstrated that each of these functions is real-analytic and thus has isolated zeros. As a consequence $(\widetilde{P_k f})^\lambda$ is almost everywhere 0. This implies $P_{\mathbf{k}}f = 0$, for all $k \in \mathbf{Z}_+$. Thus $f \equiv 0$, as claimed. This completes the proof of Theorem 4.2.

The results of Theorem 4.2 are closely associated with those of Theorem 3.1, and the methods are also closely associated. In these cases we expect the results to work for any form of the Heisenberg ball. However, the results for the function space L^∞ in the next section follow different methods and the ball requires certain properties for the analysis to work.

5. L^∞ Results

It is characteristic of L^∞ results on the Pompeiu problem to require multiple radii, often with conditions on the radii. See, for instance, [3, 5, 9, 11]. The conditions on the radii are based on avoidance of common zeros of the Gelfand transform. Once there is a set of measures for which the Gelfand transform has no common zeros, the Tauberian theorem Theorem 2.1 described in Section 2 provides the conclusion that the given sets possess the Pompeiu property. These L^∞ results of two radii require a scaling of the ball with respect to dilation. For this reason balls $B_r^{H_1}$, $B_r^{H_2}$, and $B_r^{H_3}$ can be used, but $B_r^{H_4}$ will not be used because it is not invariant under dilation.

Here we will apply the Gelfand transform to the measure associated to each of the balls considered in this paper. The functions we obtain will be fairly complicated, comparable to the functions obtained in the integrals of Sections 3 and 4. We begin with the requisite background to describe these functions. For use in the following theorems, we now define the functions F_1, and F_3 arising from the Gelfand transforms of the measures T_{1S}, T_{2S} and T_{3S}. We define $F_{1,k}$ and $F_{3,k}$ by

$$F_{1,k}^{\pm}(x) = \int_{-x}^{x} e^{\mp is/2}\Psi_k^{(n-1)}(\sqrt{x^2-s^2})ds$$
$$F_{3,k}^{\pm}(x) = \int_{-x}^{x} e^{\mp is/2}\Psi_k^{(n-1)}(x-|s|)ds,$$

and for relevant integrals involving Bessel functions, we define $Jo_{1,n}$ and $Jo_{3,n}$ as follows

$$Jo_{1,n}(x) = \int_0^{x^2} (\rho\sqrt[4]{x^4-s^2})^n J_n(\rho\sqrt[4]{x^4-s^2})ds$$
$$Jo_{3,n}(x) = \int_0^{x} (\rho\sqrt{x^2-|s|})^n J_n(\rho\sqrt{x^2-|s|})ds.$$

We remark that for each of these functions, it would be possible to make a more explicit description of the function from the integral defining the function by use of the Laplace transform and its inverse. This is similar to the method used in [9, 11] to define certain special functions.

We now address the Pompeiu problem at the L^∞ level for the three versions of a ball in $\mathbf{H}^n$ given in this paper. We begin with the integral conditions (1),

$$\int_{B_r^{H_1}} L_{\mathbf{g}}f(\mathbf{z},t)d\hat{\mu}_r(\mathbf{z},t) \qquad \text{for all } \mathbf{g}\in\mathbf{H}^n,$$

where $B_r^{H_1} = \{[\mathbf{z},t] : \|\mathbf{z}\|^4 + t^2 \le r^4\}$. We have the following theorem.

Theorem 5.1. *Consider $f \in C\cap L^\infty(\mathbf{H}^n)$ and suppose f satisfies the integral conditions*

$$\int_{B_{r_i}^{H_1}} L_{\mathbf{g}}f(\mathbf{z},t)d\hat{\mu}_r(\mathbf{z},t) = 0 \qquad \textit{for all } \mathbf{g}\in\mathbf{H}^n \tag{9}$$

for radii $r_i = r_1, r_2$. Furthermore suppose the radii satisfy the conditions

(1) $r_1/r_2 \notin \mathcal{Q}\,(Jo_{1,n}(x))$,

(2) $(r_1/r_2)^2 \notin \mathcal{Q}\left(F^{\pm}_{1,k}(x)\right)$ *for all* $k \in \mathbf{Z}_+$.

Then we may conclude $f \equiv 0$. Furthermore, if the radii do not satisfy either of these conditions, then there exists $f \not\equiv 0$ satisfying the integral conditions.

Proof. Associated to these integral conditions, we write the convolution equation $f * T_{1S} = 0$. Applying the Gelfand transform to T_{1S}, we have

$$\begin{aligned}
&\tilde{T}_{1S}(\lambda; k)\\
&= c\int_{-r^2}^{r^2}\int_{|\mathbf{z}|<\sqrt[4]{r^4-t^2}} e^{-2\pi i\lambda t} e^{-2\pi|\lambda||\mathbf{z}|^2} L_k^{(n-1)}(4\pi|\lambda||\mathbf{z}|^2)\,d\mu_{\sqrt[4]{r^4-t^2}}(\mathbf{z})\,dt\\
&= c\int_{-r^2}^{r^2} e^{-2\pi i\lambda t}\left(\int_0^{\sqrt[4]{r^4-t^2}}\int_{|\mathbf{z}|=s} e^{-2\pi|\lambda||\mathbf{z}|^2} L_k^{(n-1)}(4\pi|\lambda||\mathbf{z}|^2)\,d\sigma_s(\mathbf{z})s^{2n-1}ds\right)dt\\
&= \frac{c}{2}\int_{-r^2}^{r^2} e^{-2\pi i\lambda t}\left(\int_0^{\sqrt[4]{r^4-t^2}} e^{-2\pi|\lambda|s} L_k^{(n-1)}(4\pi|\lambda|s)s^{n-1}ds\right)dt\\
&= \frac{c}{2}\int_{-r^2}^{r^2} e^{-2\pi i\lambda t}\Psi_k^{(n-1)}(4\pi|\lambda|\sqrt{r^4-t^2})\,dt\\
&= c' F^{\pm}_{1,k}(4\pi|\lambda|r^2),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{T}_{1S}(0;\rho) &= c\int_{-r}^{r}\int_{|\mathbf{z}|<\sqrt[4]{r^4-t^2}} \frac{J_{n-1}(\rho|\mathbf{z}|)}{(\rho|\mathbf{z}|)^{n-1}}\,d\mu_{\sqrt[4]{r^4-t^2}}(\mathbf{z})\,dt\\
&= \int_{-r^2}^{r^2}\left(\int_0^{\sqrt[4]{r^4-t^2}}\int_{|\mathbf{z}|=s}\frac{J_{n-1}(\rho|\mathbf{z}|)}{(\rho|\mathbf{z}|)^{n-1}}\,d\sigma_s(\mathbf{z})s^{2n-1}ds\right)dt\\
&= \int_{-r^2}^{r^2}\left(\int_0^{\sqrt[4]{r^4-t^2}}\frac{J_{n-1}(\rho s)}{(\rho s)^{n-1}}s^{2n-1}ds\right)dt\\
&= \frac{1}{\rho^{2n}}\int_{-r^2}^{r^2}\left(\int_0^{\rho\sqrt[4]{r^4-t^2}} s^n J_n(x)\,ds\right)dt\\
&= 2\frac{1}{\rho^{2n}}\int_0^{r^2}(\rho\sqrt[4]{r^4-t^2})^n J_n(\rho\sqrt[4]{r^4-t^2})\,dt\\
&= c' Jo_{1,n}(\rho r).
\end{aligned}$$

Condition 1. of the theorem is equivalent to no common zeros for $\widetilde{T}_{1S,r_i}(0,\rho)$ for $i = 1, 2$. Condition 2. of the theorem is equivalent to no common zeros for

$\widetilde{T}_{1S,r_i}(\lambda;k)$ for $i=1,2$. Thus, for each $(0;\rho)\in\mathbf{R}_+$ and each $(\lambda;k)\in\mathbf{R}^*\times\mathbf{Z}_+$, either $\widetilde{T}_{1S,r_1}=0$ or $\widetilde{T}_{1S,r_2}=0$. Since the conditions for the Tauberian theorem Theorem 2.1 are satisfied, we have that $f*L^1_{\mathbf{0}}(\mathbf{H}^n)=0$. Thus $f\equiv 0$, as claimed.

If either of conditions 1. or 2. are not met, we will find non-zero functions satisfying the integral conditions (9). If condition 1. is not met, there exists $\rho_0\in\mathbf{R}_+$ such that $Jo_{1,n}(\rho_0 r_1)=Jo_{1,n}(\rho_0 r_2)=0$. Then letting $f(\mathbf{z},t)=\frac{J_{n-1}(\rho_0|\mathbf{z}|)}{(\rho_0|\mathbf{z}|)^{n-1}}$, the integral conditions (9), for $r_i=r_1,r_2$, become

$$\begin{aligned}\int_B L_{\mathbf{g}}f(\mathbf{z},t)d\hat{\mu}_{r_i}(\mathbf{z},t) &= c\int_{-r_i^2}^{r_i^2}\int_{|\mathbf{z}|\le\sqrt[4]{r_i^4-t^2}}\frac{J_{n-1}(\rho_0|\mathbf{z}|)}{(\rho_0|\mathbf{z}|)^{n-1}}d\mu_{\sqrt[4]{r_i^4-t^2}}(\mathbf{z})dt\\ &= c\int_{-r_i^2}^{r_i^2}\left(\int_0^{\rho_0\sqrt[4]{r_i^4-t^2}}s^nJ_n(s)ds\right)dt\\ &= c'Jo_{1,n}(\rho_0 r_i)=0.\end{aligned}$$

Thus integral conditions (9) are satisfied for $f(\mathbf{z},t)=\frac{J_{n-1}(\rho_0|\mathbf{z}|)}{(\rho_0|\mathbf{z}|)^{n-1}}$.

Similarly if 2. is not met, there exists $(\lambda_0,k_0)\in\mathbf{R}^*\times\mathbf{Z}_+$ such that $F^{\pm}_{1,k}(|\lambda_0|r_1)=F^{\pm}_{1,k}(|\lambda_0|r_2)=0$, and $\operatorname{sgn}(\lambda_0)=\pm1$. Then letting $f(\mathbf{z},t)=\psi^{-\lambda_0}_{k_0}(\mathbf{z},t)$, the integral conditions (9), for $r_i=r_1,r_2$, become

$$\begin{aligned}&\int_B L_{\mathbf{g}}f(\mathbf{z},t)d\hat{\mu}_{r_i}(\mathbf{z},t)\\ &= c\int_{-r_i^2}^{r_i^2}\int_{|\mathbf{z}|\le\sqrt[4]{r_i^4-t^2}}e^{-2\pi i\lambda_0 t}e^{-2\pi|\lambda_0||\mathbf{z}|^2}L^{\lambda_0}_{k_0}d\mu_{\sqrt[4]{r_i^4-t^2}}(\mathbf{z})dt\\ &= c\int_{-r_i^2}^{r_i^2}e^{-2\pi i\lambda_0 t}\left(\int_0^{\sqrt[4]{r_i^4-t^2}}e^{-2\pi|\lambda_0|s}L^{(n-1)}_{k_0}(4\pi|\lambda_0|s)s^{n-1}ds\right)dt\\ &= c'F^{\operatorname{sgn}(\lambda_0)}_{1,k}(4\pi|\lambda_0|r_i^2)=0.\end{aligned}$$

Thus integral conditions (9) are satisfied for $f(\mathbf{z},t)=\psi^{\lambda_0}_{k_0}(\mathbf{z},t)$. This completes the proof of Theorem 5.1.

The next case to be considered will be the ball associated with integral conditions (2),

$$\int_{B_r^{H_2}}L_{\mathbf{g}}f(\mathbf{z},t)d\hat{\mu}_r(\mathbf{z},t)\qquad\text{for all } \mathbf{g}\in\mathbf{H}^n,$$

where $B_r^{H_2}=\{[\mathbf{z},t]:\max(\mathbf{z},t)\le r\}$. We have the following theorem.

Theorem 5.2. *Consider $f\in C\cap L^\infty(\mathbf{H}^n)$ and suppose f satisfies the integral conditions*

$$(10)\qquad \int_{B_{r_i}^{H_2}}L_{\mathbf{g}}f(\mathbf{z},t)d\hat{\mu}_r(\mathbf{z},t)=0\qquad \textit{for all } \mathbf{g}\in\mathbf{H}^n$$

for radii $r_i = r_1, r_2$. Furthermore suppose the radii satisfy the conditions

(1) $r_1/r_2 \notin \mathcal{Q}\,(J_n(x))$,

(2) $(r_1/r_2)^2 \notin \mathcal{Q}\left(\Psi_k^{(n-1)}(x)\right)$ *for all* $k \in \mathbf{Z}_+$,

(3) $r_1/r_2 \notin \mathbf{Q}$.

Then we may conclude $f \equiv 0$. Furthermore, if the radii do not satisfy either of these conditions, then there exists $f \not\equiv 0$ satisfying the integral conditions.

Proof. Associated to these integral conditions, we write the convolution equation $f * T_{2S} = 0$. Applying the Gelfand transform to T_{2S}, we have

$$\begin{aligned}
&\tilde{T}_{2S}(\lambda; k)\\
&= c\int_{-r}^{r}\int_{|\mathbf{z}|<r} e^{-2\pi i\lambda t}e^{-2\pi|\lambda||\mathbf{z}|^2}L_k^{(n-1)}(4\pi|\lambda||\mathbf{z}|^2)d\mu_r(\mathbf{z})dt\\
&= c\left(\int_{-r}^{r} e^{-2\pi i\lambda t}dt\right)\left(\int_0^r\int_{|\mathbf{z}|=s} e^{-2\pi|\lambda||\mathbf{z}|^2}L_k^{(n-1)}(4\pi|\lambda||\mathbf{z}|^2)d\sigma_s(\mathbf{z})s^{2n-1}ds\right)\\
&= c\frac{1}{\pi\lambda}\sin(2\pi\lambda r)\left(\int_0^r e^{-2\pi|\lambda|s^2}L_k^{(n-1)}(4\pi|\lambda|s^2)s^{2n-1}ds\right)\\
&= c\frac{1}{\pi\lambda}\sin(2\pi\lambda r)\Psi_k^{(n-1)}(4\pi|\lambda|r^2),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{T}_{2S}(0;\rho) &= c\int_{-r}^{r}\int_{|\mathbf{z}|<r}\frac{J_{n-1}(\rho|\mathbf{z}|)}{(\rho|\mathbf{z}|)^{n-1}}d\mu_r(\mathbf{z})dt\\
&= c\left(\int_{-r}^{r}dt\right)\frac{1}{\rho^{n-1}}\int_0^r s^n J_{n-1}(\rho s)ds\\
&= 2cr\frac{1}{\rho^{2n}}\int_o^{\rho r} s^n J_{n-1}(s)ds\\
&= 2cr^{2n+1}\frac{J_n(\rho r)}{(\rho r)^n}.
\end{aligned}$$

Condition 1. of the theorem is equivalent to no common zeros for $\widetilde{T}_{2S,r_i}(0,\rho)$ for $i = 1, 2$. Conditions 2. and 3. of the theorem are equivalent to no common zeros for $\widetilde{T}_{2S,r_i}(\lambda; k)$ for $i = 1, 2$. Thus, for each $(0;\rho) \in \mathbf{R}_+$ and each $(\lambda; k) \in \mathbf{R}^* \times \mathbf{Z}_+$, either $\widetilde{T}_{2S,r_1} = 0$ or $\widetilde{T}_{2S,r_2} = 0$. Since the conditions for the Tauberian theorem Theorem 2.1 are satisfied, we have that $f * L^1_{\mathbf{0}}(\mathbf{H}^n) = 0$. Thus $f \equiv 0$, as claimed.

If any of conditions 1., 2., or 3. are not met, we will find non-zero functions satisfying the integral conditions (10). If condition 1. is not met, there exists

$\rho_0 \in \mathbf{R}_+$ such that $J_{0,n}(\rho_0 r_1) = J_{0,n}(\rho_0 r_2) = 0$. Then letting $f(\mathbf{z},t) = \frac{J_{n-1}(\rho_0|\mathbf{z}|)}{(\rho_0|\mathbf{z}|)^{n-1}}$, the integral conditions (10), for $r_i = r_1, r_2$, become

$$\begin{aligned}\int_B L_{\mathbf{g}} f(\mathbf{z},t) d\hat{\mu}_{r_i}(\mathbf{z},t) &= c\int_{-r_i}^{r_i}\int_{|\mathbf{z}|\le r_i} \frac{J_{n-1}(\rho_0|\mathbf{z}|)}{(\rho_0|\mathbf{z}|)^{n-1}} d\mu_r(\mathbf{z})dt \\ &= c\int_{-r_i}^{r_i}\left(\int_0^{\rho_0 r_i} s^n J_n(s)ds\right)dt \\ &= c'\frac{J_n(\rho_0 r_i)}{(\rho_0 r_i)^n} = 0.\end{aligned}$$

Thus integral conditions (10) are satisfied for $f(\mathbf{z},t) = \frac{J_{n-1}(\rho_0|\mathbf{z}|)}{(\rho_0|\mathbf{z}|)^{n-1}}$.

Similarly if 2. is not met, there exists $(\lambda_0, k_0) \in \mathbf{R}^* \times \mathbf{Z}_+$ such that $\Psi_k^{(n-1)}$ $(|\lambda_0|r_1) = \Psi_k^{(n-1)}(|\lambda_0|r_2) = 0$. Then letting $f(\mathbf{z},t) = \psi_{k_0}^{-\lambda_0}(\mathbf{z},t)$, the integral conditions (10), for $r_i = r_1, r_2$, become

$$\begin{aligned}\int_B L_{\mathbf{g}} f(\mathbf{z},t) d\hat{\mu}_{r_i}(\mathbf{z},t) &= c\int_{-r_i}^{r_i}\int_{|\mathbf{z}|\le r_i} e^{-2\pi i\lambda_0 t} e^{-2\pi|\lambda_0||\mathbf{z}|^2} L_{k_0}^{\lambda_0}(4\pi|\lambda_0|\mathbf{z}|^2) d\mu_{r_i}(\mathbf{z})dt \\ &= c\int_{-r_i}^{r_i} e^{-2\pi i\lambda_0 t}\left(\int_0^{r_i^2} e^{-2\pi|\lambda_0|s} L_{k_0}^{(n-1)}(4\pi|\lambda_0|s)s^{n-1}ds\right)dt \\ &= c'\frac{1}{\pi\lambda_0}\sin(2\pi\lambda_0 r_i)\Psi_{k_0}^{(n-1)}(4\pi|\lambda_0|r_i^2) = 0.\end{aligned}$$

Thus integral conditions (10) are satisfied for $f(\mathbf{z},t) = \psi_{k_0}^{\lambda_0}(\mathbf{z},t)$. The above computation also shows that $f(\mathbf{z},t) = \psi_{k_0}^{\lambda_0}(\mathbf{z},t)$ satisfies integral conditions (10) for λ_0 such that $\sin(\lambda_0 r_1) = \sin(\lambda_0 r_2) = 0$. This corresponds to conditon 3. If 3. is not met, there exists $\lambda_0 \in \mathbf{R}^*$ such that $\sin(\lambda_0 r_1) = \sin(\lambda_0 r_2) = 0$. Again letting $f(\mathbf{z},t) = \psi_{k_0}^{\lambda_0}(\mathbf{z},t)$, the integral conditions (10) are satisfied by the same computations above. This completes the proof of Theorem 5.2.

The final case to be considered will be the ball associated with integral conditions (3),

$$\int_{B_r^{H_3}} L_{\mathbf{g}} f(\mathbf{z},t) d\hat{\mu}_r(\mathbf{z},t) \qquad \text{for all } \mathbf{g} \in \mathbf{H}^n,$$

where $B_r^{H_3} = \{[\mathbf{z},t] : \|\mathbf{z}\|^2, t^2 \le r^2\}$. We have the following theorem.

Theorem 5.3. *Consider $f \in C \cap L^\infty(\mathbf{H}^n)$ and suppose f satisfies the integral conditions*

$$\int_{B_{r_i}^{H_3}} L_{\mathbf{g}} f(\mathbf{z},t) d\hat{\mu}_r(\mathbf{z},t) = 0 \qquad \textit{for all } \mathbf{g} \in \mathbf{H}^n \tag{11}$$

for radii $r_i = r_1, r_2$. Furthermore suppose the radii satisfy the conditions

(1) $r_1/r_2 \notin \mathcal{Q}\left(Jo_{3,n}(x)\right)$,
(2) $(r_1/r_2)^2 \notin \mathcal{Q}\left(F^{\pm}_{3,k}(x)\right)$ *for all* $k \in \mathbf{Z}_+$.

Then we may conclude $f \equiv 0$. *Furthermore, if the radii do not satisfy either of these conditions, then there exists* $f \not\equiv 0$ *satisfying the integral conditions.*

Proof. Associated to these integral conditions, we write the convolution equation $f * T_{3S} = 0$. Applying the Gelfand transform to T_{3S}, we have

$$\begin{aligned}
&\tilde{T}_{3S}(\lambda; k) \\
&= c\int_{-r_i^2}^{r_i^2}\int_{|\mathbf{z}|<\sqrt{r_i^2-|t|}} e^{-2\pi i\lambda t}e^{-2\pi|\lambda||\mathbf{z}|^2}L_k^{(n-1)}(4\pi|\lambda||\mathbf{z}|^2)d\mu_{\sqrt{r_i^2-|t|}}(\mathbf{z})dt \\
&= c\int_{-r_i^2}^{r_i^2} e^{-2\pi i\lambda t}\left(\int_0^{\sqrt{r_i^2-|s|}}\int_{|\mathbf{z}|=s} e^{-2\pi|\lambda||\mathbf{z}|^2}L_k^{(n-1)}(4\pi|\lambda||\mathbf{z}|^2)d\sigma_s(\mathbf{z})s^{2n-1}ds\right)dt \\
&= \frac{c}{2}\int_{-r_i^2}^{r_i^2} e^{-2\pi i\lambda t}\left(\int_0^{r_i^2-|s|} e^{-2\pi|\lambda|s}L_k^{(n-1)}(4\pi|\lambda|s)s^{n-1}ds\right)dt \\
&= \frac{c}{2(4\pi|\lambda|)^n}\int_{-r_i^2}^{r_i^2} e^{-2\pi i\lambda t}\left(\Psi_k^{(n-1)}(4\pi|\lambda|(r_i^2-t^2))\right)dt \\
&= c' F_{3,k}^{\operatorname{sgn}(\lambda)}(4\pi|\lambda|r_i^2),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{T}_{3S}(0;\rho) &= c\int_{-r_i^2}^{r_i^2}\int_{|\mathbf{z}|<\sqrt{r_i^2-|t|}} \frac{J_{n-1}(\rho|\mathbf{z}|)}{(\rho|\mathbf{z}|)^{n-1}}d\mu_{\sqrt{r_i^2-|t|}}(\mathbf{z})dt \\
&= c\int_{-r_i^2}^{r_i^2}\left(\int_0^{\sqrt{r_i^2-|t|}}\int_{|\mathbf{z}|=s} \frac{J_{n-1}(\rho|\mathbf{z}|)}{(\rho|\mathbf{z}|)^{n-1}}d\sigma_s(\mathbf{z})s^{2n-1}ds\right)dt \\
&= c\int_{-r_i^2}^{r_i^2}\left(\int_0^{\sqrt{r_i^2-|t|}} \frac{J_{n-1}(\rho s)}{(\rho s)^{n-1}}s^{2n-1}ds\right)dt \\
&= c\frac{1}{\rho^{2n}}\int_{-r_i^2}^{r_i^2}\left(\int_0^{\rho\sqrt{r_i^2-|t|}} s^n J_{n-1}(s)ds\right)dt \\
&= c\frac{2}{\rho^n}\int_0^{r_i^2}(\sqrt{r_i^2-|t|})^n J_n(\rho\sqrt{r_i^2-|t|})dt \\
&= c' Jo_{3,n}(\rho r_i).
\end{aligned}$$

Condition 1. of the theorem is equivalent to no common zeros for $\widetilde{T}_{3S,r_i}(0,\rho)$ for $i = 1, 2$. Condition 2. of the theorem is equivalent to no common zeros for

$\widetilde{T}_{3S,r_i}(\lambda;k)$ for $i = 1, 2$. Thus, for each $(0;\rho)$ and each $(\lambda;k)$, either $\widetilde{T}_{3S,r_1} = 0$ or $\widetilde{T}_{3S,r_2} = 0$. Since the conditions for the Tauberian theorem Theorem 2.1 are satisfied, we have that $f * L^1_{\mathbf{0}}(\mathbf{H}^n) = 0$. Thus $f \equiv 0$, as claimed.

If either of conditions 1. or 2. are not met, we will find non-zero functions satisfying the integral conditions (11). If condition 1. is not met, there exists $\rho_0 \in \mathbf{R}_+$ such that $Jo_{3,n}(\rho_0 r_1) = Jo_{3,n}(\rho_0 r_2) = 0$. Then letting $f(\mathbf{z}, t) = \frac{J_{n-1}(\rho_0|\mathbf{z}|)}{(\rho_0|\mathbf{z}|)^{n-1}}$, the integral conditions (11), for $r_i = r_1, r_2$, become

$$\begin{aligned}\int_B L_{\mathbf{g}} f(\mathbf{z},t)d\hat{\mu}_{r_i}(\mathbf{z},t) &= c\int_{-r_i^2}^{r_i^2}\int_{|\mathbf{z}|\leq\sqrt{r_i^2-|t|}} \frac{J_{n-1}(\rho_0|\mathbf{z}|)}{(\rho_0|\mathbf{z}|)^{n-1}} d\mu_{\sqrt{r_i^2-|t|}}(\mathbf{z})dt \\ &= c\int_{-r_i^2}^{r_i^2}\left(\int_0^{\rho_0\sqrt{r_i^2-|t|}} s^n J_n(s)ds\right)dt \\ &= c' Jo_{3,n}(\rho_0 r_i) = 0.\end{aligned}$$

Thus integral conditions (11) are satisfied for $f(\mathbf{z}, t) = \frac{J_{n-1}(\rho_0|\mathbf{z}|)}{(\rho_0|\mathbf{z}|)^{n-1}}$.

Similarly if 2. is not met, there exists $(\lambda_0, k_0) \in \mathbf{R}^* \times \mathbf{Z}_+$ such that $F^{\pm}_{3,k}(|\lambda_0|r_1) = F^{\pm}_{3,k}(|\lambda_0|r_2) = 0$, and $\operatorname{sgn}(\lambda_0) = \pm 1$. Then letting $f(\mathbf{z}, t) = \psi^{-\lambda_0}_{k_0}(\mathbf{z}, t)$, the integral conditions (11), for $r_i = r_1, r_2$, become

$$\begin{aligned}&\int_B L_{\mathbf{g}} f(\mathbf{z},t)d\hat{\mu}_{r_i}(\mathbf{z},t) \\ &= c\int_{-r_i^2}^{r_i^2}\int_{|\mathbf{z}|\leq\sqrt{r_i^2-|t|}} e^{-2\pi i\lambda_0 t}e^{-2\pi|\lambda_0||\mathbf{z}|^2} L^{\lambda_0}_{k_0} d\mu_{\sqrt{r_i^2-|t|}}(\mathbf{z})dt \\ &= c\int_{-r_i^2}^{r_i^2} e^{-2\pi i\lambda_0 t}\left(\int_0^{\sqrt{r_i^2-|t|}} e^{-2\pi|\lambda_0|s} L^{(n-1)}_{k_0}(4\pi|\lambda_0|s)s^{n-1}ds\right)dt \\ &= c' F^{\operatorname{sgn}(\lambda_0)}_{3,k}(4\pi|\lambda_0|r_i^2) = 0.\end{aligned}$$

Thus integral conditions (11) are satisfied for $f(\mathbf{z}, t) = \psi^{\lambda_0}_{k_0}(\mathbf{z}, t)$. This completes the proof of Theorem 5.3.

In this section we observed that the L^∞ results for the Pompeiu problem on $\mathbf{H}^n$ extend from sets $S \subset \mathbf{C}^n \times \{0\}$ to also yield theorems of two radii for Heisenberg balls. The results for the three concepts of the ball $B^{H_1}_r$, $B^{H_2}_r$, and $B^{H_3}_r$ are all nearly identical to cases previously considered, [1, 2, 5]. However, there is a difference in the exceptional sets for the radii between the cases for $B^{H_1}_r$, $B^{H_2}_r$, and $B^{H_3}_r$. This difference in the exceptional sets reflects the different concepts of the Heisenberg ball in $B^{H_1}_r$, $B^{H_2}_r$, and $B^{H_3}_r$.

6. Conclusions and Additional Questions

In this paper we are able to develop an approach which extends the analysis of the Pompeiu problem in $\mathbf{H}^n$ from subsets of $\mathbf{C}^n \times \{0\} \subset \mathbf{H}^n$ to several concepts of the Heisenberg ball, of codimension zero in $\mathbf{H}^n$. Although these methods are successful in allowing this extension to consider sets of codimension zero in $\mathbf{H}^n$, the setting wherein $S \subset \mathbf{C}^n \times \{0\} \subset \mathbf{H}^n$ remains the natural setting to use for the Pompeiu problem in the Heisenberg setting, $\mathbf{H}^n$. The methods used did not reveal any special properties for any one of the versions of the Heisenberg ball, $B_r^{H_1}$, $B_r^{H_2}$, and $B_r^{H_3}$, used in this paper. But rather they all arise as extensions of the case $B_r \subset \mathbf{C}^n \times \{0\}$, the complex ball that has been used in previous work on the Pompeiu problem in $\mathbf{H}^n$. It appears that the same methods should yield equivalent results in many other cases, as well.

When considering the Pompieu problem for the space $L^\infty(\mathbf{H}^n)$, Theorems 5.1, 5.2, and 5.3 all require two balls of separate radii, where the radii satisfy certain conditions. The exceptional sets of radii for the balls $B_r^{H_1}$, $B_r^{H_2}$, and $B_r^{H_3}$ are all different. Although the difference is subtle, it may relate to a more important underlying difference in the results for these cases. In the paper [9], rotations inside of $\mathbf{C}^n \times \{0\}$ are used to eliminate the exceptional set and reduce a theorem of two sets of separate radii to a theorem for a single set, including rotations. In these cases, the balls $B_r^{H_1}$, $B_r^{H_2}$, and $B_r^{H_3}$ are invariant under rotations by $U(n)$ inside of $\mathbf{C}^n \times \{0\}$, and such rotations could not provide a reduction to a theorem of one radius. We will however return in a later paper to investigate use of the larger set of rotations $SO(2n+1)$ inside of $\mathbf{C}^n \times \mathbf{R}$ and by this means achieve a reduction to a theorem for one ball with a single radius. The distinction between use of roations $U(n)$ within $\mathbf{C}^n \times \{0\}$ in [9] and use of the larger rotation group $SO(2n+1)$ within $\mathbf{C}^n \times \mathbf{R}$ for the Heisenberg ball is comparable to some of the issues in the Pompeiu problem for sets of higher codimension, as discussed in [10].

The natural setting to use for the Pompeiu problem in $\mathbf{H}^n$ is integration over a set S such that $S \subset \mathbf{C}^n \times \{0\} \subset \mathbf{H}^n$. In these cases the methods of harmonic analysis apply directly and furthermore extend to the context of the Weyl calculus, as presented in [2] and extended in [14]. In order to understand more thoroughly the Pompeiu problem for the Heisenberg ball or other sets of zero codimension, it is interesting to extend the results of the current paper to the Weyl calculus approach. This project will be taken up at a later time in another paper. For each of the results Theorems 5.1, 5.2, and 5.3, the separate conditions on the radii have the potential to be unified as the conditions for one operator-valued function. Furthermore such interpretation using the Weyl calculus is interesting because it carries an interpretation in the context of physics, as described briefly in [6] and [15]. In the current paper we considered how the methods of harmonic analysis for $\mathbf{H}^n$ proposed by Strichartz [17], as well as those using the Gelfand transform, as applied in [2] and [6], can be extended from $\mathbf{C}^n \times \{0\}$ to $\mathbf{C}^n \times \mathbf{R}$. It will be interesting to consider the same issue of extension from the point of view of the Weyl calculus.

In the paper [15] the Weyl calculus approach is used to demonstrate the closeness between the results for the Pompeiu problem in Euclidean space and the Heisenberg group $\mathbf{H}^n$. The nearness between the methods and results in the context of the Weyl calculus is used to interpret and to expand ideas and results to the Heisenberg setting. However, once we move off of $\mathbf{C}^n \times \{0\}$ to the case where $S \subset \mathbf{H}^n$, it appears there is no longer as close a connection to the Euclidean version of the Pompeiu problem. This connection with the Euclidean Pompeiu problem appears to be tied in with the Pompeiu problem as expanding upon the conception of the Radon transform as it appears in Strichartz, [17]. The methods for the harmonic analysis on $\mathbf{H}^n$ are also based on this setting. It is then also important to explore how to expand and generalize beyond sets S such that $S \subset \mathbf{C}^n \times \{0\}$, as done in this paper. An important next step will be the investigation using the Weyl calculus.

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Wayne M. Eby
Department of Mathematical Sciences
Cameron University
Lawton, OK 73505
U.S.A.
E-mail: weby@cameron.edu