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LAGRANGIAN H-UMBILICAL SUBMANIFOLDS OF PARA-KÄHLER MANIFOLDS

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Abstract. The notion of Lagrangian H-umbilical submanifolds of Kähler manifolds introduced in [3, 4] is closely related with several problems in Lagrangian geometry (cf. [7]). The classification of such submanifolds was done in a series of author's papers [3, 4, 5]. On the other hand, the study of Lagrangian submanifolds of para-Kähler manifolds was initiated very recently in [10]. In this paper we study Lagrangian H-submanifolds of para-Kähler manifolds. As results we prove several fundemental properties of such submanifolds. Moreover, we are able to classify Lagrangian H-umbilical submanifolds of the para-Kähler n-plane $(\mathbb{E}^{2n}_n, g_0, P)$ for $n \geq 3$.

1. INTRODUCTION

An almost para-Hermitian manifold is a manifold M endowed with an almost product structure $P \neq \pm I$ and a semi-Riemannian metric g such that

$$P^2 = I, \quad g(PX, PY) = -g(X, Y) \tag{1.1}$$

for vector fields X, Y tangent to M, where I is the identity map. Consequently, the dimension of M is even and the signature of g is (n, n), where $\dim M = 2n$. Let ∇ denote the Levi-Civita connection of M. An almost para-Hermitian manifold is called *para-Kähler* if it satisfies $\nabla P = 0$ identically.

Properties of para-Kähler manifolds were first studied by R. K. Rashevski in 1948 in which he considered a neutral metric of signature (n, n) defined from a potential function on a locally product $2n$-manifold [20]. He called such manifolds stratified space. Para-Kähler manifolds were explicitly defined by B. A. Rozenfeld in 1949 [21]. Rozenfeld compared Rashevskij's definition with Kähler's definition in the complex case and established the analogy between Kähler and para-Kähler ones. Such manifolds were also defined independently by H. S. Ruse in 1949 [22].

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The Levi-Civita connection of a para-Kähler manifold (M, g, P) preserves P, equivalently, its holonomy group Hol_p, $p \in M$, preserves the eigenspace decomposition $T_pM = T_p^+ \oplus T_p^-$. The parallel eigendistributions $T^{\pm}$ of P are g-isotropic integrable distributions. Moreover, they are Lagrangian distributions with respect to the Kähler form $\omega = g \circ P$, which is parallel and hence closed. The leaves of these distributions are totally geodesic submanifolds, Thus, from the standpoint of symplectic manifolds, a para-Kähler structure can be regarded as a pair of complementary integrable Lagrangian distributions (T^+, T^-) on a symplectic manifold (M, ω). Such a structure is often called a bi-Lagrangian structure or a Lagrangian 2-web (cf. [16]).

There exist many para-Kähler manifolds, for instance, a homogeneous manifold $M = G/H$ of a semisimple Lie group G admits an invariant para-Kähler structure (g, P) if and only if it is a covering of the adjoint orbit $\mathrm{Ad}_G h$ of a semisimple element h (see [19] for details).

Analogous to totally real submanifolds in an almost Hermitian manifold (cf. [11]), we call a space-like submanifold N in an almost para-Hermitian manifold (M_m^{2m}, g, P) *totally real* if P maps each tangent space T_pN, $p \in N$, into the normal space $T_p^{\perp}N$. In particular, we call N *Lagrangian* if $P(T_pN) = T_p^{\perp}N$ for each $p \in N$.

Lagrangian submanifolds in Kähler manifolds have been studied extensively since early 1970s (see [6, 7] for surveys). In contrast, no results on Lagrangian submanifolds in para-Kähler manifolds are known (see [16, Section 5: Open Problems], in particular, see Open Problem (3)). This is the reason the author initiated recently the study of Lagrangian submanifolds of para-Kähler manifolds in [10] in which two optimal inequalities for Lagrangian submanifolds in flat para-Kähler manifolds were proved. Lagrangian submanifolds satisfying the equality case of one of the two inequalities are also classified in [10].

On the other hand, the notion of Lagrangian H-umbilical submanifolds of Kähler manifolds introduced in [3, 4] is closely related with several problems in Lagrangian geometry (cf. [7]). The classification of such submanifolds was achieved in a series of author's papers [3, 4, 5].

In this paper we introduce and study Lagrangian H-submanifolds of para-Kähler manifolds. As consequences, we prove several fundamental properties of such submanifolds. Moreover, we classify Lagrangian H-umbilical submanifolds of the para-Kähler n-plane $(\mathbb{E}_n^{2n}, g_0, P)$ with $n \geq 3$.

2. Preliminaries

2.1. Para-Kähler manifolds

Definition 2.1. An *almost para-Hermitian manifold* is a manifold M endowed with an almost product structure $P \neq \pm I$ and a pseudo-Riemannian metric g such that

$$P^2 = I, \text{ and } g(Pv, Pw) = -g(v, w) \tag{2.1}$$

for vectors $v, w \in T_p(M)$, $p \in M$, where I is the identity map.

The dimension of an almost para-Hermitian manifold M is even and the metric is neutral.

Definition 2.2. An almost para-Hermitian manifold (M, g, P) is called *para-Kähler* if it satisfies $\nabla P = 0$ identically, where ∇ is the Levi-Civita connection of M.

The simplest example of para-Kähler manifolds is the pseudo-Euclidean $2n$-space $\mathbb{E}^{2n}_n$ endowed with the neutral metric:

$$g_0 = -\sum_{i=1}^{n} dx_i^2 + \sum_{j=1}^{n} dy_j^2 \tag{2.2}$$

with P being defined by

$$P\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad P\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j} \tag{2.3}$$

for $j = 1, \ldots, n$. We simply called $(\mathbb{E}^{2n}_n, g_0, P)$ the *para-Kähler n-plane*.

The following result is well-known.

Lemma 2.1. *The curvature tensor of a para-Kähler manifold satisfies*

$$R(X, Y) \circ P = P \circ R(X, Y), \tag{2.4}$$

$$R(PX, PY) = R(X, Y), \tag{2.5}$$

$$R(X, PY) = R(PX, Y). \tag{2.6}$$

For a para-Kähler manifold M, (2.1) implies that

$$g(Pv, w) + g(v, Pw) = 0, \quad v, w \in T_p(M), \ p \in M. \tag{2.7}$$

Thus $g(v, Pv) = 0$. If $\{v, Pv\}$ determines a non-degenerate plane section called a *P-section*, the sectional curvature

$$H^{P(v)} = K(v \wedge Pv)$$

of $\operatorname{Span}\{v, Pv\}$ is called a *para-sectional curvature*.

By definition a *para-Kähler space form* is a para-Kähler manifold of constant para-sectional curvature.

The para-Kähler n-plane $(\mathbb{E}^{2n}_n, g_0, P)$ is the standard model of flat para-Kähler manifolds. Models of para-Kähler space forms with nonzero para-sectional curvature were constructed in [17].

The Riemann curvature tensor of a para-Kähler space forms $M^{2n}_n(4c)$ of constant para-sectional curvature $4c$ satisfies

$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX \\ &- g(PX,Z)PY + 2g(X,PY)PZ\}. \end{aligned} \tag{2.8}$$

2.2. Basic formulas and definitions

Let $\psi : N \to M^{2n}_n$ be an isometric immersion of a Riemannian n-manifold N into a para-Kähler manifold (M^{2n}_n, g, P). Denote by ∇' and ∇ the Levi-Civita connections on N and M^{2n}_n, respectively.

For vector fields X, Y tangent to N and ξ normal to N, the formulas of Gauss and Weingarten are given respectively by (cf. [1, 2]):

$$\nabla_X Y = \nabla'_X Y + h(X,Y), \tag{2.9}$$

$$\nabla_X \xi = -A_\xi X + D_X \xi, \tag{2.10}$$

where h, A and D are the second fundamental form, the shape operator, and the normal connection of N in M^{2n}_n.

The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle, \tag{2.11}$$

where $\langle\ ,\ \rangle$ is the inner product. The *mean curvature vector* is defined by

$$H = \left(\frac{1}{n}\right) \operatorname{trace} h. \tag{2.12}$$

The equations of Gauss, Codazzi and Ricci are given respectively by

$$R'(X,Y)Z = R(X,Y)Z + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.13}$$

$$(R(X,Y)Z)^\perp = (\bar\nabla_X h)(Y,Z) - (\bar\nabla_Y h)(X,Z), \tag{2.14}$$

$$g(R^D(X,Y)\xi,\eta) = g(R(X,Y)\xi,\eta) + g([A_\xi, A_\eta]X,Y) \tag{2.15}$$

for X, Y, Z tangent to N and ξ, η normal to N, where R' (respectively, R) is the curvature tensor of N (respectively, ofM^{2n}_n), $(R(X,Y)Z)^\perp$ is the normal component of $R(X,Y)Z$, and $\bar\nabla h$ and R^D are defined by

$$(\bar\nabla_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla'_X Y, Z) - h(Y, \nabla'_X Z), \tag{2.16}$$

$$R^D(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}. \tag{2.17}$$

3. Lagrangian Submanifolds of Para-Kähler Manifolds

The following basic lemma is given in [10].

Lemma 3.2. *Let N be a Lagrangian submanifold of a para-Kähler manifold M_n^{2n}. Then we have*

(i) $P(\nabla'_X Y) = D_X(PY)$,

(ii) $A_{PX}Y = -P(h(X,Y))$,

(iii) $\langle h(X,Y), PZ\rangle = \langle h(Y,Z), PX\rangle = \langle h(Z,X), PY\rangle$,

(iv) $P(R'(X,Y)Z) = R^D(X,Y)PZ$

for X, Y, Z tangent to N.

The equations of Gauss and Codazzi for a Lagrangian submanifold N of a para-Kähler space form $M_n^{2n}(4c)$ are given respectively by

$$R'(X,Y;Z,W) = \left\langle A_{h(Y,Z)}X, W\right\rangle - \left\langle A_{h(X,Z)}Y, W\right\rangle + c\left(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\right), \tag{3.1}$$

$$(\bar\nabla_X h)(Y,Z) = (\bar\nabla_Y h)(X,Z) \tag{3.2}$$

for X, Y, Z, W tangent to N.

If we put $h = P\circ\sigma$ (equivalently $\sigma = P\circ h$), then (2.1) and Lemma 3.2(iii) imply that

$$\left\langle A_{h(Y,Z)}X, W\right\rangle = \langle h(X,W), h(Y,Z)\rangle = \langle h(X,W), P\sigma(Y,Z)\rangle = \langle h(\sigma(Y,Z),X), PW\rangle = -\langle \sigma(\sigma(Y,Z),X), W\rangle .$$

Therefore, equation (3.1) of Gauss can be rephrased as

$$R'(X,Y)Z = \sigma(\sigma(X,Z),Y) - \sigma(\sigma(Y,Z),X) + c\langle Y,Z\rangle X - c\langle X,Z\rangle Y.$$

It follows Lemma 3.2(i) that the equation of Ricci is nothing but the equation of Gauss for Lagrangian submanifolds of para-Kähler manifolds.

Now, we state the fundamental existence and uniqueness theorems for Lagrangian submanifolds in $(\mathbb{E}_n^{2n}, g_0, P)$ are given by the following.

Existence Theorem. *Let N be a simply-connected Riemannian n-manifold. If σ is a TN-valued symmetric bilinear form on N such that*

(a) $g(\sigma(X,Y),Z)$ *is totally symmetric,*

(b) $(\nabla\sigma)(X,Y,Z)$ *is totally symmetric,*

(c) $R'(X,Y)Z=\sigma(\sigma(X,Z),Y)-\sigma(\sigma(Y,Z),X)$,

then there is a Lagrangian isometric immersion $L:N\rightarrow(\mathbb{E}^{2n}_n,g_0,P)$ *whose second fundamental form is* $h=P\circ\sigma$.

Uniqueness Theorem. *Let* $L_1,L_2:N\rightarrow(\mathbb{E}^{2n}_n,g_0,P)$ *be two Lagrangian isometric immersions of a Riemannian* n*-manifold* N *with second fundamental forms* h^1 *and* h^2*, respectively. If*

$$g(h^1(X,Y),PL_{1\star}Z)=g(h^2(X,Y),PL_{2\star}Z)$$

for all vector fields X,Y,Z *tangent to* N*, then there is an isometry* Φ *of* $(\mathbb{E}^{2n}_n,g_0,P)$ *such that* $L_1=\Phi\circ L_2$.

Similar existence and uniqueness theorems also hold for Lagrangian submanifolds in para-Kähler space forms.

4. Lagrangian H-Umbilical Submanifolds

A pseudo-Riemannian submanifold N of a pseudo-Riemannian manifold is called *totally umbilical* if its second fundamental form satisfies

$$h(X,Y)=\langle X,Y\rangle\,H \tag{4.1}$$

for X,Y tangent to N.

Proposition 4.1. *The only totally umbilical Lagrangian submanifold* N *of a para-Kähler space form* $M^{2n}_n(4c)$ *with* $n\geq 2$ *is the totally geodesic ones.*

Proof. Let N be a totally umbilical Lagrangian submanifold of a para-Kähler space form $M^{2n}_n(4c)$ with $n\geq 2$. Assume that N is non-totally geodesic, then $H\neq 0$.

It follows from (4.1) that $(\bar\nabla_X h)(Y,Z)=\langle Y,Z\rangle\,D_XH$. Thus, after applying equation (3.2) of Codazzi, we find

$$\langle Y,Z\rangle\,D_XH=\langle X,Z\rangle\,D_YH \tag{4.2}$$

for X,Y,Z tangent to N. For any $X\in TN$, by choosing $0\neq Y=Z\perp X$, we get $DH=0$. Therefore, it follows from the equation of Gauss that N is of constant sectional curvature $c-||H||^2<c$, where $||H||=\sqrt{-\langle H,H\rangle}$.

Let us put $Z=PH$. Then Lemma 3.1(i) implies that $\nabla'Z=0$. Thus, Z is a nonzero parallel vector field on N, which implies that N is a flat Riemannian manifold. Hence, we get $c=-\langle H,H\rangle>0$.

Since N is totally umbilical, we have $[A_H, A_\xi] = 0$ for any normal vector ξ. Hence, by using $DH = 0$ we find from equation (2.15) of Ricci that

$$g(R(Z,Y)H, PY) = 0 \tag{4.3}$$

for $Y, Z \in TN$. On the other hand, by applying (2.8) we also have

$$g(R(Z,Y)H, PY) = c\{g(PY,H)g(PZ,PY) - g(PZ,H)g(PY,PY)\} \tag{4.4}$$

Thus, after choosing Y, Z such that $Z = PH$ and $g(Y,Z) = 0$, we find $g(H,H) = 0$. But this is a contradiction. Consequently, N must be totally geodesic. ∎

Definition 4.3. A Lagrangian submanifold N of a para-Kähler manifold is called *Lagrangian H-umbilical* if the second fundamental form takes the following simple form:

$$\begin{aligned} &h(e_1,e_1) = \lambda Pe_1,\ h(e_2,e_2) = \cdots = h(e_n,e_n) = \mu Pe_1, \\ &h(e_1,e_j) = \mu Pe_j,\ h(e_j,e_k) = 0,\ 2 \le j \ne k \le n, \end{aligned} \tag{4.5}$$

for some functions λ, μ with respect to some orthonormal local frame field.

In view of Proposition 4.1, Lagrangian H-umbilical submanifolds are the simplest Lagrangian submanifolds next to totally geodesic ones.

The following result shows that there exist many non-totally geodesic Lagrangian H-umbilical submanifolds.

Proposition 4.2. *Let $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{E}^2_1$ be a unit speed space-like curve satisfying $\langle \gamma, \gamma \rangle < 0$. Define $L : I \times \mathbf{R} \times S^{n-2}(1) \to \mathbb{E}^{2n}_n$ by*

$$\big(\gamma_1(s)\cosh t, \gamma_2(s) z \sinh t, \gamma_2(s)\cosh t, \gamma_1(s) z \sinh t\big), \tag{4.6}$$

where $z = (z_2, \ldots, z_n) \in \mathbb{E}^{n-1}$ satisfies $z_2^2 + z_3^2 + \cdots + z_n^2 = 1$. Then L defines a Lagrangian H-umbilical submanifold of $(\mathbb{E}^{2n}_n, g_0, P)$ satisfying (4.5) *with*

$$\lambda = \kappa, \quad \mu = \frac{\gamma_1'\gamma_2 - \gamma_1\gamma_2'}{||\gamma||^2}. \tag{4.7}$$

Proof. Under the hypothesis it follows from (4.6) that

$$L_s = (\gamma_1' \cosh t, \gamma_2' z \sinh t, \gamma_2' \cosh t, \gamma_1' z \sinh t), \tag{4.8}$$

$$L_t = (\gamma_1 \sinh t, \gamma_2 z \cosh t, \gamma_2 \sinh t, \gamma_1 z \cosh t), \tag{4.9}$$

$$XL = (0, \gamma_2(\sinh t)X, 0, \gamma_1(\sinh t)X), \tag{4.10}$$

$$L_{ss} = (\gamma_1'' \cosh t, \gamma_2'' z \sinh t, \gamma_2'' \cosh t, \gamma_1'' z \sinh t), \tag{4.11}$$

$$L_{st} = (\gamma_1' \sinh t, \gamma_2' z \cosh t, \gamma_2' \sinh t, \gamma_1' z \cosh t), \tag{4.12}$$

$$XL_s = (\gamma_1' \cosh t, \gamma_2'(\sinh t)X, \gamma_2' \cosh t, \gamma_1'(\sinh t)X), \tag{4.13}$$

$$XL_t = (\gamma_1 \sinh t, \gamma_2(\cosh t)X, \gamma_2 \sinh t, \gamma_1(\cosh t)X), \tag{4.14}$$

$$\begin{aligned} XYL = &(0, \gamma_2(\cosh t)\nabla'_X Y, 0, \gamma_1(\cosh t)\nabla'_X Y) \\ &- (0, \langle X, Y\rangle \gamma_2 z \cosh t, 0, \langle X, Y\rangle \gamma_1 z \cosh t) \end{aligned} \tag{4.15}$$

for X, Y tangent to $S^{n-2}(1)$. From (4.8)-(4.10) we get

$$\mathcal{P}(L_s) = (\gamma_2' \cosh t, \gamma_1' z \sinh t, \gamma_1' \cosh t, \gamma_2' z \sinh t), \tag{4.16}$$

$$\mathcal{P}(L_t) = (\gamma_2 \sinh t, \gamma_1 z \cosh t, \gamma_1 \sinh t, \gamma_2 z \cosh t), \tag{4.17}$$

$$\mathcal{P}(XL) = (0, \gamma_1(\sinh t)X, 0, \gamma_2(\sinh t)X). \tag{4.18}$$

Since $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ is a unit speed space-like curve in $\mathbb{E}^2_1$, (4.8)-(4.10) imply that the induced metric via L is given by

$$g = ds^2 + ||\gamma||^2(dt^2 + \sinh^2 t g_1), \tag{4.19}$$

where g_1 is the metric of $S^{n-2}(1)$. From (4.8)-(4.10) and (4.16)-(4.18), we know that L is Lagrangian. Because γ is unit speed and space-like, we have

$$(\gamma_1''(s), \gamma_2''(s)) = \kappa(s)(\gamma_2'(s), \gamma_1'(s)) \tag{4.20}$$

for some function κ. Thus, by (4.11)-(4.20) and $\langle z, X\rangle = 0$ for $X \in TN$, we obtain (4.5) with

$$\lambda = \kappa, \quad \mu = \frac{\gamma_1'\gamma_2 - \gamma_1\gamma_2'}{||\gamma||^2}. \tag{4.21}$$

Consequently, L defines a Lagrangian H-umbilical submanifold with the desired properties. This completes the proof of the proposition. ∎

Similarly, we also have the following.

Proposition 4.3. *Let $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{E}^2_1$ be a unit speed space-like curve satisfying $\langle \gamma, \gamma\rangle > 0$. Define $L : I \times \mathbf{R} \times S^{n-2}(1) \to \mathbb{E}^{2n}_n$ by*

$$\big(\gamma_1(s) \sin t, \gamma_1(s) z \cos t, \gamma_2(s) \sin t, \gamma_2(s) z \cos t\big), \tag{4.22}$$

where $z = (z_2, \ldots, z_n) \in \mathbb{E}^{n-1}$ satisfies $z_2^2 + z_3^2 + \cdots + z_n^2 = 1$. Then L defines a Lagrangian H-umbilical submanifold of $(\mathbb{E}^{2n}_n, g_0, P)$ satisfying (4.5) *with*

$$\lambda = \kappa, \quad \mu = \frac{\gamma_1'\gamma_2 - \gamma_1\gamma_2'}{||\gamma||^2}. \tag{4.23}$$

Proof. This can be proved in the same as Proposition 4.2 ■

Let N be a Lagrangian H-umbilical submanifold of a para-Kähler submanifold satisfying (4.5) with respect to an orthonormal frame $\{e_1, \ldots, e_n\}$. We put

$$\nabla'_X e_i = \sum_{j=1}^{n} \omega_i^j(X) e_j, \quad i = 1, \ldots, n. \tag{4.24}$$

Lemma 4.3. *Let N be a Lagrangian H-umbilical submanifold of a para-Kähler space form $M_n^{2n}(4c)$ which satisfies* (4.5) *with respect to an orthonormal frame $\{e_1, \ldots, e_n\}$. The we have*

$$e_1\mu = (\lambda - 2\mu)\omega_1^2(e_2) = \cdots = (\lambda - 2\mu)\omega_1^n(e_n), \tag{4.25}$$

$$e_j\lambda = (2\mu - \lambda)\omega_j^1(e_1), \quad j > 1, \tag{4.26}$$

$$(\lambda - 2\mu)\omega_1^j(e_k) = 0, \quad 1 < j \neq k \leq n, \tag{4.27}$$

$$e_j\mu = 3\mu\omega_1^j(e_1), \tag{4.28}$$

$$\mu\omega_1^j(e_1) = 0, \quad j > 1. \tag{4.29}$$

Proof. By applying (4.5), Lemma 3.2(i) and Codazzi's equation, we obtain this lemma by direct computation. ■

Proposition 4.4. *Let N be a Lagrangian H-umbilical submanifold of a para-Kähler space form $M_n^{2n}(4c)$ satisfying* (4.5)*. If $\lambda = 2\mu$, then μ is a constant, say b, and N is of constant sectional curvature $c - b^2$.*

Proof. Under the hypothesis, it follows from (4.25) and (4.26) that

$$e_1\mu = e_2\lambda = \cdots = e_n\lambda = 0.$$

Thus, by using $\lambda = 2\mu$ we see that μ is a constant, say b. Now, by applying the equation of Gauss and $\mu = b$ we conclude that N is of constant curvature $-b^2$. ■

Theorem 4.1. *A Lagrangian H-umbilical submanifold of $(\mathbb{E}_n^{2n}, g_0, P)$ satisfying $\lambda = 2\mu$ is either a flat totally geodesic Lagrangian submanifold or congruent to an open portion of*

$$\Big(\frac{\cosh^2(bs)\cosh t}{b}, \frac{\sinh(2bs)\sinh t}{2b} z, \frac{\sinh(2bs)\cosh t}{2b}, \frac{\cosh^2(bs)\sinh t}{b} z\Big) \tag{4.30}$$

with $b \neq 0$, where $z = (z_2, \ldots, z_n) \in \mathbb{E}^{n-1}$ satisfies $z_2^2 + z_3^2 + \cdots + z_n^2 = 1$.

Proof. Let N be a Lagrangian H-umbilical submanifold of $(\mathbb{E}_n^{2n}, \tilde{g}_0, \mathcal{P})$ satisfying $\lambda = 2\mu$. Then, by Proposition 4.4, μ is a constant, say b. If $b = 0$, then N is totally geodesic. In this case, N is a flat Lagrangian submanifold.

Next, assume b is a nonzero constant. Then N is of constant negative curvature $-b^2$. Thus, N is an open portion of a hyperbolic n-space $H^n(-b^2)$ in $\mathbb{E}^{2n}_n$ whose second fundamental form satisfies

$$\begin{aligned} &h(e_1,e_1)=2bPe_1,\ h(e_2,e_2)=\cdots=h(e_n,e_n)=bPe_1,\\ &h(e_1,e_j)=bPe_j,\ h(e_j,e_k)=0,\ 2\le j\ne k\le n, \end{aligned} \tag{4.31}$$

for some orthonormal frame $e_1,\ldots,e_n$.

On the other hand, a direct computation shows that (4.30) defines a Lagrangian H-umbilical immersion of $H^n(-b^2)$ into $(\mathbb{E}^{2n}_n, g_0, P)$ whose second fundamental form also satisfies (4.31). Therefore, by uniqueness theorem, N is congruent to an open portion of (4.30). ∎

5. Classification of Lagrangian H-Umbilical Submanifolds of $\mathbb{E}^{2n}_n$

Next, we classify Lagrangian H-umbilical submanifolds in the para-Kähler n-plane $(\mathbb{E}^{2n}_n, g_0, P)$.

Theorem 5.1. *Let $L:N\to(\mathbb{E}^{2n}_n, g_0, P)$ be a Lagrangian H-umbilical immersion of a Riemannian n-manifold N into the para-Kähler n-plane with $n\ge 3$. Then*

(i) *If N is of constant sectional curvature, then either N is flat or L is congruent to an open portion of*

$$\frac{1}{2b}\Big(2\cosh^2(bs)\cosh t, z\sinh(2bs)\sinh t, \sinh(2bs)\cosh t, 2z\cosh^2(bs)\sinh t\Big)$$

with $b\ne 0$, where $z=(z_2,\ldots,z_n)\in\mathbb{E}^{n-1}$ satisfies $z_2^2+z_3^2+\cdots+z_n^2=1$.

(ii) *If N contains no open subset of constant sectional curvature, then L is locally congruent to one of the following three types of submanifolds:*

(ii.1) *a Lagrangian submanifold defined by*

$$\Bigg(\frac{e^{2r}}{8}-\frac{e^{-2r}}{2r'^2}+a^2\sum_{j=2}^n x_j^2-\int^s\frac{2r'^2+r''}{e^{2r}r'^3}ds, \frac{1-a^2e^{2r}}{2}x_2,\ldots,\frac{1-a^2e^{2r}}{2}x_n,$$
$$-\frac{e^{2r}}{8}-\frac{e^{-2r}}{2r'^2}+a^2\sum_{j=2}^n x_j^2-\int^s\frac{2r'^2+r''}{e^{2r}r'^3}ds, \frac{1+a^2e^{2r}}{2}x_2,\ldots,\frac{1+a^2e^{2r}}{2}x_n\Bigg),$$

where $r=r(s)$ is a non-constant function and a is positive number;

(ii.2) *a Lagrangian submanifold defined by*

$$\frac{1}{2}\left(\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}+\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)\sin t,\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}+\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)z\cos t,\right.$$
$$\left.\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}-\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)\sin t,\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}-\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)z\cos t\right),$$

where $\mu(s)$ and $\varphi(s)$ are nonzero functions satisfies $\varphi\varphi'-\mu\mu'=(\mu^2-\varphi^2)\varphi$ and $\lambda=2\mu+\mu\varphi^{-1}$ and $z=(z_2,\dots,z_n)\in\mathbb{E}^{n-1}$ satisfies $z_2^2+z_3^2+\cdots+z_n^2=1$;

(ii.3) *a Lagrangian submanifold defined by*

$$\frac{1}{2}\left(\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}+\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)\cosh t,\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}-\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)z\sinh t,\right.$$
$$\left.\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}-\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)\cosh t,\left(\frac{e^{\int^s\lambda ds}}{\mu+\varphi}+\frac{e^{-\int^s\lambda ds}}{\mu-\varphi}\right)z\sinh t\right),$$

where $\mu(s)$ and $\varphi(s)$ are nonzero functions satisfies $\varphi\varphi'-\mu\mu'=(\mu^2-\varphi^2)\varphi$ and $\lambda=2\mu+\mu\varphi^{-1}$ and $z=(z_2,\dots,z_n)\in\mathbb{E}^{n-1}$ satisfies $z_2^2+z_3^2+\cdots+z_n^2=1$.

Proof. Assume that $n\geq 3$ and $L:N\to(\mathbb{E}^{2n}_n,g_0,P)$ is a Lagrangian H-umbilical submanifold of the para-Kähler n-plane which satisfies (4.5) with respect to some suitable orthonormal local frame field $e_1,\dots,e_n$.

If N is of constant curvature, then it follows from (4.5) and the equation of Gauss that $\mu(\lambda-2\mu)=0$. Thus, either $\mu=0$ or $\lambda=2\mu$ at each point. If $\mu=0$ identically, then N is flat. If $\mu\neq 0$, then $\lambda=2\mu\neq 0$ on a nonempty open subset V of N. Thus, Proposition 4.4 implies that λ and μ are nonzero constants on V. Thus, by continuity, $V=N$. Therefore, it follows from Theorem 4.1 that N is congruent to an open portion the Lagrangian submanifold given in (i).

Next, assume that N contains no open subset of constant curvature. Then

$$U:=\{\,p\in N\,:\,\mu(\lambda-2\mu)\neq 0\text{ at }p\,\}\tag{5.1}$$

is an open dense subset of N. Moreover, it follows from Lemma 4.3 that

$$\omega_1^j=\left(\frac{e_1\mu}{\lambda-2\mu}\right)\omega^j,\quad e_j\lambda=e_j\mu=0,\quad j=2,\dots,n.\tag{5.2}$$

$$\omega_1^j(e_1)=\omega_1^j(e_k)=0,\quad 2\leq j\neq k\leq n.\tag{5.3}$$

From $\omega_1^j(e_1)=0$, we find $\nabla_{e_1}e_1=0$. Thus, the integral curves of e_1 are geodesics. By using (5.2) and Cartan's structure equations, we get $d\omega^1=0$. Hence, according to Poincaré lemma, $\omega^1=ds$ for some local function s.

Let $\mathcal{D}$ denote the distribution spanned by e_1 which is clearly integrable. Using (5.3) we find

$$\langle[e_j,e_k],e_1\rangle=\omega_k^1(e_j)-\omega_j^1(e_k)=0$$

for $j,k=2,\dots,n$. Thus the complementary orthogonal distribution $\mathcal{D}^\perp$ spanned by $\{e_2,\dots,e_n\}$ is an integrable distribution. Because $\mathcal{D}$ and $\mathcal{D}^\perp$ are both integrable, there is a local coordinate system $\{s,x_2,\dots,x_n\}$ such that

(a) $\mathcal{D}$ is spanned by $\{\partial/\partial s\}$ and $\mathcal{D}^\perp$ is spanned by $\{\partial/\partial x_2,\dots,\partial/\partial x_n\}$ and

(b) $e_1=\frac{\partial}{\partial s}$, $\omega^1=ds$.

From (4.26), (4.28) and (5.3) we have $e_j\lambda=e_j\mu=0$ for $j>1$. Hence, both λ and μ depend only on s. Moreover, it follows from (5.2) and (5.3) that

$$\nabla'_X e_1=\varphi X,\ \ \varphi=\frac{\mu'}{\lambda-2\mu},\ \ X\in\mathcal{D}^\perp, \tag{5.4}$$

where $\mu'=d\mu/ds$.

It follows from (5.4) and $K_{1j}=\langle R(e_j,e_1)e_1,e_j\rangle$ that the sectional curvature K_{1j} of the plane section spanned by e_1,e_j is $K_{1j}=-\varphi'-\varphi^2$. On the other hand, (4.5) and the equation of Gauss shows that $K_{1j}=\mu^2-\lambda\mu$. Thus

$$\varphi'=\lambda\mu-\mu^2-\varphi^2. \tag{5.5}$$

Also, from (5.4) we find that

$$\left\langle\nabla'_X Y,e_1\right\rangle=-\varphi\left\langle X,Y\right\rangle. \tag{5.6}$$

This implies that the integrable distribution $\mathcal{D}^\perp$ is spherical, i.e., the leaves of $\mathcal{D}^\perp$ are totally umbilical with parallel mean curvature vector in N. Moreover, it follows from (4.6), (5.6) and Gauss' equation that each leaf of $\mathcal{D}^\perp$ (with $s=constant$) is of constant curvature $\varphi^2(s)-\mu^2(s)$. Hence, a result of [18] (see also [15, Remark 2.1]) implies that U is locally a warped product $I\times_{f(s)}R^{n-1}(c)$, where $R^{n-1}(c)$ is a Riemannian $(n-1)$-manifold of constant curvature and $f(s)$ is the warping function, where we choose $c=0$, 1 or -1 according to $\varphi^2=\mu^2$, $\varphi^2>\mu^2$, or $\varphi^2<\mu^2$, respectively. Clearly, vectors tangent to I are in $\mathcal{D}$ and vectors tangent to R^{n-1} are in $\mathcal{D}^\perp$.

The metric on $I\times_f R^{n-1}(c)$ is given by

$$g=ds^2+f^2(s)\hat g_c \tag{5.7}$$

where $\hat g_c$ is metric of $R^{n-1}(c)$. From (5.7) we obtain

$$\nabla'_{\partial/\partial s}\frac{\partial}{\partial s}=0,\ \nabla'_{\partial/\partial s}X=\frac{f'}{f}X,\ \nabla'_X Y=-ff'\left\langle X,Y\right\rangle\frac{\partial}{\partial s}+\mathcal{L}(\nabla''_X Y), \tag{5.8}$$

for vector fields X,Y tangent to $R^{n-1}(c)$, where $\mathcal{L}(\nabla''_X Y)$ is the lift of the the covariant derivative $\nabla''_X Y$ of Y with respect to X on $R^{n-1}(c)$.

Case (1). $\varphi^2 = \mu^2$. We may put $\varphi = \mu$. Also we have assume that

$$g = ds^2 + f^2(s)(dx_2^2 + dx_3^2 + \cdots + dx_n^2). \tag{5.9}$$

Thus (5.8) becomes

$$\nabla'_{\partial/\partial s}\frac{\partial}{\partial s} = 0,\ \nabla'_{\partial/\partial s}\frac{\partial}{\partial x_j} = \frac{f'}{f}\frac{\partial}{\partial x_j},\ \nabla'_{\partial/\partial x_j}\frac{\partial}{\partial x_k} = -ff'\delta_{jk}\frac{\partial}{\partial s} \tag{5.10}$$

for $j, k = 2, \ldots, n$. From (4.5), (5.10) and $(\bar{\nabla}_{\partial/\partial s}h)(\frac{\partial}{\partial s}, \frac{\partial}{\partial x_j}) = (\bar{\nabla}_{\partial/\partial x_j}h)(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$ we derive that

$$\frac{f'}{f} = \mu = \frac{\mu'}{\lambda - 2\mu}. \tag{5.11}$$

Thus there is a real number $a \neq 0$ such that

$$f(s) = a\,e^{r(s)},\ \ r(s) = \int^s \mu(x)dx. \tag{5.12}$$

From (5.11), we find

$$\lambda = 2r' + \frac{r''}{r'}. \tag{5.13}$$

Consequently, (4.5), (5.10), (5.12), (5.13) and Gauss' formula imply that the immersion $L : N \to (\mathbb{E}^{2n}_n, g_0, P)$ satisfies

$$\begin{aligned} L_{ss} &= \left(2r' + \frac{r''}{r'}\right) PL_s, \\ L_{sx_j} &= r'(L_{x_j} + PL_{x_j}), \\ L_{x_jx_k} &= a^2\delta_{jk}e^{2r}r'(PL_s - L_s). \end{aligned} \tag{5.14}$$

From $P^2 = I$ and (5.14) we have

$$\begin{aligned} PL_{ss} &= \left(2r' + \frac{r''}{r'}\right) L_s, \\ PL_{sx_j} &= r'(L_{x_j} + PL_{x_j}), \\ PL_{x_jx_k} &= a^2\delta_{jk}e^{2r}r'(L_s - PL_s). \end{aligned} \tag{5.15}$$

After solving the PDE system given by (5.14) and (5.15), we obtain

$$\begin{aligned} L(s, x_2, \ldots, x_n) = c_1e^{2r} + c_2\left(2a^2\sum_{j=2}^n x_j^2 - 2\int^s \frac{2r'^2 + r''}{e^{2r}r'^3}ds - \frac{e^{-2r}}{r'^2}\right) \\ + \sum_{i=2}^n c_{i+1}x_j + e^{2r}\sum_{j=2}^n c_{n+j}x_j, \quad r = \int^s \mu(s)ds, \end{aligned}$$

for some $\mathbb{E}^{2n}_n$-valued functions $c_1, \ldots, c_{2n}$. Consequently, after choosing suitable initial values we obtain (ii.1).

Case (2). $\varphi^2 > \mu^2$. With respect to a spherical coordinate chart $\{u_2, \ldots, u_n\}$, the metric on $I \times_f R^{n-1}(1)$ is given by

$$g = ds^2 + f^2(s)\{du_2^2 + \cos^2 u_2 du_3^2 + \cdots + \cos^2 u_2 \cdots \cos^2 u_{n-1} du_n^2\}. \tag{5.16}$$

From (5.16) we obtain

$$\begin{aligned} &\nabla'_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \;\; \nabla'_{\partial/\partial s}\frac{\partial}{\partial u_k} = \frac{f'}{f}\frac{\partial}{\partial u_k}, \;\; \nabla'_{\partial/\partial u_2}\frac{\partial}{\partial u_2} = -ff'\frac{\partial}{\partial s}, \\ &\nabla'_{\partial/\partial u_i}\frac{\partial}{\partial u_j} = -\tan u_i \frac{\partial}{\partial u_j}, \quad 2 \le i < j, \\ &\nabla'_{\partial/\partial u_j}\frac{\partial}{\partial u_j} = -ff'\prod_{\ell=2}^{j-1}\cos^2 u_\ell \frac{\partial}{\partial s} + \sum_{k=2}^{j-1}\left(\frac{\sin 2u_k}{2}\prod_{l=k+1}^{j-1}\cos^2 u_l\right)\frac{\partial}{\partial u_k}, \\ &\qquad j > 2. \end{aligned} \tag{5.17}$$

From (4.5), (5.17) and and $(\bar\nabla_{\partial/\partial s} h)(\frac{\partial}{\partial s}, \frac{\partial}{\partial u_j}) = (\bar\nabla_{\partial/\partial u_j} h)(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$ we find

$$\frac{f'}{f} = \varphi = \frac{\mu'}{\lambda - 2\mu}. \tag{5.18}$$

Thus, there is a real number $c \neq 0$ such that

$$f = a\, e^{\int^s \varphi(x)dx}. \tag{5.19}$$

By applying (5.16) and (5.19) we know that the sectional curvature K_{23} of the plane section spanned by $\partial/\partial u_2, \partial/\partial u_3$ is given by

$$K_{23} = a^{-2}e^{-2\int \varphi(s)ds} - \varphi^2. \tag{5.20}$$

On the other hand, (4.5) and Gauss' equation yields

$$K_{23} = -\mu^2. \tag{5.21}$$

Combining (5.18), (5.19), (5.20) and (5.21) gives

$$f^2 = \frac{1}{\varphi^2 - \mu^2}, \;\; \varphi = \frac{\mu'}{\lambda - 2\mu}, \;\; \lambda = 2\mu + \frac{\mu'}{\varphi}. \tag{5.22}$$

It follows from (5.5) and the last equation in (5.22) that ϕ and μ satisfy the following differential equation

$$\varphi' = \mu^2 - \varphi^2 + \frac{\mu\mu'}{\varphi}. \tag{5.23}$$

Therefore, by applying (4.5), (5.16)-(5.19), (5.22) and Gauss' formula, we obtain

$$(5.24)\quad \begin{aligned} &L_{ss} = \lambda P L_s, \\ &L_{su_j} = \varphi L_{u_j} + \mu P L_{u_j}, \\ &L_{u_i u_j} = -\tan u_i L_{u_j}, \ \ 2 \leq i < j \leq n, \\ &L_{u_j u_j} = \prod_{k=2}^{j-1} \cos^2 u_k \left(\frac{\mu}{\varphi^2 - \mu^2} P L_s - \frac{\varphi}{\varphi^2 - \mu^2} L_s \right) \\ &\qquad + \sum_{k=2}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{l=k+1}^{j-1} \cos^2 u_l \right) L_{u_k}, \ \ j = 2, \ldots, n. \end{aligned}$$

By applying $P^2 = I$, we obtain from (5.24) that

$$(5.25)\quad \begin{aligned} &P L_{ss} = \lambda L_s, \\ &P L_{su_j} = \mu L_{u_j} + \varphi P L_{u_j}, \\ &P L_{u_i u_j} = -\tan u_i P L_{u_j}, \ \ 2 \leq i < j \leq n, \\ &P L_{u_j u_j} = \prod_{k=2}^{j-1} \cos^2 u_k \left(\frac{\mu}{\varphi^2 - \mu^2} L_s - \frac{\varphi}{\varphi^2 - \mu^2} P L_s \right) \\ &\qquad + \sum_{k=2}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{l=k+1}^{j-1} \cos^2 u_l \right) P L_{u_k}, \ \ j = 2, \ldots, n. \end{aligned}$$

A direct computation shows that the compatibility condition of the PDE system (5.24)-(5.25) is (5.23).

From (5.24)-(5.25) we find

$$L_{u_2 u_2 u_2} + L_{u_2} = 0.$$

Thus

$$(5.26)\quad L = A(s, u_3, \ldots, u_n) \cos u_2 + B(s, u_3, \ldots, u_n) \sin u_2 + K(s, u_3, \ldots, u_n)$$

for some $\mathbb{E}^{2n}_n$-valued functions A, B and K. Substituting (5.26) into the third equation in (5.24) for $i = 2, j \geq 3$, we obtain $A = A(s)$ and $K = K(s)$. Thus, (5.26) reduces to

$$(5.27)\quad L = A(s, u_3, \ldots, u_n) \cos u_2 + B(s) \sin u_2 + K(s).$$

By substituting (5.27) into the last equation in (5.24) for $j = 2$ and using the first equation of (5.24), we conclude that A, B and K satisfy the following second order differential equations:

$$A_{ss} - \left(2\varphi(s) + \frac{\mu'(s)}{\mu(s)}\right) A_s + (\varphi^2(s) - \mu^2(s)) \left(2 + \frac{\mu'(s)}{\mu(s)\varphi(s)}\right) A = 0, \tag{5.28}$$

$$B_{ss} - \left(2\varphi(s) + \frac{\mu'(s)}{\mu(s)}\right) B_s + (\varphi^2(s) - \mu^2(s)) \left(2 + \frac{\mu'(s)}{\mu(s)\varphi(s)}\right) B = 0, \tag{5.29}$$

$$K_{ss} - \left(2\varphi(s) + \frac{\mu'(s)}{\mu(s)}\right) K_s = 0, \tag{5.30}$$

where μ, φ satisfy (5.23). After solving these second order differential equations we obtain

$$A = A_1(u_3, \ldots, u_n)\frac{e^{\int^s \lambda ds}}{\mu + \varphi} + A_2(u_3, \ldots, u_n)\frac{e^{-\int^s \lambda ds}}{\mu - \varphi}, \tag{5.31}$$

$$B = c_1\frac{e^{\int^s \lambda ds}}{\mu + \varphi} + c_2\frac{e^{-\int^s \lambda ds}}{\mu - \varphi}, \tag{5.32}$$

$$K = c_{-1} + c_0 \int^s \mu(s)e^{2\int^s \varphi(u)du} ds \tag{5.33}$$

for some vectors $c_{-1}, c_0, c_1, c_2 \in \mathbb{E}^{2n}_n$ and $\mathbb{E}^{2n}_n$-valued functions A_1, A_2. Thus, by combining (5.31)-(5.33) with (5.27) we conclude that, up to a suitable translation, the immersion L satisfies

$$\begin{aligned} L(s, u_2, \ldots, u_n) = &\left(\frac{e^{\int^s \lambda ds}}{\mu + \varphi}\right)(c_1 \sin u_2 + A_1(u_3, \ldots, u_n) \cos u_2) \\ &+ \left(\frac{e^{-\int^s \lambda ds}}{\mu - \varphi}\right)(c_2 \sin u_2 + A_2(u_3, \ldots, u_n) \cos u_2) \\ &+ c_0\int^s \mu(s)e^{2\int^s \varphi(u)du} ds. \end{aligned} \tag{5.34}$$

Now, by substituting (5.34) into the remaining equations of system (5.24)-(5.25), we obtain after long computation that

$$\begin{aligned} L = \frac{e^{\int^s \lambda(s)ds}}{\mu + \varphi}\Bigg\{ & c_1 \sin u_2 + \cos u_2 \Bigg(c_2 \sin u_3 + \cdots + \frac{e^{-\int^s \lambda(s)ds}}{\mu - \varphi} \\ & \Bigg\{ c_{n+1} \sin u_2 + c_{n-1} \sin u_{n-1} \prod_{\ell=3}^{n-2} \cos u_\ell + c_n \prod_{\ell=3}^{n-1} \cos u_\ell \Bigg) \Bigg\} \\ & + \cos u_2 \Bigg(c_{n+2} \sin u_3 + \cdots + c_{2n-1} \sin u_{n-1} \prod_{\ell=3}^{n-2} \cos u_\ell + c_{2n} \prod_{\ell=3}^{n-1} \cos u_\ell \Bigg) \Bigg\} \\ & + c_0 \int^s \mu(s)e^{2\int^s \varphi(u)du} ds \end{aligned}$$

for some vectors $c_1, \ldots, c_{2n} \in \mathbb{E}^{2n}_n$. Consequently, after choosing suitable initial conditions, we obtain (ii.2).

Case (3). $\varphi^2 < \mu^2$. In this case, we may assume that the metric on $I \times_f R^{n-1}(-1)$ is given by

$$g = ds^2 + f^2(s)\Big\{du_2^2 + \sinh^2 u_2(du_3^2 + \cos^2 u_3 du_4^2 + \cdots + \prod_{k=3}^{n-1} \cos^2 u_k du_{n-1}^2)\Big\}. \tag{5.35}$$

From (5.35) we obtain

$$\begin{aligned}
&\nabla'_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \quad \nabla'_{\partial/\partial s}\frac{\partial}{\partial u_k} = \frac{f'}{f}\frac{\partial}{\partial u_k},\\
&\nabla'_{\partial/\partial u_2}\frac{\partial}{\partial u_2} = -ff'\frac{\partial}{\partial s},\\
&\nabla'_{\partial/\partial u_2}\frac{\partial}{\partial u_j} = \coth u_2\frac{\partial}{\partial u_j}, \quad 3 \le j \le n,\\
&\nabla'_{\partial/\partial u_i}\frac{\partial}{\partial u_j} = -\tan u_i\frac{\partial}{\partial u_j}, \quad 3 \le i < j,\\
&\nabla'_{\partial/\partial u_j}\frac{\partial}{\partial u_j} = -\prod_{\ell=3}^{j-1}\cos^2 u_\ell\left\{ff'\sinh^2 u_2\frac{\partial}{\partial s} + \frac{\sinh 2u_2}{2}\frac{\partial}{\partial u_2}\right\}\\
&\qquad\qquad + \sum_{k=3}^{j-1}\left(\frac{\sin 2u_k}{2}\prod_{l=k+1}^{j-1}\cos^2 u_l\right)\frac{\partial}{\partial u_k}, \quad j \ge 3.
\end{aligned} \tag{5.36}$$

From (4.5), (5.36) and and $(\bar\nabla_{\partial/\partial s}h)(\frac{\partial}{\partial s}, \frac{\partial}{\partial x_j}) = (\bar\nabla_{\partial/\partial x_j}h)(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$ we also find

$$\frac{f'}{f} = \varphi = \frac{\mu'}{\lambda - 2\mu}. \tag{5.37}$$

Thus, there is a real number $c \neq 0$ such that

$$f = c\, e^{\int^s \varphi(x)dx}. \tag{5.38}$$

By applying (5.35) and (5.38) we know that the sectional curvature K_{23} of the plane section spanned by $\partial/\partial u_2, \partial/\partial u_3$ is given by

$$K_{23} = -c^{-2}e^{-2\int \varphi(s)ds} - \varphi^2. \tag{5.39}$$

On the other hand, (4.5) and Gauss' equation yields

$$K_{23} = -\mu^2. \tag{5.40}$$

Combining (5.37), (5.38), (5.39) and (5.40) gives

$$f^2 = \frac{1}{\mu^2 - \varphi^2}, \quad \varphi = \frac{\mu'}{\lambda - 2\mu}, \quad \lambda = 2\mu + \frac{\mu'}{\varphi}. \tag{5.41}$$

It follows from (5.5) and the last equation in (5.22) that ϕ and μ satisfy the following differential equation

$$\varphi' = \mu^2 - \varphi^2 + \frac{\mu\mu'}{\varphi}. \tag{5.42}$$

Therefore, by applying (4.5), (5.35)-(5.38), (5.41) and Gauss' formula, we obtain

$$\begin{aligned}
&L_{ss} = \lambda P L_s,\\
&L_{su_j} = \varphi L_{u_j} + \mu P L_{u_j}, \;\; 2 \le j \le n,\\
&L_{u_2u_2} = \frac{\mu}{\mu^2 - \varphi^2} P L_s - \frac{\varphi}{\mu^2 - \varphi^2} L_s,\\
&L_{u_2u_j} = \coth u_2 L_j, \;\; 3 \le j \le n,\\
&L_{u_iu_j} = -\tan u_i L_{u_j}, \;\; 3 \le i < j \le n,\\
&L_{u_ju_j} = \sinh^2 u_2 \prod_{\ell=3}^{j-1} \cos^2 u_\ell \left\{ \frac{\mu}{\mu^2 - \varphi^2} P L_s - \frac{\varphi}{\mu^2 - \varphi^2} L_s \right\}\\
&\quad - \frac{\sinh 2u_2}{2} \prod_{\ell=3}^{j-1} \cos^2 u_\ell L_{u_2} + \sum_{k=3}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{l=k+1}^{j-1} \cos^2 u_l \right) L_{u_k}, \; j \ge 3.
\end{aligned} \tag{5.43}$$

Now, by applying $P2 = I$ and (5.43), we get

$$\begin{aligned}
&P_{ss} = \lambda L_s,\\
&PL_{su_j} = \mu L_{u_j} + \varphi P L_{u_j}, \;\; 2 \le j \le n,\\
&PL_{u_2u_2} = \frac{\mu}{\mu^2 - \varphi^2} L_s - \frac{\varphi}{\mu^2 - \varphi^2} P L_s,\\
&PL_{u_2u_j} = \coth u_2 P L_j, \;\; 3 \le j \le n,\\
&PL_{u_iu_j} = -\tan u_i P L_{u_j}, \;\; 3 \le i < j \le n,\\
&PL_{u_ju_j} = \sinh^2 u_2 \prod_{\ell=3}^{j-1} \cos^2 u_\ell \left\{ \frac{\mu}{\mu^2 - \varphi^2} L_s - \frac{\varphi}{\mu^2 - \varphi^2} P L_s \right\}\\
&\quad - \frac{\sinh 2u_2}{2} \prod_{\ell=3}^{j-1} \cos^2 u_\ell P L_{u_2} + \sum_{k=3}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{l=k+1}^{j-1} \cos^2 u_l \right) P L_{u_k}, \; j \ge 3.
\end{aligned} \tag{5.44}$$

A direct computation shows that the compatibility condition of this system (5.43)-(5.44) is (5.23). By solving system (5.23) in a similar way as Case (2) and after long computation and using (5.23), we obtain

$$\begin{aligned} L(s,u_2,\ldots,u_n) = & \frac{e^{\int^s \lambda(s)ds}}{\mu+\varphi}\{c_1\cosh u_2+\sinh u_2\,(c_2\sin u_3+\cdots \\ & +c_{n-1}\sin u_{n-1}\prod_{\ell=3}^{n-2}\cos u_\ell + c_n\prod_{\ell=3}^{n-1}\cos u_\ell\Big)\Big\} \\ & + \frac{e^{-\int^s \lambda(s)ds}}{\mu-\varphi}\{c_{n+1}\cosh u_2+\sinh u_2\,(c_{n+2}\sin u_3+\cdots \\ & +c_{2n-1}\sin u_{n-1}\prod_{\ell=3}^{n-2}\cos u_\ell + c_{2n}\prod_{\ell=3}^{n-1}\cos u_\ell\Big)\Big\} \end{aligned}$$

for some vectors $c_1,\ldots,c_{2n}\in\mathbb{E}^{2n}_n$. Hence, after choosing suitable initial conditions, we obtain (ii.3). ∎

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