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PACKING CONSTANTS IN ORLICZ-LORENTZ SEQUENCE SPACES

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Abstract. We discussed the upper and lower bounds of packing constants in Orlicz-Lorentz sequence spaces equipped with both the Luxemburg norm and the Orlicz norm. Provided $\Phi \in \Delta_2(0)$, we showed that the Kottman constant of $\lambda_{\Phi,\omega}$ and $\lambda^o_{\Phi,\omega}$ satisfies

$$\max\left\{\frac{1}{\alpha_\Phi(0)}, \frac{1}{\alpha'_{\Phi,\omega}}\right\} \le K\left(\lambda_{\Phi,\omega}\right) \le \frac{1}{\tilde{\alpha}_{\Phi,\omega}},$$

$$\max\left\{\frac{1}{\alpha_\Phi(0)}, \frac{1}{\alpha''_{\Phi,\omega}}\right\} \le K\left(\lambda^o_{\Phi,\omega}\right) \le \frac{1}{\alpha^*_\Phi}.$$

As a corollary, the packing constant of Lorentz space $\lambda_{p,\omega}$ is $1/(1+2^{1-\frac{1}{p}})$.

The packing constants of Orlicz spaces were studied by many researchers. However, there are few results on geometric constants of Lorentz spaces as well as Orlicz-Lorentz spaces. In this paper, we shall study the packing constant in Orlicz-Lorentz sequence spaces $\lambda_{\Phi,\omega}$ and $\lambda^o_{\Phi,\omega}$ (equipped with the Luxemburg norm and the Orlicz norm respectively). We will obtain the nontrivial lower and upper bounds of the Kottman constant. Both the technical ideas and the computational methods are practical and can be employed to estimate some other geometric constants in Orlicz-Lorentz spaces.

1. Introduction

Let $\Phi(u) = \int_0^{|u|} \phi(t)dt$ be an N-function, i.e., it is even, convex, $\lim\limits_{u\to 0} \frac{\Phi(u)}{u} = 0$, and $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = \infty$, ϕ being right continuous, nondecreasing, positive for $t > 0$. The following indices have been well researched (see [8, 10, 11]):

(1.1) $$A^0_\Phi = \liminf_{t\to 0} \frac{t\phi(t)}{\Phi(t)}, \qquad B^0_\Phi = \limsup_{t\to 0} \frac{t\phi(t)}{\Phi(t)};$$

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(1.2) $$\alpha_\Phi^0 = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_\Phi^0 = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$

We say that $\Phi \in \Delta_2(0)$ if there is $u_0 > 0$ and $K > 1$ such that $\Phi(2u) \le K\Phi(u)$ whenever $u \le u_0$ and that $\Phi \in \nabla_2(0)$ if $\Psi \in \Delta_2(0)$. We define $\Psi(v) = \int_0^{|v|} \psi(s)ds$ to be the complementary N-function of Φ (ψ is the right inverse of ϕ).

The indices have the well known relationship (see [8, 13, 17]):

(1.3) $$\frac{1}{A_\Phi^0} + \frac{1}{B_\Psi^0} = 1 = \frac{1}{A_\Psi^0} + \frac{1}{B_\Phi^0};$$

(1.4) $$2\alpha_\Phi^0\beta_\Psi^0 = 1 = 2\alpha_\Psi^0\beta_\Phi^0;$$

(1.5) $$2^{-\frac{1}{A_\Phi^0}} \le \alpha_\Phi^0 \le \beta_\Psi^0 \le 2^{-\frac{1}{B_\Phi^0}}.$$

In [17], the author studied the functions which produce the quantitative indices:

(1.6) $$F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)}, \qquad G_\Phi(c,u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(cu)} \qquad (c > 1)$$

and proved that $F_\Phi(t)$ is increasing (decreasing) on $(0, \Phi^{-1}(u_0)]$ if and only if $G_\Phi(c,u)$ is increasing (decreasing) on $(0, \frac{u_0}{c}]$ for every $c > 1$. It follows that if $F_\Phi(t)$ or $G_\Phi(c,u)$ is monotonic on some interval near zero, then the inequalities (1.5) become equalities. We denote $G_\Phi(2,u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ by $G_\Phi(u)$. It is easy to see that (see [8, 12, 13]) $1 < F_\Phi(t) < \infty (0 < t < \infty)$, $\frac{1}{2} < G_\Phi(u) < 1 (0 < u < \infty)$ for any N-function Φ and $\Phi \in \Delta_2(0) \Leftrightarrow \beta_\Phi^0 < 1 \Leftrightarrow B_\Phi^0 < \infty$, $\Phi \in \nabla_2(0) \Leftrightarrow \beta_\Phi^0 > \frac{1}{2} \Leftrightarrow B_\Phi^0 > 1$.

The concept of Orlicz-Lorentz space was first introduced by A.Kamińska in 1990 [5]. Many important results concerning the qualitative geometry properties such as rotundities and H-properties were researched. However, few achievements about quantitative geometry properties (geometric constants) were obtained on comparison of the classical Orlicz spaces (see [13]) due to the difficulties from the rearrangements. So it is meaningful to explore the technique for packing constants of Orlicz-Lorentz sequence spaces.

We call a decreasing positive sequence $\omega = (\omega_1, \omega_2, \cdots)$ a weight sequence, write Π to be the set of all permutations π of natural number set $\mathcal{N}$. For a sequence of real numbers $x = (x_1, x_2, \cdots)$, we define $x^* = (x_1^*, x_2^*, \cdots)$ the rearrangement of x, i.e., it is the result of the permutation under which the sequence $(|x_1|, |x_2|, \cdots)$ becomes decreasing. We define the modula of x by

$$\rho_\Phi(x) = \sum_{i=1}^{\infty} \Phi(x^*(i))\omega_i.$$

It was proved [20] that

$$\rho_\Phi(x) = \sup_{\pi\in\Pi} \sum_{i=1}^{\infty} \Phi(x(\pi(i)) \cdot \omega_i.$$

The Orlicz-Lorentz sequence space $\lambda_{\Phi,\omega}$ is the set

$$\{x : \rho_\Phi(\lambda x) < \infty \text{ for some} \lambda > 0\}.$$

The Luxemburg norm and the Orlicz norm of x are defined respectively to be

$$\|x\| = \inf\left\{c > 0 : \rho_\Phi\left(\frac{x}{c}\right) \le 1\right\}, \tag{1.7}$$

$$\|x\|^o = \sup\left\{\sum_{i=1}^{\infty} x_i^* \cdot y_i^* \cdot \omega_i : \rho_\Psi(y) \le 1\right\}. \tag{1.8}$$

It is easy to see that $(\lambda_{\Phi,\omega}, \|\cdot\|)$ and $(\lambda_{\Phi,\omega}, \|\cdot\|^o)$ are both Banach spaces. In the sequel, we denote them by $\lambda_{\Phi,\omega}$ and $\lambda^o_{\Phi,\omega}$ respectively. It was verified [16] that for any $0 \ne x \in \lambda_{\Phi,\omega}$, we have $\|x\| < \|x\|^o \le 2\|x\|$. So the above norms are equivalent. The following two statements holds:

(1) If $\sum_{i=1}^{\infty} \omega_i < \infty$, then there exist an isometric copy of l^∞ contained in $\lambda_{\Phi,\omega}$. Therefore, it is not reflexive (see [5, 23]);

(2) If $\sum_{i=1}^{\infty} \omega_i = \infty$, then $\lambda_{\Phi,\omega}$ is reflexive if and only if $\Phi \in \Delta_2(0) \cap \bigtriangledown_2(0)$ (see [7, 20]).

Let X be an infinite dimensional Banach space, $B(X)$ and $S(X)$ be the unit ball and the unit sphere of X respectively. The *packing constant* $P(X)$ of X is

$$P(X) = \sup\{r > 0 : \text{infinitely many balls of radius } r \text{ are packed into } B(X)\ \}$$

The *Kottman constant* of an infinite dimensional Banach space X is defined as

$$K(X) = \sup_{\mathcal{X}\subset S(X)} \left\{\inf_{n\ne m} \left\|x^{(n)} - x^{(m)}\right\| : x^{(n)}, x^{(m)} \in \mathcal{X}\right\}, \tag{1.9}$$

where $\mathcal{X} = \left\{x^{(n)}\right\}_{n=1}^{\infty}$ is a point sequence in $S(X)$.

Clearly, $1 \le K(X) \le 2$. For a Banach space X, one has [6] (also see Ye [21])

$$P(X) = \frac{K(X)}{2 + K(X)}. \tag{1.10}$$

Early in 1955, Rankin [11] established $P(X)$ for an infinite dimensional Hilbert space X. Slightly later, Burlack, Rankin and Robertson [1] generalized it to $P(l^p)$. In 1994, Ye, Zhang and Pluciennik [22] found $P(d(\omega,p))$ for Lorentz sequence spaces $d(\omega,p)(1 \le p < +\infty)$. For the Orlicz sequence spaces equipped with Luxemburg norm and Orlicz norm with Φ satisfying the Δ_2-condition, Wang [14] and Ye [21] gave the expressions for Kottman constants. In 2001, the author [17] obtained some estimations for real computation which answered Rao and Ren's [12] open problem concerning the exact value of some Orlicz sequence spaces. In 2007, the author [19] showed another method to compute the value of Kottman constants of some class of Orlicz spaces. Now we devote this paper to the Orlicz-Lorentz sequence spaces with the Luxemburg norm and the Orlicz norm.

Hudzik [3] verified that $K(X) = 2$ if X is a nonreflexive Banach lattice, therefore $P(X) = \frac{1}{2}$. Since $\lambda_{\Phi,\omega}$ and $\lambda^o_{\Phi,\omega}$ are Banach lattices, it suffices to consider the Kottman constant when $\sum\limits_{i=1}^{\infty} \omega_i = \infty$ and $\Phi \in \triangle_2(0) \cap \nabla_2(0)$. So in the sequel, we always assume $\sum\limits_{i=1}^{\infty} \omega_i = \infty$.

We now show some auxiliary lemmas:

Lemma 1.1. *The following statements holds:*

(1) $\|x_n\| \to 0 \Rightarrow \rho_\Phi(x_n) \to 0$, $\rho_\Phi(x_n) \to 1 \Rightarrow \|x_n\| \to 1$. $\Phi \in \Delta_2(0)$ *if and only if* $\rho_\Phi(x_n) \to 0 \Rightarrow \|x_n\| \to 0$ *if and only if* $\|x_n\| \to 1 \Rightarrow \rho_\Phi(x_n) \to 1$ $(n \to \infty)$ (*see* [5]).

(2) *For the point* $\chi_k = (1, 1, \ldots, 1_{kth}, 0, 0, \cdots)$, *we have* (*see*[2, 15])

$$\|\chi_k\| = \frac{1}{\Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{k}\omega_i}\right)}, \qquad \|\chi_k\|^o = \sum_{i=1}^{k}\omega_i \cdot \Psi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{k}\omega_i}\right). \tag{1.11}$$

(3) *Let* $x \in \lambda_{\Phi,\omega}$, *then* (*see* [16]) $\|x\|^o = \inf\limits_{k>0} \frac{1}{k}(1 + \rho_\Phi(kx))$. *Moreover,*

$$\|x\|^o = \frac{1}{k}(1 + \rho_\Phi(kx)) \tag{1.12}$$

holds if and only if $k \in K(x) := [k^*, k^{**}]$, *where* $k^* = \inf\{k > 0 : \rho_\Psi(\phi(k|x|)) \ge 1\}$, $k^{**} = \sup\{k > 0 : \rho_\Psi(\phi(k|x|)) \le 1\}$.

Lemma 1.2. *Assume* $\Phi \in \Delta_2(0)$, $x, y \in \lambda_{\Phi,\omega}$. *Then for any* $\varepsilon_1 > 0$ *and* $L > 0$, *there exists* $\varepsilon > 0$ *such that*

$$\rho_\Phi(x) \le L, \rho_\Phi(y) \le \varepsilon \Rightarrow |\rho_\Phi(x+y) - \rho_\Phi(x)| < \varepsilon_1. \tag{1.13}$$

Proof. Since the modula functional ρ_Φ is convex in the space $\lambda_{\Phi,\omega}$. It is easy to check that the proof can be employed without any change from that of the corresponding classical Orlicz spaces (see [2, 21]). ∎

Lemma 1.3. *Let* $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$, $u \in (0, u_0]$. *Then for any* $t \in (0, \Phi^{-1}(2u_0)]$

$$\Phi\left(G_\Phi(\frac{1}{2}\Phi(t)) \cdot t\right) = \frac{1}{2}\Phi(t). \tag{1.14}$$

Proof. Set $t = \Phi^{-1}(2u)$, then $u = \frac{1}{2}\Phi(t)$, and hence $\Phi\left(G_\Phi(u) \cdot \Phi^{-1}(2u)\right) = u$. Therefore (1.14) holds. ∎

For $x = (x_1, x_2, \cdots)$, we write

$$x_{[1,i]} = (x_1, x_2, \cdots, x_i, 0, 0, \cdots),$$
$$x_{(i,\infty)} = (0, 0, \cdots, 0, x_{i+1}, x_{i+2}, \cdots),$$
$$x_{(i,j]} = (0, 0, \cdots, 0, x_{i+1}, x_{i+2}, \cdots, x_j, 0, 0, \cdots).$$

Lemma 1.4. *Let* $\Phi \in \Delta_2(0)$. *For any sequence of points* $\mathcal{X}_0 = \left\{x^{(n)}\right\}_{n=1}^{\infty} \subset S\left(\lambda_{\Phi,\omega}\right)$, *there is a subsequence* $\mathcal{Z} := \left\{z^{(k)}\right\}_{k=1}^{\infty} \subset \mathcal{X}_0$ *and a sequence of natural numbers* $i_0 < i_1 < i_2 < \cdots < i_l < \cdots < i_k < \cdots$ *satisfying the following conditions:*

$$\left\|z^{(k)}_{(i_k,\infty)}\right\| < \varepsilon, \tag{1.15}$$

$$\left\|z^{(j)}_{(i_0,i_k]}\right\| < 2\varepsilon \quad (j \geq k+1), \tag{1.16}$$

$$\left\|\left(z^{(j)} - x^{(0)}\right)_{[1,i_0]}\right\| < \varepsilon \quad (j \geq 1). \tag{1.17}$$

The above three inequalities are also holds for the Orlicz norm owing to the equivalence of the norms.

Proof. For any $x^{(n)} \in \mathcal{X}_0$, one has $\Phi\left(x_1^{(n)}\right)\omega_1 \leq \rho_\Phi(x) = 1$, which implies $x_1^{(n)} \leq \Phi^{-1}\left(\frac{1}{\omega_1}\right)$ for $n = 1, 2, \cdots$. Therefore there exists a convergent subsequence $\left\{x_1^{(n_k)}\right\}_{n=1}^{\infty}$ in $\left\{x_1^{(n)}\right\}_{n=1}^{\infty}$. In other words, there is a subsequence $\left\{x^{(n,1)}\right\} = \mathcal{X}_1 \subset \mathcal{X}_0$ such that the first coordinates of $\mathcal{X}_1$ converge to some x_1. By the same reason, there is a subsequence $\left\{x^{(n,2)}\right\} = \mathcal{X}_2 \subset \mathcal{X}_1$ whose second coordinates converge to some x_2. Inductively, there is a sequence $\mathcal{X}_0 \supset \cdots \mathcal{X}_k \supset \mathcal{X}_{k+1} \supset \cdots$ such that the first k coordinates of $\mathcal{X}_k$ are convergent.

Denotes $x^{(0)} = (x_1)_{i=1}^{\infty}$. We show $\|x^{(0)}\| \le 1$ and hence $x^0 \in \lambda_{\Phi,\omega}$. Suppose the contrary, i.e., $\|x^{(0)}\| > 1$. Then there is sufficiently large number i such that

$$\left\|x^{(0)}_{[1,i]}\right\| > 1.$$

Since for every j $(1 \le j \le i)$ in $\mathcal{X}_i$, the jth coordinate convergent to $x_j^{(0)}$. Moreover the space $\mathcal{X}_{i[1,i]} :=$ span $\left\{x^{(k)}_{[1,i]} : x^{(k)} \in \mathcal{X}_i\right\}$ is finite dimensional, so the convergence in coordinate is equivalent to the convergence in norm. Therefore, there exists N, such that $\left\|x^{(n,i)}_{[1,i]}\right\| > 1$ whenever $n > N$. This results in $\left\|x^{(n,i)}\right\| \ge \left\|x^{(n,i)}_{[1,i]}\right\| > 1$ which contradicts the fact that $\left\|x^{(n,i)}\right\| = 1$.

Let $\varepsilon > 0$ be an arbitrary small number. Since $\Phi \in \Delta_2(0)$, there exist i_0 such that $\left\|x^{(0)}_{(i_0,\infty)}\right\| < \varepsilon$. Delete some finite elements in $\mathcal{X}_{i_0}$ if necessary we obtain a subsequence $\mathcal{Y}_1 \in \mathcal{X}_{i_0}$ such that

$$\left\|\left(y - x^{(0)}\right)_{[1,i_0]}\right\| < \varepsilon.$$

for every $y \in \mathcal{Y}_1$. We denote the first element in $\mathcal{Y}_1$ by $z^{(1)}$. Next we select $i_1 > i_0$ such that $\left\|z^{(1)}_{(i_1,\infty)}\right\| < \varepsilon$. The elements in $\mathcal{X}_{i_1[1,i_1]}$ are convergent to $x^{(0)}_{[1,i_1]}$ in coordinate. Moreover, $\left\|x^{(0)}_{(i_0,i_1]}\right\| \le \left\|x^{(0)}_{(i_0,\infty)}\right\| < \varepsilon$. Therefore, we obtain a subsequence $\mathcal{Y}_2 \subset \mathcal{Y}_1$ from $\mathcal{X}_{i_1}$ by deleting some finite elements such that

$$\left\|y_{(i_0,i_1]}\right\| \le \left\|\left(y - x^{(0)}\right)_{(i_0,i_1]}\right\| + \left\|x^{(0)}_{(i_0,i_1]}\right\| < 2\varepsilon.$$

We denote the first element in $\mathcal{Y}_2$ after $z^{(1)}$ by $z^{(2)}$. Then we select $i_2 > i_1$, such that $\left\|z^{(2)}_{(i_2,\infty)}\right\| < \varepsilon$, obtaining a subsequence $\mathcal{Y}_3 \subset \mathcal{Y}_2$ from $\mathcal{X}_{i_2}$ satisfying $\left\|y_{(i_0,i_2]}\right\| < 2\varepsilon$.

Inductively, if $\mathcal{Y}_k$ and $z^{(k)}$ was selected such that $\left\|z^{(k)}_{(i_k,\infty)}\right\| < \varepsilon$, we obtain a subsequence $\mathcal{Y}_{k+1} \subset \mathcal{Y}_k$ from $\mathcal{X}_{i_k}$ satisfying $\left\|y_{(i_0,i_{k+1}]}\right\| < 2\varepsilon$. Then we denote the first element in $\mathcal{Y}_{k+1}$ after $z^{(k)}$ by $z^{(k+1)}$. In this way we obtain a sequence $\mathcal{Z} := \left\{z^{(k)}\right\}_{k=1}^{\infty} \subset \mathcal{X}_0$ and a sequence of natural numbers $i_0 < i_1 < i_2 < \cdots < i_l < \cdots < i_k < \cdots$ satisfying (1.15)-(1.17). ■

For any given $k \ge 1$, let

$$Z_k = (0, 0, \cdots, 0) \text{ and } X_{\Phi,k} = \Phi^{-1}\left(\frac{1}{\sum_{i=1}^{k} \omega_i}\right)(1, 1, \cdots, 1) \tag{1.18}$$

with $\dim Z_k = \dim X_k = k$. We produce a sequence $\mathcal{Y} = \{y^{(n)}\}_{n=1}^{\infty} \subset S(\lambda_{\Phi,\omega})$, where

$$
\begin{aligned}
y^{(1)} &= (X_{\Phi,1}, Z_2, Z_3, 0, \cdots) = \left(\Phi^{-1}\left(\frac{1}{\omega_1}\right), 0, 0, \cdots\right),\\
y^{(2)} &= (Z_1, X_{\Phi,2}, Z_3, 0, \cdots) = \left(0, \Phi^{-1}\left(\frac{1}{\omega_1+\omega_2}\right), \Phi^{-1}\left(\frac{1}{\omega_1+\omega_2}\right), 0, 0, \cdots\right),\\
&\vdots \qquad\qquad \vdots\\
y^{(n)} &= (Z_1, Z_2, \cdots, Z_{n-1}, X_{\Phi,n}, Z_{n+1}, 0, \cdots)\\
&= \left(\underbrace{0, \cdots, 0}_{\frac{n(n-1)}{2}}, \underbrace{\Phi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega_i}\right), \cdots, \Phi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega_i}\right)}_{n}, 0, 0, \cdots\right),\\
&\vdots \qquad\qquad \vdots
\end{aligned}
\tag{1.19}
$$

We claim that for every $\varepsilon > 0$, there exists a sequence $\{z^{(n)}\}_{n=1}^{\infty} \subset S(\lambda_{\Phi,\omega})$, such that

$$
\rho_\Phi\left(z^{(n)} - z^{(m)}\right) > 2 - \varepsilon. \quad (n \neq m) \tag{1.20}
$$

In fact, the candidate can be selected from the above $\mathcal{Y}$.

Take $n_1 = 1$, i.e., $y^{(n_1)} = \left(\Phi^{-1}\left(\frac{1}{\omega_1}\right), 0, 0, \cdots\right)$, we denote it by $z^{(1)}$. Since $\sum_{i=1}^{\infty}\omega_i = \infty$, there is n_2 such that $\frac{\omega_1}{\sum_{i=1}^{n_2}\omega_i} < \varepsilon$. Let $z^{(2)} = y^{(n_2)}$, then

$$
\rho_\Phi\left(z^{(1)} - z^{(2)}\right) = \frac{1}{\omega_1}\omega_1 + \frac{1}{\sum_{i=1}^{n_2}\omega_i}\sum_{i=2}^{n_2+1}\omega_i = 2 - \frac{\omega_1 - \omega_{n_2+1}}{\sum_{i=1}^{n_2}\omega_i} > 2 - \varepsilon.
$$

Choose $n_3 > n_2$, such that $\sum_{i=1}^{n_2}\omega_i / \sum_{i=1}^{n_3}\omega_i < \varepsilon$, then let $z^{(3)} = y^{(n_3)}$. Inductively, choose $n_k > n_{k-1}$, such that $\sum_{i=1}^{n_{k-1}}\omega_i / \sum_{i=1}^{n_k}\omega_i < \varepsilon$, then let $z^{(k)} = y^{(n_k)}$. Now for any pair of $k > l$, we have

$$
\rho_\Phi\left(z^{(k)} - z^{(l)}\right) = \frac{1}{\sum_{i=1}^{n_l}\omega_i}\sum_{i=1}^{n_l}\omega_i + \frac{1}{\sum_{i=1}^{n_k}\omega_i}\sum_{i=n_l+1}^{n_l+n_k}\omega_i
$$

$$= 2 - \frac{\sum_{i=1}^{n_l} \omega_i - \sum_{i=n_k+1}^{n_k+n_l} \omega_i}{\sum_{i=1}^{n_k} \omega_i} > 2 - \frac{\sum_{i=1}^{n_l} \omega_i}{\sum_{i=1}^{n_k} \omega_i} \geq 2 - \frac{\sum_{i=1}^{n_{k-l}} \omega_i}{\sum_{i=1}^{n_k} \omega_i} > 2 - \varepsilon.$$

2. Estimation of $K(\lambda_{\Phi,\omega})$

In this section, we deal with the Kottman constant of Orlicz-Lorentz sequence space quipped with the Luxemburg norm, namely, $K(\lambda_{\Phi,\omega})$.

The lower bounds have the following result:

Theorem 2.1. *For any N-function Φ, we have*

$$K(\lambda_{\Phi,\omega}) \geq \max\left\{\frac{1}{\alpha_\Phi(0)}, \frac{1}{\alpha'_{\Phi,\omega}}\right\} \tag{2.1}$$

where $\alpha_\Phi(0)$ is defined as in (1.2), $\alpha'_{\Phi,\omega} := \inf_{k\geq 1}\left\{\frac{\Phi^{-1}\left(\frac{1}{S(2k)}\right)}{\Phi^{-1}\left(\frac{1}{S(k)}\right)}\right\}$, $S(k) := \sum_{i=1}^{k} \omega_i$.

Proof. We first prove

$$K(\lambda_{\Phi,\omega}) \geq \frac{1}{\alpha_\Phi(0)}.$$

Construct the sequence $\mathcal{Y} = \{y^{(n)}\}_{n=1}^{\infty} \subset S(\lambda_{\Phi,\omega})$ of (1.19). For any $\delta > 0$ there is a sequence $\mathcal{U} = \{u_n\}_{n=1}^{\infty}$ with $u_n \searrow 0 (n \to \infty)$ such that $\frac{\Phi^{-1}(u_n)}{\Phi^{-1}(2u_n)} < \alpha_\Phi(0) + \delta$.

Given $\varepsilon > 0$, because $\sum_{i=1}^{\infty} \omega_i = \infty$, there is a nature number N_0 such that for every $n \geq N_0$ we have

$$\frac{\sum_{i=1}^{n} \omega_i}{\sum_{i=1}^{n+1} \omega_i} \geq 1 - \frac{\omega_1}{\sum_{i=1}^{n+1} \omega_i} > 1 - \varepsilon. \tag{2.2}$$

The set of the intervals $\left\{\left(\frac{1}{2\sum_{i=1}^{j+1} \omega_i}, \frac{1}{2\sum_{i=1}^{j} \omega_i}\right]\right\}_{j=N_0}^{\infty}$ forms a division of the interval $(0, t_0]$ where $t_0 = \frac{1}{2\sum_{i=1}^{N_0} \omega_i}$. So we can select a $u_{n_1} \in \mathcal{U}$ such that $\frac{1}{2\sum_{i=1}^{n_1+1} \omega_i} < u_{n_1} \leq \frac{1}{2\sum_{i=1}^{n_1} \omega_i}$ satisfying $n_1 \geq N_0$. Define $z^{(1)} = y^{(n_1)}$.

Since $\sum_{i=1}^{\infty}\omega_i=\infty$, there is a natural number N_1 such that for every $n\geq N_1$ one has $\sum_{i=1}^{n_1}\omega_i/\sum_{i=1}^{n}\omega_i<\varepsilon$. Select $n_2\geq N_1$ such that some $u_{n_2}\in\mathcal{U}$ located on the interval $\left(\frac{1}{2\sum_{i=1}^{n_2+1}\omega_i},\frac{1}{2\sum_{i=1}^{n_2}\omega_i}\right]$. Then define $z^{(2)}=y^{(n_2)}$ like $z^{(1)}$.

Inductively, choose $n_k>n_{k-1}$, such that $\sum_{i=1}^{n_{k-1}}\omega_i/\sum_{i=1}^{n_k}\omega_i<\varepsilon$, and some $u_{n_k}\in\mathcal{U}$ located on the interval $\left(\frac{1}{2\sum\limits_{i=1}^{n_k+1}\omega_i},\frac{1}{2\sum\limits_{i=1}^{n_k}\omega_i}\right]$. Then define

$$z^{(k)}=y^{(n_k)}=\left(\underbrace{0,\cdots,0}_{\frac{(n_k-1)n_k}{2}},\underbrace{\Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{n_k}\omega_i}\right),\cdots,\Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{n_k}\omega_i}\right)}_{n_k},0,0,\cdots\right).$$

We obtain a subsequence $\mathcal{Z}=\{z^{(k)}\}_{k=1}^{\infty}\subset S\left(\lambda_{\Phi,\omega}\right)$ and $u_{n_k}\in\mathcal{U}$.

For any pair of k,l $(k>l)$, we have

$$\begin{aligned}
&\rho_\Phi\left(\left(\alpha_\Phi(0)+\delta\right)\left(z^{(k)}-z^{(l)}\right)\right)\\
&=\sum_{i=1}^{n_l}\Phi\left[\left(\alpha_\Phi(0)+\delta\right)\cdot\Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{n_l}\omega_i}\right)\right]\omega_i\\
&\qquad+\sum_{i=n_l+1}^{n_l+n_k}\Phi\left[\left(\alpha_\Phi(0)+\delta\right)\cdot\Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{n_k}\omega_i}\right)\right]\omega_i\\
&\geq\sum_{i=1}^{n_l}\Phi\left[\frac{\Phi^{-1}(u_{n_l})}{\Phi^{-1}(2u_{n_l})}\cdot\Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{n_l}\omega_i}\right)\right]\omega_i\\
&\qquad+\sum_{i=n_l+1}^{n_l+n_k}\Phi\left[\frac{\Phi^{-1}(u_{n_k})}{\Phi^{-1}(2u_{n_k})}\cdot\Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{n_k}\omega_i}\right)\right]\omega_i
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{n_l} \Phi\left[\frac{\Phi^{-1}(u_{n_l})}{\Phi^{-1}(2u_{n_l})} \cdot \Phi^{-1}(2u_{n_l})\right]\omega_i + \sum_{i=n_l+1}^{n_l+n_k} \Phi\left[\frac{\Phi^{-1}(u_{n_k})}{\Phi^{-1}(2u_{n_k})} \cdot \Phi^{-1}(2u_{n_k})\right]\omega_i \\
&= \sum_{i=1}^{n_l} \Phi\left(\Phi^{-1}(u_{n_l})\right)\omega_i + \sum_{i=n_l+1}^{n_l+n_k} \Phi\left(\Phi^{-1}(u_{n_k})\right)\omega_i \\
&= \sum_{i=1}^{n_l} u_{n_l}\omega_i + \sum_{i=n_l+1}^{n_l+n_k} u_{n_k}\omega_i \\
&> \sum_{i=1}^{n_l} \frac{1}{2\sum\limits_{i=1}^{n_l+1}\omega_i} \cdot \omega_i + \sum_{i=n_l+1}^{n_l+n_k} \frac{1}{2\sum\limits_{i=1}^{n_k+1}\omega_i} \cdot \omega_i \\
&= \frac{1}{2}\left[\frac{1}{\sum\limits_{i=1}^{n_l+1}\omega_i}\sum_{i=1}^{n_l}\omega_i + \frac{1}{\sum\limits_{i=1}^{n_k+1}\omega_i}\sum_{i=n_l+1}^{n_l+n_k}\omega_i\right] > \frac{1}{2}\left[1-\varepsilon+(1-\varepsilon)\frac{\sum\limits_{i=n_l+1}^{n_l+n_k}\omega_i}{\sum\limits_{i=1}^{n_k+1}\omega_i}\right] \\
&= \frac{1-\varepsilon}{2}\left[1+\frac{\sum\limits_{i=1}^{n_k}\omega_i - \sum\limits_{i=1}^{n_l}\omega_i + \sum\limits_{i=n_k+1}^{n_l+n_k}\omega_i}{\sum\limits_{i=1}^{n_k}\omega_i}\right] > \frac{1-\varepsilon}{2}\left[2-\frac{\sum\limits_{i=1}^{n_{k-1}}\omega_i}{\sum\limits_{i=1}^{n_k}\omega_i}\right] \\
&> \frac{1-\varepsilon}{2}(2-\varepsilon) > (1-\varepsilon)^2.
\end{aligned}
$$

We know from Lemma 1.1.1 that for every $\varepsilon_1 > 0$, there is $\varepsilon > 0$ such that $\rho_\Phi(x) > 1-\varepsilon$ implies $\|x\| > 1-\varepsilon_1$. Therefore,

$$
\left\|z^{(k)} - z^{(l)}\right\| \geq \frac{1-\varepsilon_1}{\alpha_\Phi(0)+\delta}. \tag{2.3}
$$

By the arbitrariness of δ and ε_1 we obtain

$$
K\left(\lambda_{\Phi,\omega}\right) \geq \sup_{\mathcal{Z}_{\varepsilon_1,\delta}} \inf_{k\neq l} \left\|z^{(k)} - z^{(l)}\right\| \geq \frac{1}{\alpha_\Phi(0)} \tag{2.4}
$$

where $\mathcal{Z}_{\varepsilon_1,\delta}$ denotes the sequence $\left\{z^{(n)}\right\}_1^\infty$ depending on ε_1, δ.

Next we show

$$
K\left(\lambda_{\Phi,\omega}\right) \geq \frac{1}{\alpha'_{\Phi,\omega}}.
$$

Let Z_k and $X_{\Phi,k}$ be described as in (1.18), and define

$$
x^{(n)} = (Z_k, Z_k, \cdots, Z_k, X_{\Phi,k}, Z_k, Z_k, \cdots)(n \geq 1),
$$

with $X_{\Phi,k}$ being at the nth position. Then $\left\|x^{(n)}\right\| = 1 \quad (n \geq 1)$, and for $n \neq m$,

$$\left\|x^{(n)} - x^{(m)}\right\| = \Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{k}\omega_i}\right) \cdot \left[\Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{2k}\omega_i}\right)\right]^{-1} = \frac{\Phi^{-1}\left(\frac{1}{S(k)}\right)}{\Phi^{-1}\left(\frac{1}{S(2k)}\right)}.$$

It follows that

$$K(\lambda_{\Phi,\omega}) \geq \sup_{k\geq 1}\inf_{n\neq m}\left\|x^{(n)} - x^{(m)}\right\| = \sup_{k\geq 1}\frac{\Phi^{-1}\left(\frac{1}{S(k)}\right)}{\Phi^{-1}\left(\frac{1}{S(2k)}\right)} = \frac{1}{\alpha'_{\Phi,\omega}} \tag{2.5}$$

and we finishes the proof. ∎

To show the upper bounds of $K(\lambda_{\Phi,\omega})$, we cite the index in $\lambda_{\Phi,\omega}$ as in [21]:

$$d_\lambda = \sup_{x\in S\left(\lambda_{\Phi,\omega}\right)}\left\{C_x > 0 : \rho_\Phi\left(\frac{x}{C_x}\right) = \frac{1}{2}\right\}. \tag{2.6}$$

It is easy to see that $1 \leq d_\lambda \leq 2$. For every $x \in S(\lambda_{\Phi,\omega})$ we have

$$\rho_\Phi\left(\frac{x}{d_\lambda}\right) \leq \frac{1}{2}. \tag{2.7}$$

Theorem 2.2. *If $\Phi \in \Delta_2(0)$, then*

$$K(\lambda_{\Phi,\omega}) \leq d_\lambda. \tag{2.8}$$

Proof. Suppose $\mathcal{X}_0$ to be the arbitrary sequence in Lemma 1.4, we have a subsequence $\mathcal{Z}$ satisfying (1.15)-(1.17). For every pair of k, l $(k > l)$, we have

$$\begin{aligned}
\rho_\Phi\left(\frac{z^{(k)} - z^{(l)}}{d_\lambda}\right) &= \sup_{\pi\in\Pi}\left[\sum_{i=1}^{\infty}\Phi\left(\frac{\left|z^{(k)}_{\pi(i)} - z^{(l)}_{\pi(i)}\right|}{d_\lambda}\right)\omega_i\right] \\
&= \sup_{\pi\in\Pi}\left[\sum_{i=1}^{\infty}\Phi\left(\frac{\left|z^{(k)}_i - z^{(l)}_i\right|}{d_\lambda}\right)\omega_{\pi(i)}\right] \\
&= \sup_{\pi\in\Pi}\left[\sum_{i=1}^{i_0}\Phi\left(\frac{\left|z^{(k)}_i - z^{(l)}_i\right|}{d_\lambda}\right)\omega_{\pi(i)} + \sum_{i=i_0+1}^{i_l}\Phi\left(\frac{\left|z^{(k)}_i - z^{(l)}_i\right|}{d_\lambda}\right)\omega_{\pi(i)}\right. \\
&\qquad \left. + \sum_{i=i^l+1}^{\infty}\Phi\left(\frac{\left|z^{(k)}_i - z^{(l)}_i\right|}{d_\lambda}\right)\omega_{\pi(i)}\right].
\end{aligned}$$

By (1.15) and (1.16) we have

$$\begin{aligned}
I_1 &:= \sup_{\pi\in\Pi}\left[\sum_{i=1}^{i_0}\Phi\left(\frac{\left|z_i^{(k)}-z_i^{(l)}\right|}{d_\lambda}\right)\omega_{\pi(i)}\right]\\
&\le \sup_{\pi\in\Pi}\left[\sum_{i=1}^{i_0}\Phi\left(\left|z_i^{(k)}-z_i^{(l)}\right|\right)\omega_{\pi(i)}\right] \le \left\|\left(z_i^{(k)}-z_i^{(l)}\right)_{[1,i_0]}\right\|\\
&\le \left\|\left(z_i^{(k)}-x_i^{(0)}\right)_{[1,i_0]}\right\| + \left\|\left(x_i^{(0)}-z_i^{(l)}\right)_{[1,i_0]}\right\| < 2\varepsilon,\\
&\sup_{\pi\in\Pi}\left[\sum_{i=i_0+1}^{i_l}\Phi\left(\frac{\left|z_i^{(k)}\right|}{d_\lambda}\right)\omega_{\pi(i)}\right] \le \sup_{\pi\in\Pi}\left[\sum_{i=i_0+1}^{i_l}\Phi\left(\left|z_i^{(k)}\right|\right)\omega_{\pi(i)}\right]\\
&\le \left\|z^{(k)}_{(i_0,i_l]}\right\| < 2\varepsilon.
\end{aligned}$$

It follows from Lemma 1.2 that

$$\begin{aligned}
I_2 &:= \sup_{\pi\in\Pi}\sum_{i=i_0+1}^{i_l}\Phi\left(\frac{\left|z_i^{(k)}-z_i^{(l)}\right|}{d_\lambda}\right)\omega_{\pi(i)} = \rho_\Phi\left(\frac{z^{(k)}_{(i_0,i_l]}}{d_\lambda}-\frac{z^{(l)}_{(i_0,i_l]}}{d_\lambda}\right)\\
&\le \rho_\Phi\left(\frac{z^{(l)}_{(i_0,i_l]}}{d_\lambda}\right)+\varepsilon_1 \le \frac{1}{2}+\varepsilon_1.
\end{aligned}$$

Secondly, by (1.17), we have

$$\sup_{\pi\in\Pi}\left[\sum_{i=i_l+1}^{\infty}\Phi\left(\frac{\left|z_i^{(l)}\right|}{d_\lambda}\right)\omega_{\pi(i)}\right] \le \left\|z^{(l)}_{(i_0+1,\infty}\right\| < \varepsilon.$$

Therefore,

$$\begin{aligned}
I_3 &:= \sup_{\pi\in\Pi}\sum_{i=i_l+1}^{\infty}\Phi\left(\frac{\left|z_i^{(k)}-z_i^{(l)}\right|}{d_\lambda}\right)\omega_{\pi(i)} = \rho_\Phi\left(\frac{z^{(k)}_{(i_l,\infty)}}{d_\lambda}-\frac{z^{(l)}_{(i_l,\infty)}}{d_\lambda}\right)\\
&\le \rho_\Phi\left(\frac{z^{(k)}_{(i_l,\infty)}}{d_\lambda}\right)+\varepsilon_1 \le \frac{1}{2}+\varepsilon_1.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\rho_\Phi\left(\frac{z^{(k)}-z^{(l)}}{d_\lambda}\right) &\le I_1+I_2+I_3\\
&\le 2\varepsilon+1+2\varepsilon_1.
\end{aligned}$$

Observing $\Phi \in \Delta_2(0)$, we can see that

$$\left\|\frac{z^{(k)} - z^{(l)}}{d_\lambda}\right\| \leq 1 + \varepsilon_2$$

for some arbitrary $\varepsilon_2 > 0$. It follows that

$$\inf_{x \in \mathcal{X}_0} \left\|x^{(k)} - x^{(l)}\right\| \leq \inf_{z \in \mathcal{Z}} \left\|z^{(k)} - z^{(l)}\right\| \leq (1 + \varepsilon_2) d_\lambda. \tag{2.9}$$

By the arbitrariness of $\mathcal{X}_0$ we have

$$K\left(\lambda_{\Phi,\omega}\right) \leq (1 + \varepsilon_2) d_\lambda. \tag{2.10}$$

Thus, (3.8) holds in view of the arbitrariness of ε_2. ■

Corollary 2.3. *If* $\Phi \in \Delta_2(0)$, *then*

$$K\left(\lambda_{\Phi,\omega}\right) \leq \frac{1}{\tilde{\alpha}_{\Phi,\omega}} \tag{2.11}$$

where $\tilde{\alpha}_{\Phi,\omega} := \inf\limits_{u \in (0, 1/2\omega_1]} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$.

Proof. If $u \leq \frac{1}{2\omega_1} = u_0$, then $t = \Phi^{-1}(2u) \leq \Phi^{-1}\left(\frac{1}{\omega_1}\right) = t_0$ which is the upper bound of every coordinate of arbitrary $x \in \lambda_{\Phi,\omega}$. From the definition of $\tilde{\alpha}_{\Phi,\omega}$ we deduce from Lemma 1.3 that

$$\rho_\Phi\left(\tilde{\alpha}_{\Phi,\omega} x\right) \leq \frac{1}{2} \rho_\Phi(x) = \frac{1}{2} \tag{2.12}$$

which implies

$$d_\lambda \leq \frac{1}{\tilde{\alpha}_{\Phi,\omega}}, \tag{2.13}$$

whence we finishes the proof. ■

Corollary 2.4. *Let* $\Phi \in \Delta_2(0)$, $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$. *We have:*

(1) *If* $G_\Phi(u)$ *is increasing on the interval* $(0, \Phi^{-1}(\frac{1}{2\omega_1})]$, *then*

$$K\left(\lambda_{\Phi,\omega}\right) = \frac{1}{\gamma_\Phi(0)} \tag{2.14}$$

where $\gamma_\Phi(0) := \lim\limits_{u \to 0} G_\Phi(u)$.

(2) *If* $G_\Phi(u)$ *is decreasing on* $(0, \Phi^{-1}(\frac{1}{2\omega_1})]$, *then*

$$\max\left\{\frac{1}{\gamma_\Phi(0)}, \frac{\Phi^{-1}(1/\omega_1)}{\Phi^{-1}\left(1/(\omega_1+\omega_2)\right)}\right\} \le K\left(\lambda_{\Phi,\omega}\right) \le \frac{\Phi^{-1}(1/\omega_1)}{\Phi^{-1}(1/2\omega_1)}. \tag{2.15}$$

Particularly, if $\omega_1 = \omega_2$, *then* $K\left(\lambda_{\Phi,\omega}\right) = \frac{\Phi^{-1}(1/\omega_1)}{\Phi^{-1}(1/2\omega_1)}$.

Proof.

(1) Observe that $\tilde{\alpha}_{\Phi,\omega} = \alpha_\Phi(0)$ under the condition, besides, the limit $\gamma_\Phi(0)$ exists. So (2.14) holds.

(2) If $G_\Phi(u)$ is decreasing on $(0, \Phi^{-1}(\frac{1}{\omega_1})]$, then $\tilde{\alpha}_{\Phi,\omega} = \frac{\Phi^{-1}(1/\omega_1)}{\Phi^{-1}(1/2\omega_1)}$, while $\alpha'_{\Phi,\omega} \le \frac{\Phi^{-1}(1/(\omega_1+\omega_2))}{\Phi^{-1}(1/\omega_1)}$. Note that $\alpha'_{\Phi,\omega}$ need not be less than $\gamma_\Phi(0)$ if $\omega_2 < \omega_1$. ■

Corollary 2.5. *For any weight sequence* ω, *the Lorentz space* $\lambda_{p,\omega}$ $(p > 1)$ *have the constants:*

$$K\left(\lambda_{p,\omega}\right) = 2^{\frac{1}{p}}, \text{ whence } P\left(\lambda_{p,\omega}\right) = \frac{1}{1+2^{1-\frac{1}{p}}}. \tag{2.16}$$

Proof. In fact, The space $\lambda_{p,\omega}$ is generated by the $N-$function $\Phi(u) = \frac{1}{p}|u|^p$. The function $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u} = 2^{-\frac{1}{p}}$ is a constant, so $\gamma_\Phi(0) = 2^{-\frac{1}{p}}$. ■

Example 2.6. For a pair of complementary $N-$functions

$$M(u) = e^{|u|} - |u| - 1 \text{ and } N(v) = (1+|v|)\ln(1+|v|) - |v|, \tag{2.17}$$

since $F_M(t) = \frac{tM'(t)}{M(t)}$ is increasing on $(0,\infty)$, $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is also increasing on $(0,\infty)$. One can simply count out $\lim\limits_{t\to 0} F_M(t) = 2 = A_M(0) = B_M(0)$, which implies by (1.5) that $\gamma_M(0) = 2^{-\frac{1}{2}}$. Thus, by Corollary 2.4.1 we have

$$K\left(\lambda_{M,\omega}\right) = \sqrt{2} \tag{2.18}$$

and hence the packing sphere constants, according to (1.10), is

$$P\left(\lambda_{M,\omega}\right) = \frac{1}{1+\sqrt{2}}. \tag{2.19}$$

Meanwhile, $F_N(t)$ is decreasing on $(0,\infty)$ which implies $G_N(u)$ is also decreasing on $(0,\infty)$. We select a weight sequence $\omega = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. In that case,

$$\frac{N^{-1}\left(\frac{1}{\omega_1}\right)}{N^{-1}\left(\frac{1}{\omega_1+\omega_2}\right)} = \frac{N^{-1}(1)}{N^{-1}(\frac{2}{3})} \approx 1.26320,$$

while $\frac{1}{\gamma_N(0)} = \sqrt{2} \approx 1.41421$ according to (1.5). We estimate by (2.15) that

$$1.41421 \approx \frac{1}{\gamma_N(0)} \leq K\left(\lambda_{N,\omega}\right) \leq \frac{N^{-1}(1)}{N^{-1}(\frac{1}{2})} \approx 1.48699. \tag{2.20}$$

3. Estimation of $K\left(\lambda^o_{\Phi,\omega}\right)$

We now estimate the Kottman constant of Orlicz-Lorentz sequence space quipped with the Orlicz norm, namely, $K\left(\lambda^o_{\Phi,\omega}\right)$. We first give the lower bound.

Theorem 3.7. *For any N-function Φ, we have*

$$K\left(\lambda^o_{\Phi,\omega}\right) \geq \max\left\{\frac{1}{\alpha_\Phi(0)}, \frac{1}{\alpha''_{\Phi,\omega}}\right\} \tag{3.1}$$

where $\alpha_\Phi(0)$ is defined as in (1.2), Denoting $S(k) = \sum\limits_{i=1}^{k} \omega_i$, then

$$\alpha''_{\Phi,\omega} := \inf_{k\geq 1}\left\{\frac{S(k)}{S(2k)} \frac{\Psi^{-1}\left(\frac{1}{S(k)}\right)}{\Psi^{-1}\left(\frac{1}{S(2k)}\right)}\right\}. \tag{3.2}$$

Proof. For any given $k \geq 1$, let

$$Z_k = (0, 0, \cdots, 0), \qquad Y_{\Psi,k} = \frac{1}{\sum\limits_{i=1}^{k} \omega_i \Psi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{k}\omega_i}\right)}(1, 1, \cdots, 1), \tag{3.3}$$

with $\dim Z_k = \dim Y_{\Psi,k} = k$. We first construct a sequence $\mathcal{Y} = \{y^{(n)}\}_{n=1}^{\infty} \subset S\left(\lambda^o_{\Phi,\omega}\right)$, where

$$y^{(1)} = (Y_{\Psi,1}, Z_2, Z_3, 0, 0, \cdots) = \left(\frac{1}{\Psi^{-1}\left(\frac{1}{\omega_1}\right)\omega_1}, 0, 0, \cdots\right),$$

$$y^{(2)} = (Z_1, Y_{\Psi,2}, Z_3, 0, 0, \cdots)$$

$$= \left(0, \frac{1}{\Psi^{-1}\left(\frac{1}{\omega_1+\omega_2}\right)(\omega_1+\omega_2)}, \frac{1}{\Psi^{-1}\left(\frac{1}{\omega_1+\omega_2}\right)(\omega_1+\omega_2)}, 0, 0, \cdots\right),$$

$$\vdots \qquad\qquad \vdots$$

$$y^{(n)} = (Z_1, \cdots, Z_{n-1}, Y_{\Psi,n}, Z_{n+1}, 0, 0, \cdots)$$

$$= \left(\underbrace{0, \cdots, 0}_{\frac{(n-1)n}{2}}, \underbrace{\frac{1}{\Psi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega_i}\right)\left(\sum_{i=1}^{n}\omega_i\right)}, \cdots, \frac{1}{\Psi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega_i}\right)\left(\sum_{i=1}^{n}\omega_i\right)}}_{n}, 0, 0, \cdots \right),$$

$$\vdots \qquad \vdots$$

Then $\left\|y^{(n)}\right\|^o = 1$ for $n = 1, 2, \cdots$(see (1.11)).

Let $\delta > 0$ be arbitrary. By the definition of $\beta_\Psi(0)$, there is a sequence $\mathcal{V} = \{v_n\}_{n=1}^{\infty}$ with $v_n \searrow 0 (n \to \infty)$ such that $\frac{\Psi^{-1}(v_n)}{\Psi^{-1}(2v_n)} > \beta_\Psi(0) - \delta$.

Analogous to the way in Theorem 2.1. Let $\varepsilon > 0$ be arbitrary. For the sequence

$$\xi^{(1)} = (X_{\Psi,1}, Z_2, Z_3, 0, 0, \cdots) = \left(\Psi^{-1}\left(\frac{1}{\omega_1}\right), 0, 0, \cdots\right),$$

$$\xi^{(2)} = (Z_1, X_{\Psi,2}, Z_3, 0, 0, \cdots) = \left(0, \Psi^{-1}\left(\frac{1}{\omega_1+\omega_2}\right), \Psi^{-1}\left(\frac{1}{\omega_1+\omega_2}\right), 0, 0, \cdots\right),$$

$$\vdots \qquad \vdots$$

$$\xi^{(n)} = (Z_1, \cdots Z_{n-1}, X_{\Psi,n}, Z_{n+1}, 0, \cdots)$$

$$= \left(0, \cdots, 0, \Psi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega_i}\right), \Psi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega_i}\right), \cdots, \Psi^{-1}\left(\frac{1}{\sum_{i=1}^{n}\omega_i}\right), 0, 0, \cdots\right),$$

$$\vdots \qquad \vdots$$

find a sequence $\{n_k\} \subset \mathcal{N}$ satisfying $\sum_{i=1}^{n_{k-1}} \omega_i / \sum_{i=1}^{n_k} \omega_i < \varepsilon$, and some $v_{n_k} \in \mathcal{V}$ located on the interval $\left(\frac{1}{2\sum_{i=1}^{n_k+1}\omega_i}, \frac{1}{2\sum_{i=1}^{n_k}\omega_i}\right]$. Then define $\eta^{(k)} = \xi^{(n_k+1)}$.

For any pair of k, l $(k > l)$, we have

$$\rho_\Psi\left((\beta_\Psi(0) - \delta)\left(\eta^{(k)} - \eta^{(l)}\right)\right)$$

$$= \sum_{i=1}^{n_l+1} \Psi\left[(\beta_\Psi(0) - \delta) \cdot \Psi^{-1}\left(\frac{1}{\sum_{i=1}^{n_l+1}\omega_i}\right)\right]\omega_i$$

$$
\begin{aligned}
&\quad + \sum_{i=n_l+2}^{n_l+n_k+2} \Phi \left[(\beta_\Psi(0)-\delta) \cdot \Psi^{-1} \left(\frac{1}{\sum\limits_{i=1}^{n_k+1} \omega_i} \right) \right] \omega_i \\
&\leq \sum_{i=1}^{n_l+1} \Psi \left[\frac{\Psi^{-1}(v_{n_l})}{\Psi^{-1}(2v_{n_l})} \cdot \Psi^{-1} \left(\frac{1}{\sum\limits_{i=1}^{n_l+1} \omega_i} \right) \right] \omega_i \\
&\quad + \sum_{i=n_l+2}^{n_l+n_k+2} \Psi \left[\frac{\Psi^{-1}(v_{n_k})}{\Psi^{-1}(2v_{n_k})} \cdot \Psi^{-1} \left(\frac{1}{\sum\limits_{i=1}^{n_k+1} \omega_i} \right) \right] \omega_i \\
&< \sum_{i=1}^{n_l+1} \Psi \left[\frac{\Psi^{-1}(v_{n_l})}{\Psi^{-1}(2v_{n_l})} \cdot \Psi^{-1}(2v_{n_l}) \right] \omega_i + \sum_{i=n_l+2}^{n_l+n_k+2} \Psi \left[\frac{\Psi^{-1}(v_{n_k})}{\Psi^{-1}(2v_{n_k})} \cdot \Psi^{-1}(2v_{n_k}) \right] \omega_i \\
&= \sum_{i=1}^{n_l+1} \Psi \left(\Psi^{-1}(v_{n_l}) \right) \omega_i + \sum_{i=n_l+2}^{n_l+n_k+2} \Psi \left(\Psi^{-1}(v_{n_k}) \right) \omega_i \\
&= \sum_{i=1}^{n_l} v_{n_l} \omega_i + \sum_{i=n_l+1}^{n_l+n_k} v_{n_k} \omega_i \leq \sum_{i=1}^{n_l+1} \frac{1}{2 \sum\limits_{i=1}^{n_l} \omega_i} \cdot \omega_i + \sum_{i=n_l+2}^{n_l+n_k+2} \frac{1}{2 \sum\limits_{i=1}^{n_k} \omega_i} \cdot \omega_i \\
&= \frac{1}{2} \left[\frac{1}{\sum\limits_{i=1}^{n_l} \omega_i} \sum_{i=1}^{n_l+1} \omega_i + \frac{1}{\sum\limits_{i=1}^{n_k} \omega_i} \sum_{i=n_l+2}^{n_l+n_k+2} \omega_i \right] \\
&< \frac{1}{2(1-\varepsilon)} \left[1 + \frac{\sum\limits_{i=1}^{n_k+1} \omega_i + \sum\limits_{i=n_k+2}^{n_l+n_k+2} \omega_i - \sum\limits_{i=1}^{n_l+1} \omega_i}{\sum\limits_{i=1}^{n_k+1} \omega_i} \right] \leq \frac{1}{1-\varepsilon}.
\end{aligned}
$$

It follows that

$$\rho_\Psi \left((1-\varepsilon)(\beta_\Psi(0)-\delta) \left(\eta^{(k)} - \eta^{(l)} \right) \right) \leq (1-\varepsilon) \rho_\Psi \left((\beta_\Psi(0)-\delta) \left(\eta^{(k)} - \eta^{(l)} \right) \right) < 1.$$

Now we select a subsequence $\mathcal{Z} = \{z^{(k)}\}_{k=1}^{\infty} \subset \mathcal{Y}$ such that $z^{(k)} = y^{(n_k+1)}$ where $\{n_k\}$ is defined as the above succession. From the definition of the Orlicz norm, we have for $l < k$ that

$$\left\| z^{(k)} - z^{(l)} \right\|^o = \sup \left\{ \sum_{i=1}^{\infty} \left(z^{(k)} - z^{(l)} \right)_i^* \cdot \zeta_i^* \cdot \omega_i : \rho_\Psi(\zeta) \leq 1 \right\}$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{\infty}\left(z^{(k)}-z^{(l)}\right)_i^* \cdot\left((1-\varepsilon)(\beta_\Psi(0)-\delta)\left(\eta^{(k)}-\eta^{(l)}\right)\right)_i^* \cdot \omega_i \\
&= (1-\varepsilon)(\beta_\Psi(0)-\delta)\sum_{i=1}^{\infty}\left(z^{(k)}-z^{(l)}\right)_i^* \cdot\left(\eta^{(k)}-\eta^{(l)}\right)_i^* \cdot \omega_i \\
&= (1-\varepsilon)(\beta_\Psi(0)-\delta)\left[\sum_{i=1}^{n_l+1}\frac{1}{\sum\limits_{i=1}^{n_l+1}\omega_i}\cdot\omega_i+\sum_{i=n_l+2}^{n_l+n_k+2}\frac{1}{\sum\limits_{i=1}^{n_k+1}\omega_i}\cdot\omega_i\right] \\
&= (1-\varepsilon)(\beta_\Psi(0)-\delta)\left[\frac{1}{\sum\limits_{i=1}^{n_l+1}\omega_i}\sum_{i=1}^{n_l+1}\omega_i+\frac{1}{\sum\limits_{i=1}^{n_k+1}\omega_i}\sum_{i=n_l+2}^{n_l+n_k+2}\omega_i\right] \\
&= (1-\varepsilon)(\beta_\Psi(0)-\delta)\left[1+\frac{\sum\limits_{i=1}^{n_k+1}\omega_i+\sum\limits_{i=n_k+2}^{n_l+n_k+2}\omega_i-\sum\limits_{i=1}^{n_l+1}\omega_i}{\sum\limits_{i=1}^{n_k+1}\omega_i}\right] \\
&> (1-\varepsilon)(\beta_\Psi(0)-\delta)\,(2-\varepsilon).
\end{aligned}
$$

By the arbitrariness of δ and ε we obtain

$$K\left(\lambda^o_{\Phi,\omega}\right) \geq \sup_{\mathcal{Z}_{\varepsilon,\delta}} \inf_{k\neq l}\left\|z^{(k)}-z^{(l)}\right\|^o \geq 2\beta_\Psi(0). \tag{3.4}$$

where $\mathcal{Z}_{\varepsilon,\delta}$ denotes the sequence $\left\{z^{(n)}\right\}_1^\infty$ depending on ε, δ. In view of the first author's equality (1.4) we obtain

$$K\left(\lambda^o_{\Phi,\omega}\right) \geq \frac{1}{\alpha_\Phi(0)}. \tag{3.5}$$

Finally, for any $k \geq 1$, define

$$x^{(n)} = (Z_k, Z_k, \cdots, Z_k, Y_{\Psi,k}, Z_k, Z_k, \cdots) \quad (n \geq 1)$$

where Z_k and $Y_{\Psi,k}$ are defined in (3.3), $Y_{\Psi,k}$ being at the nth position. Then $\left\|x^{(n)}\right\|^o = 1 (n \geq 1)$ and for $n \neq m$,

$$\left\|x^{(n)}-x^{(m)}\right\|^o = \frac{1}{\sum\limits_{i=1}^{k}\omega_i\Psi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{k}\omega_i}\right)}\cdot\left[\sum_{i=1}^{2k}\omega_i\Psi^{-1}\left(\frac{1}{\sum\limits_{i=1}^{2k}\omega_i}\right)\right]$$

$$= \frac{S(2k)}{S(k)} \frac{\Psi^{-1}\left(\frac{1}{S(2k)}\right)}{\Phi^{-1}\left(\frac{1}{S(k)}\right)}.$$

It follows that

$$K(\lambda_{\Phi,\omega}) \geq \sup_{k\geq 1} \inf_{n\neq m} \left\| x^{(n)} - x^{(m)} \right\|^{o} = \sup_{k\geq 1} \frac{S(2k)}{S(k)} \frac{\Psi^{-1}\left(\frac{1}{S(2k)}\right)}{\Psi^{-1}\left(\frac{1}{S(k)}\right)} = \frac{1}{\alpha''_{\Phi,\omega}}$$

and we finishes the proof. ∎

To discuss the upper bound, we revise Wang's work (see [14]) to the space $\lambda^{o}_{\Phi,\omega}$. For any fixed $x \in \lambda^{o}_{\Phi,\omega}$ with $\|x\|^{o} = 1$ and any given $k > 1$, the function of d, $\rho_\Phi\left(\frac{kx}{d}\right)$, is continuous and strictly decreasing on $[1,\infty)$ if $\rho_\Phi(kx) < \infty$. The fact that $\rho_\Phi(kx) \geq \|kx\|^{o} - 1 = k - 1$ and $\lim_{d\to\infty} \rho_\Phi\left(\frac{kx}{d}\right) = 0$ implies that there exists unique $d_{x,k} > 1$ such that $\rho_\Phi\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}$. Denote $d_x = \inf_{k>1} d_{x,k}$. Since there exists $k' > 1$ such that $1 = \|x\|^{o} = \frac{1}{k'}(1 + \rho_\Phi(k'x))$, i.e., $\rho_\Phi(k'x) = k' - 1$ and hence $\rho_\Phi\left(\frac{k'x}{2}\right) < \frac{k'-1}{2}$, we have $d_{x,k'} < 2$ which implies $d_x < 2$. If $\rho_\Phi(kx) = \infty$ and $d_{x,k}$ exists, then $d_{x,k} > 1$ since $\rho_\Phi(kx) \geq \|kx\|^{o} - 1 = k - 1$ always holds. For the final case that $\rho_\Phi(kx) = \infty$ and $d_{x,k}$ does not exist, we leave $d_{x,k}$ a vacancy. Define

$$d^{o}_{\lambda} = \sup_{\|x\|^{o}=1} d_x = \sup_{\|x\|^{o}=1} \inf_{k>1} \left\{ d_{x,k} : \rho_\Phi\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2} \right\}. \tag{3.6}$$

Obviously $1 \leq d^{o}_{\lambda} \leq 2$. It should be observed that the definition of d^{o}_{λ} appears the same as Wang's [14], however, here it is independent of the Δ_2-condition.

Theorem 3.8. *If $\Phi \in \Delta_2(0)$, we have*

$$K\left(\lambda_{\Phi,\omega}\right) \leq d^{o}_{\lambda}. \tag{3.7}$$

Proof. Let $\mathcal{X}_0 = \{x^{(n)}\}_{n=1}^{\infty} \subset S\left(\lambda^{o}_{\Phi,\omega}\right)$ be arbitrary. For every $\varepsilon > 0$, $\inf_{k>1} d_{x^{(n)},k} = d_{x^{(n)}} \leq d^{o}_{\lambda}$ $(n = 1, 2, \cdots)$, there is $k_{(x^{(n)})} > 1$ such that $d_{x^{(n)},k_{(x^{(n)})}} < d^{o}_{\lambda} + \varepsilon$ $(n = 1, 2, \cdots)$.

We first assume that the set $\left\{k_{(x^{(n)})}\right\}_{n=1}^{\infty}$ is unbounded. Select a subsequence $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ such that $k_n := k_{(y^{(n)})} \to \infty$ $(n \to \infty)$. We show $d^{o}_{\lambda} + \varepsilon \geq 2$. Suppose the contrary, i.e., $d^{o}_{\lambda} + \varepsilon < 2$, then

$$\begin{aligned} \frac{k_n - 1}{2} &= \rho_\Phi\left(\frac{k_n y^{(n)}}{d_{y^{(n)},k_n}}\right) > \rho_\Phi\left(\frac{k_n y^{(n)}}{d^{o}_{\lambda} + \varepsilon}\right) > \frac{2}{d^{o}_{\lambda} + \varepsilon} \rho_\Phi\left(\frac{k_n y^{(n)}}{2}\right) \\ &\geq \frac{2}{d^{o}_{\lambda} + \varepsilon}\left(\left\|\frac{k_n y^{(n)}}{2}\right\|^{o} - 1\right) = \frac{2}{d^{o}_{\lambda} + \varepsilon}\left(\frac{k_n}{2} - 1\right). \end{aligned}$$

It follows that

$$1 > \frac{2}{d_\lambda^o + \varepsilon}\frac{k_n - 2}{k_n - 1} \to \frac{2}{d_\lambda^o + \varepsilon} > 1 (n \to \infty),$$

a contradiction. Thus,

$$\inf_{n \neq m}\left\{\left\|x^{(n)} - x^{(m)}\right\|^o\right\} \leq 2 \leq d_\lambda^o + \varepsilon. \tag{3.8}$$

We then assume that $\left\{k_{(x^{(n)})}\right\}_{n=1}^\infty$ is bounded. There exists a subsequence, still denoted by $\left\{k_{(x^{(n)})}\right\}_{n=1}^\infty$, such that $h_n := \left\{k_{(x^{(n)})}\right\}_{n=1}^\infty \to h_0 \geq 1\ (n \to \infty)$. For every $\varepsilon > 0$, By Lemma 1.4, there is a subsequence $\mathcal{Z} := \left\{z^{(k)}\right\}_{k=1}^\infty \subset \mathcal{X}_0$ and a sequence of natural numbers $i_0 < i_1 < i_2 < \cdots < i_l < \cdots < i_k < \cdots$ satisfying the conditions (1.15)-(1.17) in the Orlicz norm. Note that for a pair of sufficiently large number $k, l (k > l)$, one has

$$|h_k - h_0| < \varepsilon, \qquad |h_l - h_0| < \varepsilon. \tag{3.9}$$

$\Phi(x_i)\omega_1 \leq \rho_\Phi(x) \leq \|x\|^o = 1$ implies that $x_i \leq \Phi^{-1}\left(\frac{1}{\omega_1}\right) := u_0\ (i \geq 1)$ for any $x \in S\left(\lambda_{\Phi,\omega}^o\right)$. Considering $\Phi \in \Delta_2(0)$, there exists $K > 1$ such that

$$\Phi(2h_0 u) \leq K\Phi(u) \tag{3.10}$$

for $u \in (0, u_0]$. Let us observe the inequality

$$\begin{aligned}
\left\|\frac{z^{(k)} - z^{(l)}}{d_\lambda^o + \varepsilon}\right\|^o &\leq \frac{1}{h_0}\left[1 + \rho_\Phi\left(\frac{h_0\left(z^{(k)} - z^{(l)}\right)}{d_\lambda^o + \varepsilon}\right)\right] \\
&= \frac{1}{h_0}\left\{1 + \sup_{\pi \in \Pi}\left[\sum_{i=1}^\infty \Phi\left(\frac{h_0\left|z_{\pi(i)}^{(k)} - z_{\pi(i)}^{(l)}\right|}{d_\lambda^o + \varepsilon}\right)\omega_i\right]\right\} \\
&= \frac{1}{h_0}\left\{1 + \sup_{\pi \in \Pi}\left[\sum_{i=1}^\infty \Phi\left(\frac{h_0\left|z_i^{(k)} - z_i^{(l)}\right|}{d_\lambda^o + \varepsilon}\right)\omega_{\pi(i)}\right]\right\}.
\end{aligned}$$

where

$$\begin{aligned}
&\sup_{\pi \in \Pi}\left[\sum_{i=1}^\infty \Phi\left(\frac{h_0\left|z_i^{(k)} - z_i^{(l)}\right|}{d_\lambda^o + \varepsilon}\right)\omega_{\pi(i)}\right] \\
&= \sup_{\pi \in \Pi}\left[\sum_{i=1}^{i_0} \Phi\left(\frac{h_0\left|z_i^{(k)} - z_i^{(l)}\right|}{d_\lambda^o + \varepsilon}\right)\omega_{\pi(i)} + \sum_{i=i_0+1}^{i_l} \Phi\left(h_0\frac{\left|z_i^{(k)} - z_i^{(l)}\right|}{d_\lambda^o + \varepsilon}\right)\omega_{\pi(i)}\right.
\end{aligned}$$

$$+ \sum_{i=i^l+1}^{\infty} \Phi \left(\frac{h_0 \left| z_i^{(k)} - z_i^{(l)} \right|}{d_\lambda^o + \varepsilon} \right) \omega_{\pi(i)} \Bigg].$$

By (3.10) and (1.15) we have

$$\begin{aligned} J_1 :&= \sup_{\pi \in \Pi} \left[\sum_{i=1}^{i_0} \Phi \left(\frac{h_0 \left| z_i^{(k)} - z_i^{(l)} \right|}{d_\lambda^o + \varepsilon} \right) \omega_{\pi(i)} \right] \\ &\leq \sup_{\pi \in \Pi} \left[\sum_{i=1}^{i_0} K \Phi \left(\left| z_i^{(k)} - z_i^{(l)} \right| \right) \omega_{\pi(i)} \right] \leq K \left\| \left(z_i^{(k)} - z_i^{(l)} \right)_{[1,i_0]} \right\|^o < 2K\varepsilon. \end{aligned}$$

Secondly, by (1.16) and (3.9) we have

$$\begin{aligned} &\sup_{\pi \in \Pi} \left[\sum_{i=i_0+1}^{i_l} \Phi \left(\frac{h_0 \left| z_i^{(k)} \right|}{d_\lambda^o + \varepsilon} \right) \omega_{\pi(i)} \right] \leq \sup_{\pi \in \Pi} \left[\sum_{i=i_0+1}^{i_l} K \Phi \left(\left| z_i^{(k)} \right| \right) \omega_{\pi(i)} \right] \\ &\leq K \left\| z_{(i_0,i_l]}^{(k)} \right\|^o < 2K\varepsilon, \\ &\sup_{\pi \in \Pi} \left[\sum_{i=i_0+1}^{i_l} \Phi \left(\frac{\left| (h_l - h_0) z_i^{(l)} \right|}{d_\lambda^o + \varepsilon} \right) \omega_{\pi(i)} \right] \leq \sup_{\pi \in \Pi} \left[\sum_{i=i_0+1}^{i_l} \varepsilon \Phi \left(\left| z_i^{(l)} \right| \right) \omega_{\pi(i)} \right] \leq \varepsilon. \end{aligned}$$

It follows that (by means of Lemma 1.2 twice),

$$\begin{aligned} J_2 :&= \sup_{\pi \in \Pi} \sum_{i=i_0+1}^{i_l} \Phi \left(\frac{h_0 \left| z_i^{(k)} - z_i^{(l)} \right|}{d_\lambda^o + \varepsilon} \right) \omega_{\pi(i)} \\ &\leq \sup_{\pi \in \Pi} \sum_{i=i_0+1}^{i_l} \Phi \left(\frac{h_l \left| z_i^{(l)} \right| + \left| (h_l - h_0) z_i^{(l)} \right| + h_0 \left| z_i^{(k)} \right|}{d_\lambda^o + \varepsilon} \right) \omega_{\pi(i)} \\ &\leq \rho_\Phi \left(\frac{h_l z^{(l)}}{d_\lambda^o + \varepsilon} \right) + 2\varepsilon_1 = \frac{h_l - 1}{2} + 2\varepsilon_1 \\ &= \frac{h_0 - 1}{2} + \frac{h_l - h_0}{2} + 2\varepsilon_1 < \frac{h_0 - 1}{2} + \frac{\varepsilon}{2} + 2\varepsilon_1. \end{aligned}$$

Thirdly, by (1.17), we have

$$\sup_{\pi \in \Pi} \left[\sum_{i=i_l+1}^{\infty} \Phi \left(\frac{h_0 \left| z_i^{(l)} \right|}{d_\lambda^o + \varepsilon} \right) \omega_{\pi(i)} \right] \leq K \left\| z_{(i_0+1,\infty)}^{(l)} \right\|^o < K\varepsilon.$$

It follows also from Lemma 1.2 that

$$J_3 := \sup_{\pi\in\Pi}\sum_{i=i_l+1}^{\infty}\Phi\left(\frac{h_0\left|z_i^{(k)}-z_i^{(l)}\right|}{d_\lambda^o+\varepsilon}\right)\omega_{\pi(i)} = \rho_\Phi\left(\frac{h_0 z^{(k)}_{(i_l,\infty)}}{d_\lambda^o+\varepsilon}-\frac{h_0 z^{(l)}_{(i_l,\infty)}}{d_\lambda^o+\varepsilon}\right)$$

$$\leq \rho_\Phi\left(\frac{h_0 z^{(k)}_{(i_l,\infty)}}{d_\lambda^o+\varepsilon}\right)+\varepsilon_1 \leq \frac{h_0-1}{2}+\varepsilon_1.$$

Consequently, we have

$$\left\|\frac{z^{(k)}-z^{(l)}}{d_\lambda^o+\varepsilon}\right\|^o \leq \frac{1}{h_0}\left[1+J_1+J_2+J_3\right]$$

$$\leq \frac{1}{h_0}\left[1+2K\varepsilon+\frac{h_0-1}{2}+\frac{\varepsilon}{2}+2\varepsilon_1+\frac{h_0-1}{2}+\varepsilon_1\right]$$

$$= 1+\varepsilon_2$$

where $\varepsilon_2 \to 0$ as $\varepsilon \to 0$. Thus, we deduce that

$$\left\|z^{(k)}-z^{(l)}\right\|^o \leq \left(d_\lambda^o+\varepsilon\right)(1+\varepsilon_2). \tag{3.11}$$

By the arbitrariness of $\mathcal{X}_0$ and ε we conclude that

$$K\left(\lambda^o_{\Phi,\omega}\right) \leq d_\lambda^o. \tag{3.12}$$

The proof is finished. ■

Corollary 3.9. *Let* $\Phi \in \Delta_2(0)$. *Then*

$$K(l^o_{\Phi,\lambda}) \leq \frac{1}{\alpha^*_\Phi}, \tag{3.13}$$

where

$$\alpha^*_\Phi = \inf\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u \leq \frac{1}{2\omega_1}(b^*_\Psi-1)\right\}, \tag{3.14}$$

with

$$b^*_\Psi = \sup\left\{\frac{s\psi(s)}{\Psi(s)} : 0 < s \leq \Psi^{-1}\left(\frac{1}{\omega_1}\right)\right\}. \tag{3.15}$$

Proof. Define $Q_\Phi = \sup\limits_{\|x\|^o=1}\{k_x > 1 : k_x \in K(x)\}$ and

$$\widetilde{\alpha}^*_\Phi = \inf\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u \leq \frac{1}{2\omega_1}(Q_\Phi-1)\right\}. \tag{3.16}$$

We first assume that $Q_\Phi < \infty$.

Let $u_i = \frac{1}{2}\Phi(k_x|x_i|)$ for all $x_i \neq 0$. Then $\Phi(k_x|x_i|)\omega_1 \le \rho_\Phi(k_x x) = k_x - 1 \le Q_\Phi - 1$. It follows from Lemma 1.3 that

$$\begin{aligned}\rho_\Phi\left(\widetilde{\alpha}_\Phi^* k_x x\right) &= \sup_{\pi\in\Pi}\sum_{i=1}^\infty \Phi\left(\widetilde{\alpha}_\Phi^* k_x|x_i|\right)\omega_{\pi(i)}\\ &\le \sup_{\pi\in\Pi}\sum_{i=1}^\infty \Phi\left[G_\Phi\left(\frac{1}{2}\Phi(k_x|x_i|)\right)k_x|x_i|\right]\omega_{\pi(i)}\\ &= \sup_{\pi\in\Pi}\frac{1}{2}\sum_{i=1}^\infty \Phi\left(k_x|x_i|\right)\omega_{\pi(i)} = \frac{1}{2}\rho_\Phi(k_x x) = \frac{1}{2}(k_x - 1).\end{aligned}$$

Therefore, we deduce from Theorem 3.2 that

$$K(l_{\Phi,\lambda}^o) \le d_\lambda^o \le \sup_{\|x\|^o=1}\inf_{k_x}\left\{d_{x,k_x} : \rho_\Phi\left(\frac{k_x x}{d_{x,k_x}}\right) = \frac{k_x-1}{2}\right\} \le \frac{1}{\widetilde{\alpha}_\Phi^*}. \tag{3.17}$$

The estimation (3.17) is not practical for usage unless we estimate the upper bound of Q_Φ. We now show

$$Q_\Phi \le b_\Psi^* \tag{3.18}$$

so that (3.13) holds.

For a sufficiently small $\varepsilon > 0$, by the definition in Lemma 1.1.3,

$$\Psi\left(\phi((k_x^{**}-\varepsilon)|x(i)|)\right)\omega_1 \le \sum_{i=1}^\infty \Psi\left[\phi((k_x^{**}-\varepsilon)|x(i)|)\right]^*\omega_i \le 1 \quad (i\ge 1).$$

So $(k_x^{**}-\varepsilon)|x(i)| \le \psi\left(\Psi(\frac{1}{\omega_1})\right)$ for any $x = \{x_i\}_{i=1}^\infty \in S\left(\lambda_{\Phi,\omega}^o\right)$. Define

$$a_\Phi^* = \inf\left\{\frac{t\phi(t)}{\Phi(t)} : 0 < t \le \psi\left[\Psi^{-1}\left(\frac{1}{\omega_1}\right)\right]\right\}, \tag{3.19}$$

then (see [17])

$$\frac{1}{a_\Phi^*} + \frac{1}{b_\Psi^*} = 1. \tag{3.20}$$

Moreover, by the definition of a_Φ^*,

$$a_\Phi^*\Phi\left[(k_x^{**}-\varepsilon)|x(i)|\right] \le (k_x^{**}-\varepsilon)|x(i)|\phi\left[(k_x^{**}-\varepsilon)|x(i)|\right], \quad (i\ge 1).$$

It follows that

$$1 \geq \rho_\Psi\left(\phi\left[(k_x^{**}-\varepsilon)x\right]\right) = \sup_{\pi\in\Pi}\sum_{i=1}^{\infty}\Psi\left\{\phi\left[(k_x^{**}-\varepsilon)|x(i)|\right]\right\}\omega_{\pi(i)}$$
$$= \sup_{\pi\in\Pi}\sum_{i=1}^{\infty}\left\{(k_x^{**}-\varepsilon)|x(i)|\phi\left[(k_x^{**}-\varepsilon)|x(i)|\right] - \Phi\left[(k_x^{**}-\varepsilon)|x(i)|\right]\right\}\omega_{\pi(i)}$$
$$\geq (a_\Phi^*-1)\sup_{\pi\in\Pi}\sum_{i=1}^{\infty}\Phi\left[(k_x^{**}-\varepsilon)|x(i)|\right]\omega_{\pi(i)}$$
$$\geq (a_\Phi^*-1)\left[\|(k_x^{**}-\varepsilon)x\|^o - 1\right] = (a_\Phi^*-1)\left[(k_x^{**}-\varepsilon)-1\right],$$

i.e.,

$$k_x^{**} \leq \frac{a_\Phi^*}{a_\Phi^*-1} + \varepsilon = b_\Psi^* + \varepsilon$$

which proves (3.18) since ε is arbitrary.

If $Q_\Phi = \infty$, then $\Phi \notin \nabla_2(0)$ (see Theorem 1.35(2) in [2]), which implies $\alpha_\Phi^0 = \frac{1}{2}$ and $b_\Psi^* = \infty$. Therefore, $\alpha_\Phi^* = \widetilde{\alpha}_\Phi^* = \alpha_\Phi^0 = \frac{1}{2}$ since $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} \geq \frac{1}{2}$. It follows that

$$K(l_{\Phi,\lambda}^o) \leq 2 = \frac{1}{\widetilde{\alpha}_\Phi^*} = \frac{1}{\alpha_\Phi^*}.$$

The proof is completed. ∎

Corollary 3.10. *Let* $\Phi \in \Delta_2(0)$, $u_0 = \frac{1}{2\omega_1}(b_\Psi^*-1)$, *then*

(1) *If* $G_\Phi(t)$ *is increasing on* $(0, u_0]$, *then*

$$K\left(\lambda_{\Phi,\omega}^o\right) = \frac{1}{\gamma_\Phi(0)} \tag{3.21}$$

where $\gamma_\Phi(0) := \lim_{u\to 0} G_\Phi(u)$.

(2) *If* $G_\Phi(u)$ *is decreasing on* $(0, u_0]$, *then*

$$\max\left\{\frac{1}{\gamma_\Phi(0)}, \frac{\omega_1+\omega_2}{\omega_1}\frac{\Psi^{-1}\left(\frac{1}{\omega_1+\omega_2}\right)}{\Psi^{-1}\left(\frac{1}{\omega_1}\right)}\right\} \leq K\left(\lambda_{\Phi,\omega}^o\right) \leq \frac{\Phi^{-1}(2u_0)}{\Phi^{-1}(u_0)}. \tag{3.22}$$

Proof. It directly results from (3.1)(3.2) and (3.13). ∎

Example 3.11. For the pair of complementary $N-$functions

$$M(u) = e^{|u|} - |u| - 1 \text{ and } N(v) = (1+|v|)\ln(1+|v|) - |v|$$

as described in Example 2.6, we have shown that $F_M(t)$ is increasing on $(0, +\infty)$ while $F_N(t)$ is decreasing on $(0, +\infty)$, and that $\gamma_M(0) = \gamma_N(0) = 2^{-\frac{1}{2}}$. Furthermore, for the weight sequence $\omega = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, it is not difficult to count out that

$$\frac{\omega_1+\omega_2}{\omega_1}\frac{M^{-1}\left(\frac{1}{\omega_1+\omega_2}\right)}{M^{-1}\left(\frac{1}{\omega_1}\right)}=\frac{3M^{-1}(\frac{2}{3})}{2M^{-1}(1)}\approx 1.26840<\sqrt{2}=\frac{1}{\gamma_N(0)},$$

and $u_0 \approx 0.64619$ (see Corollary 3.4). Therefore, we obtain:

$$K\left(\lambda^o_{M,\omega}\right)=\frac{1}{\gamma_M(0)}=\sqrt{2}; \tag{3.23}$$

$$1.41421\approx\frac{1}{\gamma_N(0)}\leq K\left(\lambda^o_{N,\omega}\right)\leq\frac{N^{-1}(2\times 0.64619)}{N^{-1}(0.64619)}\approx 1.49434. \tag{3.24}$$

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