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DIFFERENTIATION ON VILENKIN GROUPS USING A MATRIX

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Abstract. Given a Vilenkin group G, a scalar matrix $\Lambda = [\lambda_{ij}]_{i\in\mathbb{N},j\in\mathbb{N}_0}$,a function $f \in L^1(G)$, and a point $x \in G$ we introduce, for each $\alpha \in \mathbb{R}$, the $(\Lambda, \alpha)-$ derivative f at x denoted by $f^{(\Lambda,\alpha)}(x)$. We also introduce the sets:

$$M_\alpha = M(G, \Lambda, \alpha, x) := \left\{ f \in L^1(G) : \exists f^{(\Lambda,\alpha)}(x) \right\},$$
$$M = M(G, \Lambda, x) := \left\{ f \in L^1(G) : \exists f^{\Lambda}(x) \right\};$$

where $f^\Lambda(x)$ derivative in [8], which is a generalization of Onneweer's derivative $f^{[1]}(x)$ in [6]. We proved:

(a) Five theorems which express essential characteristics of $(\Lambda, \alpha)-$ derivative,
(b) $M = M_0$,
(c) $(\forall\alpha, \beta \in \mathbb{R}) \wedge (\alpha < \beta) \Rightarrow (M_\alpha \subseteq M_\beta) \wedge (M_\beta \diagdown M_\alpha \neq \theta)$.

Statement $b)$ states that the method $(\Lambda, \alpha)-$ differentiation, for $\alpha = 0$, is equal to $\Lambda-$ differentiation and statement $c)$ says that $(\Lambda, \alpha)-$ differentiation increases with increasing $\alpha \in \mathbb{R}$.

1. Introduction and Preliminaries

By a Vilenkin group G we mean an infinite, totally unconnected, compact Abelian group which satisfies the second axiom of countability. Vilenkin $[10]$ has shown that the topology in G can be given by basic chain of neighborhoods of zero

(1) $$G = G_0 \supset G_1 \supset \cdots \supset G_n \supset \ldots, \cap_{n=0}^{\infty} G_n = \{0\}$$

consisting of open subgroups of the group G, such that quotient group $G_n \diagup G_{n+1}$ is cyclic group of prime order $p_{n+1}, \forall n \in \mathbb{N}_0$. G is called **bounded** iff a sequence

$$(p_n)_{n\in\mathbb{N}} = (p_1, p_2, \ldots),$$

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is bounded.

Classical example of Vilenkin group is product space

$$\prod_{k=0}^{\infty} G_k,$$

where $G_k = \{0,1\}$ is a cyclic group of the second order for all $k \in \mathbb{N}_0$, equipped by discrete topology, with the component adding (note that adding in each component is done by module 2). It's direct generalization is group

$$G = \prod_{k=0}^{\infty} \mathbb{Z}(n_k),$$

where $\mathbb{Z}(n_k) := \{0,1,2,\ldots,n_k-1\}$, $n_k \geq 2$, is cyclic group of order n_k $(k \in \mathbb{N}_0)$ equipped by discrete topology.

It is possible to supply G with a normalized Haar measure μ such that $\mu(G_n) = m_n^{-1}$, where $m_n := p_1 p_2 \ldots p_n$ $(m_0 := 1)$. For every $1 \leq p < \infty$ let $L^p(G)$ denote the L^p space on G with respect to the measure μ. The class of all continuous complex functions on G will be denoted by $C(G)$. If $1 \leq p_1 < p_2 < \infty$, then $L^{p_2}(G) \subset L^{p_1}(G)$. Let Γ denote the (multiplicative) group of characters of the group G, and let $\Gamma_n = G_n^{\perp}$ denote the annihilator of G_n in Γ. The dual group $(\Gamma, \cdot)$ is a discrete countable Abelian group with torzion $[5, (24.15)$ and $(24.26)]$. Vilenkin [10] has proved that there exists a $Paley-tupe$ ordering of the elements in Γ : let us chose a $\chi \in \Gamma_{k+1} \diagdown \Gamma_k$ and denote by χ_{m_k}. Every $n \in \mathbb{N}$ has a unique representation as

$$n = \sum_{i=0}^{N} a_i m_i, \; a_i \in \{0,1,2,\ldots,p_{i+1}-1\} \wedge a_N \neq 0 \wedge N = N(n). \tag{2}$$

Therefore, $m_N \leq n < m_{N+1}$ and $n \to \infty \Leftrightarrow N \to \infty$. Let χ_n character defined by

$$\chi_n = \prod_{i=0}^{N} \chi_{m_i}^{a_i} = \prod_{i=0}^{N} r_i^{a_i}, \; r_i := \chi_{m_i} (\forall i \in \mathbb{N}_0) \tag{3}$$

It is straightforward that

$$(\forall n \in \mathbb{N}_0) \Gamma_n = \{\chi_j : 0 \leq j < m_n\} \tag{4}$$

The sequence $(\chi_n)_{n \in \mathbb{N}_0}$ is a called a **Vilenkin system**. For every $n \in \mathbb{N}_0$ there exists $x_n \in G_n \diagdown G_{n+1}$ such that $r_n(x_n) = e^{\frac{2\pi}{p_{n+1}} i}$. Every $x \in G$ can be represented in unique way as

$$x = \sum_{n=0}^{\infty} a_n x_n, \;\; a_n \in \{0,1,2,\ldots,p_{n+1}-1\} \tag{5}$$

Then

$$G_n = \left\{ x \in G : \sum_{i=0}^{\infty} a_i x_i,\ a_i = 0,\ 0 \leq i < n \right\} \tag{6}$$

A Vilenkin series $\sum\limits_{n=0}^{\infty} c_n \chi_n$ is a Fourier series iff there is a function $f \in L^1(G)$ such that

$$c_n = \hat{f}(\chi_n) = \hat{f}(n) := \int\limits_G f\overline{\chi_n},\ \forall n \in \mathbb{N}_0 \tag{7}$$

where $\overline{z}$ denotes the complex-conjugate of z. In that case, the $n-th$ partial sum of the series is given by

$$S_n(f) = \sum_{k=0}^{n-1} \hat{f}(k)\chi_k = f * D_n \tag{8}$$

where D_n defined by

$$D_n := \sum_{k=0}^{n-1} \chi_k \tag{9}$$

is the **Dirichlet kernel** of index n on G and

$$f * \varphi(x) := \int_G f(x-h)\varphi(h)d\mu(h) \tag{10}$$

is **convolution** of function f and φ on G.

Spelling J. E. Gibs [3] and [4] first introduced the **duadic derivative** " [1] " with the following property

$$[\omega_k(x)]^{[1]} = k \cdot \omega_k(x),$$

where ω_k is Walsh (J. L. Walsh) function of index k. This derivative was further studied by P. L. Butzer and H. J. Wagner [2], and also F. Schipp [9] who proved that $k \cdot a_k \to 0$ yields

$$\left[\sum_{k=0}^{\infty} a_k \omega_k(x)\right]^{[1]} = \sum_{k=0}^{\infty} k a_k \omega_k(x).$$

V. A. Skvortsov and W. R. Wade have proved the analogue result for the series over arbitrary system of characters from 0-dimensional groups under more general assumptions and have simplified the proof. J. Pal and P. Simon [7] have defined the derivative of a function defined on an arbitrary 0-dimensional compact commutative group. C. V. Onneweer [6] has studied differentiation of functions (with complex values) defined on dyadic group **D**. In [6] he has given three definitions of dyadic differentiation where the Leibniz differentiation formula does not hold. His main idea was that the derivative on a duadic group should be defined in such a way that relations between a function defined on **D** (manly relations between characters on **D**) and its derivative be as simple and natural as possible. For example, the natural relation the character

$$e^{ikx} = \cos(kx) + i\sin(kx)$$

on the torus group $\mathbf{T} = \mathbb{R}/2\pi\mathbf{Z}$ and its derivative

$$(e^{ikx})' = ike^{ikx}$$

should be in some way preserved for a dyadic derivative of a character on **D**. M. Pepić, in [8], starting with [6, $Definition$ 3], applied to Vilenkin groups, gave a matrix interpretation of Onneweer's derivative $f^{\,[1]}(x)$, of a function $f \in L^1(G)$, defined by

(11) $$f^{[1]}(x) := \lim_{n\to\infty} E_n f(x)$$

where

$$E_n f(x) := \sum_{k=0}^{n-1} (m_{k+1} - m_k)\left[f(x) - S_{m_k}(x)\right].$$

This derivative is represented by the matrix $\Lambda = [\lambda_{ij}]_{i\in\mathbb{N}, j\in\mathbb{N}_0}$, where

(12) $$\lambda_{ij} := \begin{cases} 1 & \text{for } m_n \le i < m_{n+1} \wedge 0 \le j < m_n,\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

That motivated him to introduce the following definition.

Definition 0. *Let G be a given Vilenkin group and let $\Lambda = [\lambda_{ij}]_{i\in\mathbb{N}, j\in\mathbb{N}_0}$ be a scalar matrix. For $f \in L^1(G)$ and $x \in G$ and $i, n \in \mathbb{N}$ let*

(13) $$L_i(f, \Lambda, x) := \sum_{j=0}^{\infty} \lambda_{ij} \hat{f}(j) \chi_j(x)$$

and

(14) $$E_n(f, \Lambda, x) := \sum_{j=-1}^{n-1} \sum_{i=m_j}^{m_{j+1}-1} \left[\sigma(f, \Lambda, x) - L_i(f, \Lambda, x)\right]$$

($m_{-1} := 0$) with the condition that

$$L_i(f, \Lambda, x) \to \sigma(f, \Lambda, x), i \to \infty.$$

Then the

(15) $$\lim_{n\to\infty} E_n(f, \Lambda, x) \text{ (if it exists)}$$

*is called $\Lambda-$**derivative** of the function f at $x \in G$ and denoted by $f^{\,\Lambda}(x)$.*

The $\Lambda-$derivative $f^{\Lambda}(x)$ is a generalization of Onneweer's derivative $f^{[1]}(x)$ [8, Re$mark$ 1.2]. Also in [8] five theorems that express the essential characteristic of the $\Lambda-$derivative are given. In this paper, for any $\alpha \in \mathbb{R}$ we to introduce the new notion $(\Lambda, \alpha)-$derivative by a following Definition 1.

Definition 1.

(a) *Let* $\alpha \in \mathbb{R}$ *be a given. Let* G *be a given Vilenkin group, and let* $\Lambda = [\lambda_{ij}]_{i\in\mathbb{N}, j\in\mathbb{N}_0}$ *be a given scalar matrix. For* $f \in L^1(G)$ *and* $x \in G$ *and* $i, n \in \mathbb{N}$ *let*

(16) $$L_i(G, f, \Lambda, x) := \sum_{j=0}^{\infty} \lambda_{ij}\hat{f}(j)\chi_j(x)$$

and

(17) $$E_n(G, f, \Lambda, \alpha, x) \coloneq \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^{\alpha}} \sum_{k=m_i}^{m_{i+1}-1} [\sigma(G, f, \Lambda, x) - L_k(G, f, \Lambda, x)]$$

with the condition that

$$L_k(G, f, \Lambda, x) \to \sigma(G, f, \Lambda, x), k \to \infty,$$

where n *is given by* (2). *Then the*

(18) $$\lim_{n\to\infty} E_n(G, f, \Lambda, \alpha, x) \text{ if it exists)}$$

is called the $(\mathbf{\Lambda}, \alpha)-$*derivative of the function* f *at* x *and denoted*

$$f^{(\Lambda,\alpha)}(x).$$

(b) *Suppose the condition in* $a)$ *holds*

$$\lim_{n\to\infty} E_n(G, f, \Lambda, \alpha, x) = g(x), \forall x \in G,$$

then function $g \in L^1(G)$ *called* $(\mathbf{\Lambda}, \alpha)-$***derivative*** *of the* f *and we write*

$$g = f^{\ (\Lambda,\alpha)}.$$

(c) *If* G, f, Λ, α, x *be are as in* $a)$, *then we use following notation:*

(19) $$M_{\alpha} = M(G, \Lambda, \alpha, x) := \left\{ f \in L^1(G) : \text{ exists } f^{(\Lambda,\alpha)}(x) \right\}$$

(20) $$M = M(G, \Lambda, x) := \left\{ f \in L^1(G) : \text{ exists } f^{\Lambda}(x) \right\}$$

The results in this paper are the following statements about the main properties of the $(\Lambda, \alpha)-$ derivative of the functions on Vilenkin group G.

2. Results

Theorem 1. *Let $G, f, \Lambda, \alpha, x, L_i(G, f, \Lambda, x)$ and $E_n(G, f, \Lambda, \alpha, x)$ be as in Definition 1. Then the following statements are true:*

(a) $(\forall i \in \mathbb{N})(\forall s \in \mathbb{N}_0)(\forall \chi_s \in \Gamma)(\forall x \in G) L_i(G, \chi_s, \Lambda, x) = \lambda_{is}\chi_s(x)$. *Therefore,* $L_i(G, \chi_s, \Lambda, x) \to \lambda_{\infty s}\chi_s(x), i \to \infty$; *where*

$$\lambda_{\infty s} := \lim_{i\to\infty} \lambda_{is} \tag{21}$$

and

$$(\forall s \in \mathbb{N}_0)(\forall\ \chi_s \in \Gamma)(\forall x \in G)\sigma(G, \chi_s, Lambda, x) = \lambda_{\infty s}\chi_s(x).$$

(b) $(\forall s \in \mathbb{N}_0)(\forall \alpha \in \mathbb{N})(\forall x \in G) E_n(G, \chi_s, \Lambda, \alpha, x) = \Lambda_n(G, s, \alpha)\cdot\chi_s(x)$, *where*

$$\Lambda_n(G, s, \alpha) := \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^\alpha} \sum_{k=m_i}^{m_{i+1}-1} (\lambda_{\infty s} - \lambda_{ks}) \tag{22}$$

(c) *For arbitrary $s \in \mathbb{N}_0, \chi_s$ is a $(\Lambda, \alpha)-$ differentiable function at every $x \in G$ iff the following limit exists*

$$\Lambda_\infty(G, s, \alpha) = \lim_{n\to\infty} \Lambda_n(G, s, \alpha) \tag{23}$$

In that case

$$\chi_s^{(\Lambda,\alpha)}(x) = \Lambda_\infty(G, s, \alpha) \cdot \chi_s(x)$$

holds.

(d) $(\forall k \in \mathbb{N})(\forall s \in \mathbb{N}_0)(\forall x \in G) \left[L_k(G, f, \Lambda, x)\right]^\wedge (s) = \lambda_{ks} \cdot \hat{f}(s)$, *under the condition that the series that appears in the proof may be integrated term by term.*

(e) $(\forall n \in \mathbb{N})(\forall s \in \mathbb{N}_0)(\forall \alpha \in \mathbb{R})(\forall f \in L^1(G)(\forall x \in G)$

$$\left[E_n(G, f, \Lambda, \alpha, x)\right]^\wedge (s) = \Lambda_n(G, s, \alpha) \cdot \hat{f}(s),$$

under the condition that the series that appears in the proof may be integrated term by term.

Corollary 1. *If in Theorem 1. we take $\lambda_{is} = C, \forall i \in \mathbb{N}$ $(C - constant)$, where $s \in \mathbb{N}_0$ is given. Then the following holds:*

1. $(\forall n \in \mathbb{N})\Lambda_n(G, s, \alpha) = 0 \wedge \Lambda_\infty(G, s, \alpha) = 0.$
2. $(\forall i \in \mathbb{N})L_i(G, \chi_s, \Lambda, x) = C \cdot \chi_s(x).$

3. $(\forall n \in \mathbb{N})(\forall x \in G)E_n(G, \chi_s, \Lambda, \alpha, x) = 0$ *(particular* $\chi_s^{(\Lambda,\alpha)}(x) = 0, \forall x \in G)$.

4. $(\forall i \in \mathbb{N})\,[L_i(G, f, \Lambda, x)]^{\curlywedge}(s) = C\cdot \hat{f}(s)$.

5. $(\forall n \in \mathbb{N})\,[E_n(G, f, \Lambda, \alpha, x)]^{\curlywedge}(s) = 0$.

Corollary 2. *If in Theorem 1. we take* $(\forall i \in \mathbb{N})(\forall j \geq i)\lambda_{ij} = 0$*, then*

$$L_i(G, f, \Lambda, x) = \sum_{i=0}^{i-1} \lambda_{ij} \cdot \hat{f}(j)\chi_j(x).$$

In that case the following statements holds:

1. $(\forall i \in \mathbb{N})(\forall s \in \mathbb{N}_0)L_i(G, \chi_s, \Lambda, x) = \lambda_{is} \cdot \delta^*(i, s) \cdot \chi_s(x)$, *where*

(24) $$\delta^*(i, s) := \begin{cases} 1 & , i > s \\ 0 & , \textit{otherwise} \end{cases}$$

2. $(\forall k \in \mathbb{N})(\forall s \in \mathbb{N}_0)\,[L_k(G, f, \Lambda, x)]^{\curlywedge}(s) = \lambda_{ks} \cdot \delta^*(k, s) \cdot \hat{f}(s)$.

3. $(\forall n \in \mathbb{N})(\forall s \in \mathbb{N}_0)E_n(G, \chi_s, \Lambda, \alpha, x) = \Lambda_n^*(G, s, \alpha) \cdot \chi_s(x)$, *where*

(25) $$\Lambda_n^*(G, s, \alpha) := \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^\alpha} \sum_{k=m_i}^{m_{i+1}-1} [\lambda_{\infty s} - \lambda_{ks} \cdot \delta^*(k, s)]$$

   In particular if n *satisfies* $m_{N+1} < s$*, then*

$$\Lambda_n^*(G, s, \alpha) = \lambda_{\infty s} \cdot \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^{\alpha-1}}.$$

4. $(\forall n \in \mathbb{N})(\forall s \in \mathbb{N}_0)\,[E_n(G, f, \Lambda, \alpha, x)]^{\curlywedge}(s) = \Lambda_n^*(G, s, \alpha) \cdot \hat{f}(s)$.

Theorem 2. *Let* $G, f, \Lambda, \alpha, x, L_i(G, f, \Lambda, x)$ *and* $E_n(G, f, \Lambda, \alpha, x)$ *be as in Definition 1. Then the following statements hold:*

(a) *If* $(\forall x \in G)f(x) = C$ $(C - constant)$*, then* f *is* $(\Lambda, \alpha)-$ *differentiable at every point* $x \in G$ *iff the limit*

$$\Lambda_\infty(G, 0, \alpha) = \lim_{n\to\infty} \Lambda_n(G, 0, \alpha)$$

exists. In that case

$$(\forall x \in G)f^{\,(\Lambda,\alpha)}(x) = C \cdot \Lambda_\infty(G, 0, \alpha)$$

and particular, $C \neq 0$*, then*

$$f^{\,(\Lambda,\alpha)}(x) = 0 (\forall x \in G) \textit{ iff } \Lambda_\infty(G, 0, \alpha) = 0.$$

(b) *If f and g are $(\Lambda,\alpha)-$ differentiable functions at a point $x\in G$, then the function $F := f+g$ is $(\Lambda,\alpha)-$differentiable at x and*

$$(f+g)^{(\Lambda,\alpha)}(x) = f^{(\Lambda,\alpha)}(x) + g^{(\Lambda,\alpha)}(x) \tag{26}$$

(c) *If f $(\Lambda,\alpha)-$differentiable functions at a point $x\in G$, and C is constant, then the function $\varphi := C\cdot f$ is $(\Lambda,\alpha)-$differentiable at x and*

$$(C\cdot f\ )^{(\Lambda,\alpha)}(x) = C\cdot f^{\ (\Lambda,\alpha)}(x) \tag{27}$$

(d) *If f and g are $(\Lambda,\alpha)-$differentiable functions at a point $x\in G$, then the function $\Psi := f*g$ is $(\Lambda,\alpha)-$differentiable at x and*

$$(f*g)^{(\Lambda,\alpha)}(x) = \sum_{i=-1}^{\infty}\frac{1}{(m_{i+1}-m_i)^{\alpha}}\sum_{k=m_i}^{m_{i+1}-1}\sum_{j=0}^{\infty}(\lambda_{\infty j}-\lambda_{kj})\hat{f}(j)\hat{g}(j)\chi_j(x)$$

holds (under condition that the indicated limit exists).

(e) *The Leibniz differentiation formula does not hold (generally) for the $(\Lambda,\alpha)-$derivative.*

The well known fact that differentiability implies continuity in the classical case, hold in some sense in the case of the $(\Lambda,\alpha)-$derivative. That fact is made precise by the following theorem.

Theorem 3. *Let $(\forall k\in\mathbb{N})L_k(G,f,\Lambda,x)$ be a continuous function in some neighborhood x_0+G_s of the point x_0 (this condition is automatically fulfilled when Λ is a triangular matrix). Then: If f is $(\Lambda,\alpha)-$differentiable functions in x_0+G_s and*

$$\sigma(G,f,\Lambda,x) = \lim_{k\to\infty} L_k(G,f,\Lambda,x)$$

uniformly on x_0+G_s, then $\sigma(G,f,\Lambda,x)$ is a continuous function in x_0+G_s.

Remark 1. *Let us notice for every function*

$$f\in L^p(G)(1\le p\le\infty)\,\|S_{m_n}(f)-f\|_p\to 0(n\to\infty)\,[1,p.133].$$

If $\alpha=0$ and $\Lambda=[\lambda_{ij}]_{i\in\mathbb{N},j\in\mathbb{N}_0}$ is the matrix in Onneweer's definition of differentiation, then

$$(\forall x\in G)(\forall i\in\mathbb{N})(\forall k: m_i\le k< m_{i+1})L_k(G,f,\Lambda,x) = S_{m_k}f(x)$$

and

$$\sigma(G,f,\Lambda,x) = f(x).$$

In that case Theorem 3 be comes: If f is $(\Lambda,0)-$ differentiable functions in some neighborhood x_0+G_s of the point x_0 and $S_{m_k}f(x)\to f(x)(k\to\infty)$ uniformly on x_0+G_s, then f is continuous function in x_0.

Theorem 4. *Suppose g is the $(\Lambda, \alpha)-$derivative of $f \in L^1(G)$. If the Lebesgue dominated convergence theorem can be applied to the seguence*

$$(E_n)_{n\in\mathbb{N}}, E_n = E_n(G, f, \Lambda, \alpha, x)$$

and function g, then $g \in L^1(G)$ and for each $j \in \mathbb{N}_0$

$$\overset{\curlywedge}{g}(j) = \Lambda_\infty(G, j, \alpha) \cdot \hat{f}(j) \tag{28}$$

Theorem 5. *If G, f, Λ, α, x are as in Definition 1, then we following statements hold:*

(a) $\exists f^{\,\Lambda}(x) \Leftrightarrow \exists f^{\,(\Lambda,0)}(x)$. *In that case $f^\Lambda(x) = f^{(\Lambda,0)}(x)$ holds. Therefore*

$$M = M_0 \tag{29}$$

(b) $(\forall\alpha, \beta \in \mathbb{R}) \wedge (\alpha < \beta) \Rightarrow (M_\alpha \subseteq M_\beta) \wedge (M_\beta \setminus M_\alpha \neq \theta)(30)$

Remark 2. *Statement a) in Theorem 5 says that the $\Lambda-$derivative is equal to the $(\Lambda, 0)-$ derivative and Statement b) in Theorem 5 says that the $(\Lambda, \alpha)-$ derivative, for each $0 < \alpha$ is a strict generalization of the $\Lambda-$ derivative.*

3. Proofs

3.1. Proof of the Theorem 1.

1. Knowing that

$$\overset{\curlywedge}{\chi}_m(n) = \int\limits_G \chi_m \overline{\chi_n} = \delta(m, n) := \begin{cases} 1 & , m = n \\ 0 & , m \neq n \end{cases},$$

one obtains

$$L_i(G, \chi_s, \Lambda, x) = \sum_{j=0}^{\infty} \lambda_{ij} \overset{\curlywedge}{\chi}_s(j) \chi_j(x) = \lambda_{is} \cdot \chi_s(x)$$

and

$$L_i(G, \chi_s, \Lambda, x) \to \lambda_{\infty s} \cdot \chi_s(x)(i \to \infty).$$

Therefore,

$$\sigma(G, \chi_s, \Lambda, x) = \lambda_{\infty s} \cdot \chi_s(x).$$

2. From statement 1 one obtains

$$\begin{aligned}
&E_n(G,\chi_s,\Lambda,\alpha,x)\\
&:=\sum_{i=-1}^{N}\frac{1}{(m_{i+1}-m_i)^{\alpha}}\sum_{k=m_i}^{m_{i+1}-1}[\sigma(G,\chi_s,\Lambda,x)-L_k(G,\chi_s,\Lambda,x)]\\
&=\sum_{i=-1}^{N}\frac{1}{(m_{i+1}-m_i)^{\alpha}}\sum_{k=m_i}^{m_{i+1}-1}[\lambda_{\infty s}\chi_s(x)-\lambda_{ks}\chi_s(x)]=\Lambda_n(G,s,\alpha)\cdot\chi_s(x).
\end{aligned}$$

3. Follows from Definition 1 and statement 2.

4.
$$\begin{aligned}
[L_k(G,f,\Lambda,x)]^{\curlywedge}(s) &= \int_G\left(\sum_{j=0}^{\infty}\lambda_{kj}\hat{f}(j)\chi_j\right)\overline{\chi_s}\\
&=\sum_{j=0}^{\infty}\lambda_{kj}\hat{f}(j)\int_G\chi_j\overline{\chi_s}=\lambda_{ks}\hat{f}(s)
\end{aligned}$$

(under the condition that the series can be integrated term by term).

5.
$$\begin{aligned}
&[E_n(G,f,\Lambda,x)]^{\curlywedge}(s)\\
&=\int_G\left\{\sum_{i=-1}^{N}\frac{1}{(m_{i+1}-m_i)^{\alpha}}\sum_{k=m_i}^{m_{i+1}-1}[\sigma(G,f,\Lambda,x)-L_k(G,f,\Lambda,x)]\right\}\overline{\chi_s}\\
&=\sum_{i=-1}^{N}\frac{1}{(m_{i+1}-m_i)^{\alpha}}\sum_{k=m_i}^{m_{i+1}-1}\left[\int_G\sigma(G,f,\Lambda,x)\overline{\chi_s}-\int_G L_k(G,f,\Lambda,x)\overline{\chi_s}\right]\\
&=\sum_{i=-1}^{N}\frac{1}{(m_{i+1}-m_i)^{\alpha}}\sum_{k=m_i}^{m_{i+1}-1}\left(\lambda_{\infty s}\hat{f}(s)-\lambda_{ks}\hat{f}(s)\right)=\Lambda_n(G,s,\alpha)\hat{f}(s).
\end{aligned}$$

(under the condition that the series $\sigma(G,f,\Lambda,x)\overline{\chi_s}$ and $L_k(G,f,\Lambda,x)\overline{\chi_s}$ can be integrated term by term). ∎

3.2. Proof of the Corolary 1.

The proof is evident. ∎

3.3. Proof of the Corolary 2.

1. Knowing that

$$\overset{\curlywedge}{\chi}_m(n)=\delta(m,n)\text{ and }(\forall i\in\mathbb{N})(\forall j\in\mathbb{N}_0)\lambda_{ij}=\lambda_{ij}\cdot\delta^*(i,j),$$

one obtains

$$L_i(G,\chi_s,\Lambda,x)=\sum_{j=0}^{i-1}\lambda_{ij}\overset{\curlywedge}{\chi}_s(j)\chi_j(x)=\lambda_{is}.\chi_s(x).\delta^*(i,s).$$

2. $[L_i(G,f,\Lambda,x)]^{\curlywedge}(s)=\int\limits_G\left(\sum\limits_{j=0}^{i-1}\lambda_{ij}\hat{f}(j)\chi_j\right)\overline{\chi_s}=\lambda_{is}.\delta^*(i,s).\hat{f}(s).$

3. From 1. we have

$$\begin{aligned}E_n(G,\chi_s,\Lambda,\alpha,x) &= \sum_{i=-1}^{N}\frac{1}{(m_{i+1}-m_i)^\alpha}\\ &\qquad\sum_{k=m_i}^{m_{i+1}-1}[\lambda_{\infty s}.\chi_s(x)-\lambda_{ks}.\delta^*(k,s).\chi_s(x)]\\ &= \Lambda_n^*(G,s,\alpha)\cdot\chi_s(x).\end{aligned}$$

4.

$$\begin{aligned}&[E_n(G,f,\Lambda,\alpha,x)]^{\curlywedge}(s)\\ &=\int\limits_G\left\{\sum_{i=-1}^{N}\frac{1}{(m_{i+1}-m_i)^\alpha}\sum_{k=m_i}^{m_{i+1}-1}[\sigma(G,f,\Lambda,x)-L_k(G,f,\Lambda,x)]\right\}\overline{\chi_s}\\ &=\sum_{i=-1}^{N}\frac{1}{(m_{i+1}-m_i)^\alpha}\sum_{k=m_i}^{m_{i+1}-1}\int\limits_G\left[\sum_{j=0}^{\infty}\lambda_{\infty j}\hat{f}(j)\chi_j-\sum_{j=0}^{k-1}\lambda_{kj}\hat{f}(j)\chi_j\right]\overline{\chi_s}\\ &=\sum_{i=-1}^{N}\frac{1}{(m_{i+1}-m_i)^\alpha}\sum_{k=m_i}^{m_{i+1}-1}\left[\lambda_{\infty s}\hat{f}(s)-\lambda_{ks}\delta^*(k,s)\hat{f}(s)\right]\\ &=\Lambda_n^*(G,s,\alpha)\cdot\hat{f}(s)\end{aligned}$$

(under the condition that the series

$$\sum_{j=0}^{\infty}\lambda_{\infty j}\hat{f}(j)\chi_j(x)\overline{\chi_s(x)}$$

can be integrated term by term) ■

3.4. Proof of the Theorem 2.

(a) Let

$$(\forall x\in G)f(x)=C\ (C-constnt).$$

Then, knowing that

$$\overset{\curlywedge}{C}(j) = \int\limits_G C\overline{\chi_j} = \begin{cases} C & , j = 0 \\ 0 & , j \neq 0 \end{cases} = C \cdot \delta(0, j),$$

one obtains

$$L_k(G, C, \Lambda, x) = C\lambda_{k0} \text{ and } \sigma(G, C, \Lambda, x) = C\lambda_{\infty 0}.$$

Therefore,

$$\begin{aligned} E_n(G, C, \Lambda, \alpha, x) &= \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^{\alpha}} \sum_{k=m_i}^{m_{i+1}-1} (C.\lambda_{\infty s} - C.\lambda_{ks}) \\ &= C.\Lambda_n(G, 0, \alpha) \text{ and } f^{\,(\Lambda,\alpha)}(x) = C.\Lambda_{\infty}(G, 0, \alpha), \forall x \in G. \end{aligned}$$

Particular, if $C \neq 0$, then

$$f^{(\Lambda,\alpha)}(x) = 0 (\forall x \in G) \Leftrightarrow \Lambda_{\infty}(G, 0, \alpha) = 0.$$

(b) Let f and g are $(\Lambda, \alpha)-$differentiable functions at a point $x \in G$ and

$$F := f + g.$$

Then

$$\begin{aligned} & E_n(G, F, \Lambda, \alpha, x) \\ = & \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^{\alpha}} \sum_{k=m_i}^{m_{i+1}-1} \left[\begin{array}{c} \sigma(G, f, \Lambda, x) + \sigma(G, g, \Lambda, x) \\ -L_k(G, f, \Lambda, x) - L_k(G, g, \Lambda, x) \end{array} \right] \\ = & f^{(\Lambda,\alpha)}(x) + g^{(\Lambda,\alpha)}(x). \end{aligned}$$

(c) Let f be a $(\Lambda, \alpha)-$ differentiable functions at a point $x \in G$ and C a constant. Let

$$\varphi := C \cdot f.$$

Then

$$\begin{aligned} & E_n(G, F, \Lambda, \alpha, x) \\ = & \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^{\alpha}} \sum_{k=m_i}^{m_{i+1}-1} [C \cdot \sigma(G, f, \Lambda, x) - C \cdot L_k(G, f, \Lambda, x)] \\ = & C \cdot \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^{\alpha}} \sum_{k=m_i}^{m_{i+1}-1} [\sigma(G, f, \Lambda, x) - L_k(G, f, \Lambda, x)] \\ \to & C \cdot f^{\,(\Lambda,\alpha)}(x) \, (N \to \infty \Leftrightarrow n \to \infty) \, . \end{aligned}$$

(d) Let f and g are $(\Lambda, \alpha)-$differentiable functions at a point $x \in G$ and

$$\Psi := f * g.$$

Then

$$L_k(G, \Psi, \Lambda, \alpha, x) = \sum_{j=0}^{\infty} \lambda_{kj} \overset{\curlywedge}{f}(j) \overset{\curlywedge}{g}(j) \chi_j(x) \to \sum_{j=0}^{\infty} \lambda_{\infty j} \overset{\curlywedge}{f}(j) \overset{\curlywedge}{g}(j) \chi_j(x),$$

$$\begin{aligned} & E_n(G, \Psi, \Lambda, \alpha, x) \\ &= \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^{\alpha}} \sum_{k=m_i}^{m_{i+1}-1} [\sigma(G, \Psi, \Lambda, x) - L_k(G, \Psi, \Lambda, x)] \\ &= \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^{\alpha}} \sum_{k=m_i}^{m_{i+1}-1} \left[\sum_{j=0}^{\infty} \lambda_{\infty j} \overset{\curlywedge}{f}(j) \overset{\curlywedge}{g}(j) \chi_j(x) - \sum_{j=0}^{\infty} \lambda_{kj} \overset{\curlywedge}{f}(j) \overset{\curlywedge}{g}(j) \chi_j(x) \right] \\ &= \sum_{i=-1}^{N} \frac{1}{(m_{i+1} - m_i)^{\alpha}} \sum_{k=m_i}^{m_{i+1}-1} (\lambda_{\infty j} - \lambda_{kj}) \overset{\curlywedge}{f}(j) \overset{\curlywedge}{g}(j) \chi_j(x), \end{aligned}$$

under the condition the series

$$\sum_{j=0}^{\infty} \lambda_{\infty j} \overset{\curlywedge}{f}(j) \overset{\curlywedge}{g}(j) \chi_j(x) \text{ and } \sum_{j=0}^{\infty} \lambda_{kj} \overset{\curlywedge}{f}(j) \overset{\curlywedge}{g}(j) \chi_j(x)$$

converge at the point $x \in G$. Therefore,

$$\begin{aligned} & \Psi^{(\Lambda, \alpha)}(x) \\ &= \lim_{n \to \infty} E_n(G, \Psi, \Lambda, \alpha, x) = \lim_{N \to \infty} E_n(G, \Psi, \Lambda, \alpha, x) \\ &= \sum_{i=-1}^{\infty} \frac{1}{(m_{i+1} - m_i)^{\alpha}} \sum_{k=m_i}^{m_{i+1}-1} (\lambda_{\infty j} - \lambda_{kj}) \overset{\curlywedge}{f}(j) \overset{\curlywedge}{g}(j) \chi_j(x) \text{(if the limit exists).} \end{aligned}$$

(e) Let $f = \chi_{m_n} = r_n$ and $g = \chi_{m_{n+1}} = r_{n+1}$. Then $f \cdot g = \chi_{m_n + m_{n+1}}$. From Theorem 1, statement 3, we have

$$\begin{aligned} (f \cdot g)^{(\Lambda, \alpha)}(x) &= \chi_{m_n + m_{n+1}}(x) \cdot \Lambda_{\infty}(G, m_n + m_{n+1}, \alpha) \\ &= \chi_{m_n}(x) . \chi_{m_{n+1}}(x) \cdot \Lambda_{\infty}(G, m_n + m_{n+1}, \alpha) \end{aligned}$$

and

$$\begin{aligned} & f^{(\Lambda, \alpha)}(x) \cdot g(x) + g^{(\Lambda, \alpha)}(x) \cdot f(x) \\ &= \chi_{m_n}(x) . \chi_{m_{n+1}}(x) \cdot [\Lambda_{\infty}(G, m_n, \alpha) + \Lambda_{\infty}(G, m_{n+1}, \alpha)] . \end{aligned}$$

Therefore,

$$(f \cdot g)^{(\Lambda, \alpha)}(x) \neq f^{(\Lambda, \alpha)}(x) \cdot g(x) + g^{(\Lambda, \alpha)}(x) \cdot f(x) \square$$

3.5. Proof of the Theorem 3.

The theorem follows from the fact that the uniform limit of a sequence of continuous functions is continuous ■

3.6. Proof of the Theorem 4.

Applying the Lebesgue dominated convergence theorem of the sequence

$$(E_n)_{n\in\mathbb{N}}, E_n = E_n(G, f, \Lambda, \alpha, x),$$

and its limit

$$g = f^{\ (\Lambda,\alpha)}$$

we conclude

$$g \in L^1(G) \wedge \|E_n - g\|_1 \to 0(n \to \infty).$$

Therefore,

$$(\forall j \in \mathbb{N}_0) \left| \overset{\curlywedge}{E_n}(j) - \overset{\curlywedge}{g}(j) \right| \to 0(n \to \infty).$$

But by Theorem 1 (statement 5)

$$\lim_{n\to\infty} \overset{\curlywedge}{E_n}(j) = \Lambda_\infty(G, j, \alpha) \cdot \overset{\curlywedge}{f}(j)$$

holds. Therefore,

$$(\forall j \in \mathbb{N}_0) \overset{\curlywedge}{g}(j) = \Lambda_\infty(G, j, \alpha) \cdot \overset{\curlywedge}{f}(j)$$

holds ■

3.7. Proof of the Theorem 5.

(a) The proof is evident.

(b) For arbitrary $\alpha, \beta \in \mathbb{R} \wedge \alpha < \beta$, we designate $\gamma := \beta - \alpha \in \mathbb{R}^+$. If $f \in M_\alpha = M(G, \Lambda, \alpha, x)$, then series

$$\sum_{i=-1}^{\infty} \frac{1}{(m_{i+1} - m_i)^\alpha} \sum_{k=m_i}^{m_{i+1}-1} [\sigma(G, f, \Lambda, x) - L_k(G, f, \Lambda, x)] = \sum_{i=-1}^{\infty} A_i$$

is convergent. From that and the series

$$\sum_{i=-1}^{\infty} \frac{1}{(m_{i+1} - m_i)^\beta} \sum_{k=m_i}^{m_{i+1}-1} [\sigma(G, f, \Lambda, x) - L_k(G, f, \Lambda, x)] = \sum_{i=-1}^{\infty} B_i$$

is convergent, because

$$\frac{B_i}{A_i} = \frac{1}{(m_{i+1} - m_i)^\gamma} \to 0(i \to \infty).$$

Therefore, $f \in M_\beta = (G, \Lambda, \beta, x)$ and $M_\alpha \subseteq M_\beta$. Hence holds $M_\beta \diagdown M_\alpha \neq \theta$, is proved for following Example.

Example 1. Let

$$\lambda_{\infty s} - \lambda_{ks} := (m_{i+1} - m_i)^{\alpha-1}, \forall i \in \mathbb{N}, \forall k \in [m_i, m_{i+1}).$$

Then $\chi_s \in M_\beta$ because then the series

$$\chi_s(x) \cdot \sum_{i=-1}^{\infty} \frac{1}{(m_{i+1} - m_i)^\beta} \sum_{k=m_i}^{m_{i+1}-1} (\lambda_{\infty s} - \lambda_{ks}) = \chi_s(x) \cdot \sum_{i=-1}^{\infty} \frac{1}{(m_{i+1} - m_i)^\gamma}$$

is convergent and $\chi_s \notin M_\alpha$, because then the series

$$\chi_s(x) \cdot \sum_{i=-1}^{\infty} \frac{1}{(m_{i+1} - m_i)^\alpha} \sum_{k=m_i}^{m_{i+1}-1} (\lambda_{\infty s} - \lambda_{ks}) = \chi_s(x) \cdot \sum_{i=-1}^{\infty} 1$$

is not convergent. ■

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