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COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Abstract. In this paper, we introduce and investigate each of the following subclasses:

$$\mathcal{S}_g(\lambda,\gamma) \quad \text{and} \quad \mathcal{K}_g(\lambda,\gamma,m;u) \quad \Big(0 \leqq \lambda \leqq 1; u \in \mathbb{R} \setminus (-\infty,-1];\ m \in \mathbb{N} \setminus \{1\}\Big)$$

of analytic functions of complex order $\gamma \in \mathbb{C} \setminus \{0\}$, $g : \mathbb{U} \to \mathbb{C}$ being some suitably constrained convex function in the open unit disk $\mathbb{U}$. We obtain coefficient bounds and coefficient estimates involving the Taylor-Maclaurin coefficients of the function $f(z)$ when $f(z)$ is in the class $\mathcal{S}_g(\lambda,\gamma)$ or in the class $\mathcal{K}_g(\lambda,\gamma,m;u)$. The various results, which are presented in this paper, would generalize and improve those in related works of several earlier authors.

1. Introduction, Definitions and Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} = \{1,2,3,\cdots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers. We also let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

A function $f(z) \in \mathcal{A}$ is said to belong to the class $\mathcal{S}^*(\gamma)$ of *starlike functions of complex order* γ if it satisfies the following inequality:

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$$\Re\left(1+\frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)}-1\right]\right)>0 \qquad (z\in\mathbb{U};\ \gamma\in\mathbb{C}^*:=\mathbb{C}\setminus\{0\}). \tag{2}$$

Furthermore, a function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{C}(\gamma)$ of *convex functions of complex order* γ if it satisfies the following inequality:

$$\Re\left(1+\frac{1}{\gamma}\left[\frac{zf''(z)}{f'(z)}\right]\right)>0 \qquad (z\in\mathbb{U};\ \gamma\in\mathbb{C}^*). \tag{3}$$

The function classes $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ were investigated earlier by Nasr and Aouf [14] (see also [15]) and Wiatrowski [20], respectively, and (more recently) by Altintaş *et al.* ([1] to [10]), Deng [11], Murugusundaramoorthy and Srivastava [13], Srivastava *et al.* [19], and others (see, for example, [12] and [18]).

For two functions f and g, analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and we write $f\prec g$ or, more precisely,

$$f(z)\prec g(z) \qquad (z\in\mathbb{U})$$

if there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0)=0 \qquad \text{and} \qquad |\mathfrak{w}(z)|<1 \qquad (z\in\mathbb{U}),$$

such that

$$f(z)=g(\mathfrak{w}(z)) \qquad (z\in\mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to the following relationships:

$$f(0)=g(0) \qquad \text{and} \qquad f(\mathbb{U})\subset g(\mathbb{U}).$$

Recently, Srivastava *et al.* [17] introduced the subclasses $\mathcal{S}(\lambda,\gamma,A,B)$ and $\mathcal{K}(\lambda,\gamma,A,B,m;u)$ of analytic functions of *complex* order $\gamma\in\mathbb{C}^*$ by using the above subordination principle between analytic functions, and obtained the coefficient bounds for the Taylor-Maclaurin coefficients for functions in each of these new sublasses $\mathcal{S}(\lambda,\gamma,A,B)$ and $\mathcal{K}(\lambda,\gamma,A,B,m;u)$ of complex order $\gamma\in\mathbb{C}^*$, which are given by Definitions 1 and 2 below.

Definition 1. (see [17]). Let $\mathcal{S}(\lambda,\gamma,A,B)$ denote the class of functions given by

$$\mathcal{S}(\lambda,\gamma,A,B)=\left\{f: f\in\mathcal{A} \text{ and } 1+\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)\right.$$
$$\left.\prec\frac{1+Az}{1+Bz}\ \ (z\in\mathbb{U})\right\} \tag{4}$$
$$(0\leqq\lambda\leqq 1;\gamma\in\mathbb{C}^*;-1\leqq B<A\leqq 1).$$

Definition 2. (see [17]). A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{K}(\lambda, \gamma, A, B, m; u)$ if it satisfies the following *nonhomegenous Cauchy-Euler type differential equation of order m*:

$$z^m \frac{d^m w}{dz^m} + \binom{m}{1}(u+m-1) z^{m-1} \frac{d^{m-1} w}{dz^{m-1}} + \cdots + \binom{m}{m} w \prod_{j=0}^{m-1} (u+j)$$

$$= g(z) \prod_{j=0}^{m-1} (u+j+1) \tag{5}$$

$$\big(w = f(z) \in \mathcal{A};\ g(z) \in \mathcal{S}(\lambda, \gamma, A, B);$$

$$u \in \mathbb{R} \setminus (-\infty, -1];\ m \in \mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2, 3, 4, \cdots\}\big).$$

Making use of Definitions 1 and 2, Srivastava *et al.* [17] proved the following coefficient bounds for the Taylor-Maclaurin coefficients for functions in the sublasses $\mathcal{S}(\lambda, \gamma, A, B)$ and $\mathcal{K}(\lambda, \gamma, A, B, m; u)$ of analytic functions of complex order $\gamma \in \mathbb{C}^*$.

Theorem 1. (see [17]). *Let the function* $f(z)$ *be defined by* (1). *If* $f \in \mathcal{S}(\lambda, \gamma, A, B)$, *then*

$$|a_n| \leqq \frac{\prod_{k=0}^{n-2} \left(k + \frac{2|\gamma|(A-B)}{1-B} \right)}{(n-1)![1+\lambda(n-1)]} \qquad (n \in \mathbb{N}^*). \tag{6}$$

Theorem 2. (see [17]). *Let the function* $f(z)$ *be defined by* (1). *If* $f \in \mathcal{K}(\lambda, \gamma, A, B, m; u)$, *then*

$$|a_n| \leqq \frac{\prod_{k=0}^{n-2} \left(k + \frac{2|\gamma|(A-B)}{1-B} \right) \prod_{j=0}^{m-2} (u+j+1)}{(n-1)![1+\lambda(n-1)] \prod_{j=0}^{m-1} (u+j+n)} \qquad (m, n \in \mathbb{N}^*). \tag{7}$$

Here, in our present sequel to some of the aforecited works (especially [17]), we first introduce the following subclasses of analytic functions of complex order $\gamma \in \mathbb{C}^*$.

Definition 3. Let $g : \mathbb{U} \to \mathbb{C}$ be a convex function such that

$$g(0) = 1 \qquad \text{and} \qquad \Re[g(z)] > 0 \qquad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}_g(\lambda, \gamma)$ the class of functions given by

$$\mathcal{S}_g(\lambda, \gamma)$$

$$= \left\{ f : f \in \mathcal{A} \text{ and } 1 + \frac{1}{\gamma} \left(\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)} - 1 \right) \in g(\mathbb{U})\ (z \in \mathbb{U}) \right\} \tag{8}$$

$$(0 \leqq \lambda \leqq 1;\ \gamma \in \mathbb{C}^*).$$

Definition 4. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_g(\lambda, \gamma, m; u)$ if it satisfies the following *nonhomogenous Cauchy-Euler differential equation*:

$$z^m \frac{d^m w}{dz^m} + \binom{m}{1}(u+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}} + \cdots + \binom{m}{m} w \prod_{j=0}^{m-1}(u+j)$$
$$= h(z) \prod_{j=0}^{m-1}(u+j+1) \tag{9}$$
$$\big(w = f(z) \in \mathcal{A};\ h(z) \in \mathcal{S}_g(\lambda, \gamma);\ u \in \mathbb{R} \setminus (-\infty, -1];\ m \in \mathbb{N}^*\big).$$

Remark 1. There are many choices of the function $g(z)$ which would provide interesting subclasses of analytic functions of complex order $\gamma \in \mathbb{C}^*$. In particular, if we let

$$g(z) = \frac{1+Az}{1+Bz} \qquad (-1 \leqq B < A \leqq 1;\ z \in \mathbb{U}), \tag{10}$$

it is fairly easy to verify that $g(z)$ is a convex function in $\mathbb{U}$ and satisfies the hypotheses of Definition 3. Clearly, therefore, the function class $\mathcal{S}_g(\lambda, \gamma)$, with the function $g(z)$ given by (10), coincides with the function class $\mathcal{S}(\lambda, \gamma, A, B)$ given by Definition 1.

Remark 2. In view of Remark 1, if the function $g(z)$ is given by (10), it is easily observed that the function classes

$$\mathcal{S}_g(\lambda, \gamma) \qquad \text{and} \qquad \mathcal{K}_g(\lambda, \gamma, m; u)$$

reduce to the aforementioned function classes

$$\mathcal{S}(\lambda, \gamma, A, B) \qquad \text{and} \qquad \mathcal{K}(\lambda, \gamma, A, B, m; u),$$

respectively (see Definitions 1 and 2).

In this paper, by using the subordination principle between analytic functions, we obtain coefficient bounds for the Taylor-Maclaurin coefficients for functions in the substantially more general function classes

$$\mathcal{S}_g(\lambda, \gamma) \qquad \text{and} \qquad \mathcal{K}_g(\lambda, \gamma, m; u)$$

of analytic functions of complex order $\gamma \in \mathbb{C}^*$. The various results presented here would generalize and improve the corresponding results obtained by (for example) Srivastava *et al.* [17].

2. Main Results and Their Derivations

In order to prove our main results, we will need the following lemma due to Rogosinski [16].

Lemma (see [16]). *Let the function* $\mathfrak{g}(z)$ *given by*

$$\mathfrak{g}(z) = \sum_{k=1}^{\infty} \mathfrak{b}_k z^k \qquad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. *Also let the function* $\mathfrak{f}(z)$ *given by*

$$\mathfrak{f}(z) = \sum_{k=1}^{\infty} \mathfrak{a}_k z^k \qquad (z \in \mathbb{U})$$

be holomorphic in $\mathbb{U}$. *If*

$$\mathfrak{f}(z) \prec \mathfrak{g}(z) \qquad (z \in \mathbb{U}),$$

then

$$|\mathfrak{a}_k| \leqq |\mathfrak{b}_1| \qquad (k \in \mathbb{N}). \tag{11}$$

Our first main result is now stated as Theorem 3 below.

Theorem 3. *Let the function* $f(z)$ *be defined by* (1). *If* $f \in \mathcal{S}_g(\lambda, \gamma)$, *then*

$$|a_n| \leqq \frac{\prod_{k=0}^{n-2} (k + |g'(0)| \cdot |\gamma|)}{(n-1)![1 + \lambda(n-1)]} \qquad (n \in \mathbb{N}^*). \tag{12}$$

Proof. Let the function $\mathcal{F}(z)$ be defined by

$$\mathcal{F}(z) = \lambda z f'(z) + (1 - \lambda) f(z) \qquad (z \in \mathbb{U}).$$

Then, clearly, $\mathcal{F}(z)$ is an analytic function in $\mathbb{U}$, $\mathcal{F}(0) = 1$, and a simple computation shows that the function $\mathcal{F}(z)$ has the following Taylor-Maclaurin series expansion:

$$\mathcal{F}(z) = z + \sum_{j=2}^{\infty} A_j z^j \qquad (z \in \mathbb{U}), \tag{13}$$

where, for convenience,

$$A_j = (1 - \lambda + j\lambda) a_j \qquad (j \in \mathbb{N}^*). \tag{14}$$

Now, from Definition 3, we have

$$1 + \frac{1}{\lambda}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1\right) \in g(\mathbb{U}).$$

Also, by setting

$$p(z) = 1 + \frac{1}{\lambda}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1\right), \tag{15}$$

we deduce that

$$p(0) = g(0) = 1 \qquad \text{and} \qquad p(z) \in g(\mathbb{U}) \qquad (z \in \mathbb{U}).$$

Therefore, we have

$$p(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

Thus, according to the above Lemma based upon the principle of subordination between analytic functions, we obtain

$$\left|\frac{p^{(m)}(0)}{m!}\right| \leqq |g'(0)| \qquad (m \in \mathbb{N}). \tag{16}$$

On the other hand, we find from (15) that

$$z\mathcal{F}'(z) = \big(1 + \lambda[p(z) - 1]\big)\mathcal{F}(z) \qquad (z \in \mathbb{U}). \tag{17}$$

Further, we let

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}). \tag{18}$$

Since $A_1 = 1$, in view of (13), (17) and (18), we deduce that

$$(j-1)A_j = (p_1 A_{j-1} + p_2 A_{j-2} + \cdots + p_{j-1}) \qquad (j \in \mathbb{N}^*). \tag{19}$$

By combining (16) and (19), for $j = 2, 3, 4$, we obtain

$$|A_2| \leqq |g'(0)||\lambda|,$$

$$|A_3| \leqq \frac{|g'(0)| \cdot |\lambda|(1 + |g'(0)| \cdot |\lambda|)}{2!}$$

and

$$|A_4| \leqq \frac{|g'(0)| \cdot |\lambda|(1 + |g'(0)| \cdot |\lambda|)(2 + |g'(0)| \cdot |\lambda|)}{3!},$$

respectively. By appealing to the principle of mathematical induction, we thus obtain

$$|A_n| \leqq \frac{\prod_{k=0}^{n-2}(k + |g'(0)| \cdot |\lambda|)}{(n-1)!} \qquad (n \in \mathbb{N}^*).$$

We now easily find from (14) that

$$|a_n| \leqq \frac{\prod_{k=0}^{n-2}(k + |g'(0)| \cdot |\lambda|)}{(n-1)![1 + \lambda(n-1)]} \qquad (n \in \mathbb{N}^*),$$

as asserted by Theorem 3. This evidently completes the proof of Theorem 3. ∎

Theorem 4. *Let the function* $f(z) \in \mathcal{A}$ *be defined by* (1). *If* $f \in \mathcal{K}_g(\lambda, \gamma, m; u)$, *then*

$$|a_n| \leqq \frac{\prod_{k=0}^{n-2}(k + |g'(0)| \cdot |\lambda|) \prod_{j=0}^{m-2}(u + j + 1)}{(n-1)![1 + \lambda(n-1)] \prod_{j=0}^{m-1}(u + j + n)} \qquad \big(m, n \in \mathbb{N}^*\big) \tag{20}$$

$$\Big(0 \leqq \lambda \leqq 1;\ \gamma \in \mathbb{C}^*;\ u \in \mathbb{R} \setminus (-\infty, -1]\Big).$$

Proof. Let the function $f(z) \in \mathcal{A}$ be given by (1). Also let

$$h(z) = z + \sum_{k=2}^{\infty} h_k z^k \in \mathcal{S}_g(\lambda, \gamma). \tag{21}$$

Hence, from (9), we deduce that

$$a_n = \left(\frac{\prod_{j=0}^{m-1}(u + j + 1)}{\prod_{j=0}^{m-1}(u + j + n)} \right) h_n \qquad \Big(n \in \mathbb{N}^*;\ u \in \mathbb{R} \setminus (-\infty, -1]\Big). \tag{22}$$

Using Theorem 3 in conjunction with (22), we arrive at the assertion (20) of Theorem 4. The proof of Theorem 4 is thus completed. ∎

3. Corollaries and Consequences

In view of Remarks 1 and 2, if we let the function $g(z)$ in Theorems 3 and 4 be given by (10), we can readily deduce the following Corollaries 1 and 2, respectively, which we choose to merely state here *without* proofs.

Corollary 1. *Let the function* $f \in \mathcal{A}$ *be defined by* (1). *If* $f \in \mathcal{S}(\lambda, \gamma, A, B)$, *then*

$$|a_n| \leqq \frac{\prod_{k=0}^{n-2}(k + |\gamma|(A - B))}{(n-1)![1 + \lambda(n-1)]} \qquad (n \in \mathbb{N}^*). \tag{23}$$

Corollary 2. *Let the function* $f \in \mathcal{A}$ *be defined by* (1). *If* $f \in \mathcal{K}(\lambda, \gamma, A, B, m; u)$, *then*

$$|a_n| \leqq \frac{\prod_{k=0}^{n-2}(k+|\lambda|(A-B))\prod_{j=0}^{m-2}(u+j+1)}{(n-1)![1+\lambda(n-1)]\prod_{j=0}^{m-1}(u+j+n)} \qquad (m, n \in \mathbb{N}^*) \tag{24}$$

$$\Big(0 \leqq \lambda \leqq 1;\ \gamma \in \mathbb{C}^*;\ u \in \mathbb{R} \setminus (-\infty, -1]\Big).$$

Remark 3. It is easy to see that

$$\big(k+|\gamma|(A-B)\big) \leqq \left(k + \frac{2|\gamma|(A-B)}{1-B}\right)$$
$$\Big(k \in \mathbb{N}_0;\ -1 \leqq B < A \leqq 1;\ \gamma \in \mathbb{C}^*\Big),$$

which, in conjunction with Corollaries 1 and 2, would obviously yield significant improvements over Theorems 1 and 2 (see also the earlier work by Srivastava *et al.* [17] for several *further* corollaries and consequences Theorems 1 and 2).

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