TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 5, pp. 2337-2349, October 2011
This paper is available online at http://tjm.math.ntu.edu.tw/index.php/TJM

MODULES CHARACTERIZED BY THEIR SIMPLE SUBMODULES

Lixin Mao

Abstract. M is said to be a min-coherent (resp. PS, FS) module if its every simple submodule is finitely presented (resp. projective, flat). In this article, we study the properties of min-coherent, PS and FS modules. Some known results are generalized.

1. Introduction

According to Nicholson and Watters [13], M is called a PS *module* if its every simple submodule is projective, equivalently if its socle $\mathrm{Soc}(M)$ is projective. R is said to be a *left PS ring* if ${}_RR$ is a PS module. Examples of PS modules include nonsingular modules, regular modules in the sense of Zelmanowitz and modules with zero socle. As a generalization of PS modules and PS rings, Liu and Xiao introduced the concept of FS modules and FS rings in [9, 19]. Recall that M is an FS *module* if every simple submodule of M is flat, equivalently if $\mathrm{Soc}(M)$ is flat. R is called a *left FS ring* if ${}_RR$ is an FS module. PS and FS modules (rings) have been studied extensively (see e.g. [9, 10, 11, 12, 13, 16, 19, 20]). Although PS modules are FS, the converse is false (see [9, Example 2.2] or [19, Example 1]). So it is important to further clarify the connection between PS and FS modules.

In the present paper, we introduce the concept of min-coherent modules. We will call a module *min-coherent* if its every simple submodule is finitely presented. It is clear that M is a PS module if and only if M is a min-coherent and FS module. On the other hand, recall that M is a *coherent module* [2] if its every finitely generated submodule is finitely presented. So the definition of min-coherent modules is a generalization of both PS and coherent modules. The main aim of this paper is to characterize and investigate min-coherent, PS and FS modules.

Received February 16, 2009, accepted July 8, 2010.
Communicated by Bernd Ulrich.
2010 *Mathematics Subject Classification*: 16P70, 16D40, 16D50.
Key words and phrases: Min-coherent module, PS module, FS module, M-min-flat module, M-min-injective module, Preenvelope, Precover.

In Section 2, we show, among other things, that the following conditions are equivalent for a left R-module M: (1) M is a min-coherent module. (2) If L is a maximal left ideal of R, then either $r_M(L) = 0$ or L is finitely generated. (3) If K is a simple left R-module, then $\text{Hom}(K, M) = 0$ or K is finitely presented. We also prove that the following conditions are equivalent for a left R-module M: (1) M is a PS module. (2) $\text{Soc}({}_RR)K = K$ for any simple submodule K of M. (3) $\text{Soc}({}_RR)\text{Soc}({}_RM) = \text{Soc}({}_RM)$.

In Section 3, we introduce the concept of M-min-flat and M-min-injective modules. After several elementary properties of M-min-flat and M-min-injective modules are obtained, we prove that the following conditions are equivalent for a finitely presented left R-module M: (1) M is a min-coherent module. (2) The class of M-min-flat right R-modules is closed under direct products. (3) Every right R-module has an M-min-flat preenvelope. (4) A left R-module N is M-min-injective if and only if N^+ is M-min-flat. (5) The class of M-min-injective left R-modules is closed under direct limits. We also show that a flat left R-module M is FS if and only if the class of M-min-flat right R-modules is closed under submodules, and a projective left R-module M is PS if and only if the class of M-min-injective left R-modules is closed under quotient modules.

Throughout this paper, R is an associative ring with identity and all modules are unitary. We denote by ${}_RM$ (resp. M_R) a left (resp. right) R-module. For a nonempty subset T of R and $x \in {}_RM$, $r_M(T) = \{m \in M : tm = 0 \text{ for all } t \in T\}$, $l_R(x) = \{r \in R : rx = 0\}$. $\text{Soc}(M)$ stands for the socle of M, and the character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M is denoted by M^+. Let M and N be R-modules. $\text{Hom}(M, N)$ means $\text{Hom}_R(M, N)$ and $M \otimes N$ denotes $M \otimes_R N$. For unexplained concepts and notations, we refer the reader to [1, 4, 8, 15, 18].

2. Min-coherent and PS Modules

We start with the following definition.

Definition 2.1. Let R be a ring. A left R-module M is called *min-coherent* if every simple submodule of M is finitely presented.

Remark 2.2.

(1) If $\text{Soc}(M) = 0$, then M is clearly a min-coherent module.

(2) PS modules are clearly min-coherent. But the converse is false in general. For example, let $R = \mathbb{Z}_4$. Then R is a min-coherent R-module, but it is not a PS R-module because the simple ideal $\{0, \overline{2}\}$ is not projective.

(3) Coherent modules are obviously min-coherent, and the converse is not true in general (see [6, p.110]). However, a semisimple module is min-coherent if and only if it is coherent.

(4) In [11, 12], the author introduced and studied min-coherent rings. R is called a *left min-coherent ring* if every simple left ideal of R is finitely presented. Clearly, R is a left min-coherent ring if and only if ${}_RR$ is a min-coherent module if and only if Soc$({}_RR)$ is a min-coherent left R-module.

Proposition 2.3. *The class of min-coherent (PS, FS) left R-modules is closed under extensions, direct products, direct sums and submodules.*

Proof. First, we will prove that the class of min-coherent left R-modules is closed under extensions.

Let $0 \to A \to B \xrightarrow{g} C \to 0$ be an exact sequence of left R-modules with A and C min-coherent. Let N be a simple submodule of B. Then $g(N) = 0$ or $g(N) \cong N$.

(1) If $g(N) = 0$, then $N \subseteq \ker(g) = A$. So N is finitely presented since A is min-coherent.

(2) If $g(N) \cong N$, then $g(N)$ is a simple submodule of C. Thus $g(N)$ is finitely presented since C is min-coherent. Hence N is finitely presented.

It follows that B is a min-coherent module.

Now we will prove that the class of min-coherent left R-modules is closed under direct products.

Let N be a simple submodule of $\Pi_{i\in\Lambda}M_i$, where every M_i is a min-coherent left R-module. Let $\lambda : N \to \Pi_{i\in\Lambda}M_i$ be the inclusion and $\pi_i : \Pi_{i\in\Lambda}M_i \to M_i$ be the ith projection. We claim that there exists $j \in \Lambda$ such that $\pi_j\lambda$ is a monomorphism. Otherwise, if $\ker(\pi_i\lambda) \neq 0$ for any $i \in \Lambda$, then $\pi_i\lambda = 0$ since N is simple, and so $\lambda = 0$, a contradiction. Thus N embeds in M_j for some $j \in \Lambda$. Hence N is finitely presented. That is to say, $\Pi_{i\in\Lambda}M_i$ is min-coherent.

The rest are similar. ■

As an immediate consequence of Proposition 2.3, we have

Corollary 2.4. *R is a left min-coherent (resp. PS, FS) ring if and only if every projective left R-module is min-coherent (resp. PS, FS).*

Remark 2.5. If $F : R$-Mod $\to S$-Mod defines a Morita equivalence, then by [1, Lemma 21.3 and Proposition 21.8], a left R-module M is min-coherent if and only if $F(M)$ is min-coherent. In particular, R is a left min-coherent ring if and only if S is a left min-coherent ring by Corollary 2.4 and [1, Proposition 21.6].

Proposition 2.6. *Let R be a ring.*

(1) *Every simple left R-module is finitely presented if and only if every left R-module is min-coherent.*

(2) *Every simple left R-module is flat if and only if every left R-module is FS.*

(3) *R is a semisimple Artinian ring if and only if every left R-module is PS.*

Proof. (1) " $\Rightarrow$ " is obvious.

" $\Leftarrow$ " Let K be any simple left R-module. Since K embeds in its injective envelope $E(K)$ and $E(K)$ is min-coherent, K is finitely presented by hypothesis.

The proofs of (2) and (3) are similar. ■

Now we give several characterizations of min-coherent modules.

Theorem 2.7. *The following conditions are equivalent for a left R-module M:*

(1) *M is a min-coherent module.*

(2) *If Rx is a simple submodule of M, then $l_R(x)$ is finitely generated.*

(3) *If L is a maximal left ideal of R, then either $r_M(L) = 0$ or L is finitely generated.*

(4) *If K is a simple left R-module, then* $\mathrm{Hom}(K, M) = 0$ *or K is finitely presented.*

Proof.

(1) $\Leftrightarrow$ (2) is clear.

(1) $\Rightarrow$ (3) Let L be a maximal left ideal of R and $r_M(L) \neq 0$. Then there exists $0 \neq x \in r_M(L)$. Since $L \subseteq l_R(x) \neq R$, we have $L = l_R(x)$. So $Rx \cong R/l_R(x) = R/L$ is simple. By (1), Rx is finitely presented. Thus L is finitely generated.

(3) $\Rightarrow$ (4) Let L be a maximal left ideal of R. Define $\alpha : \mathrm{Hom}(R/L, M) \rightarrow r_M(L)$ via $\alpha(f) = f(\overline{1})$ for $f \in \mathrm{Hom}(R/L, M)$, and define $\beta : r_M(L) \rightarrow \mathrm{Hom}(R/L, M)$ via $\beta(x)(\overline{t}) = tx$ for $x \in r_M(L)$ and $\overline{t} \in R/L$. It is easy to verify that α and β are well-defined and $\mathrm{Hom}(R/L, M) \cong r_M(L)$. So (4) follows.

(4) $\Rightarrow$ (1) Let K be any simple submodule of M. Then $\mathrm{Hom}(K, M) \neq 0$, and so K is finitely presented by (4). ■

Corollary 2.8. *The following conditions are equivalent for a ring R:*

(1) *R is a left min-coherent ring.*

(2) *If L is a maximal left ideal of R, then either $r_R(L) = 0$ or L is finitely generated.*

(3) *If K is a simple left R-module, then* $\mathrm{Hom}(K, R) = 0$ *or K is finitely presented.*

(4) *R has a faithful min-coherent left R-module.*

Proof.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Theorem 2.7 by letting $M = {}_RR$.

(1) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (2) Suppose that M is a faithful min-coherent left R-module. Let L be a maximal left ideal of R and $r_R(L) \neq 0$. Then there exists $0 \neq t \in r_R(L)$. So there is $x \in M$ such that $tx \neq 0$ since M is faithful. Thus $L \subseteq l_R(tx) \neq R$, and so $L = l_R(tx)$. Hence $R/L = R/l_R(tx) \cong R(tx)$ is finitely presented by hypothesis. Thus L is finitely generated. ■

Example 2.9. Let $A = \oplus_{i=1}^{\infty}\mathbb{Z}_2$ be the direct sum of countably infinite copies of $\mathbb{Z}_2$ and $R = \mathbb{Z}_2 \propto A = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in \mathbb{Z}_2, b \in A \right\}$ be the trivial extension of $\mathbb{Z}_2$ by A. Then R is a commutative ring with Soc(R) $= 0 \propto A$. Note that Soc(R) is the unique maximal ideal (which is not finitely generated) and the annihilator of Soc(R) in R is Soc(R) itself. Thus R is not an FS ring (see [21, p. 3328]). Moreover we claim that R is not a min-coherent ring by Corollary 2.8.

The following theorem generalizes [13, Theorem 2.4] and [20, Proposition 1].

Theorem 2.10. *Consider the following conditions for a left R-module M:*

(1) *M is a PS module.*

(2) Soc$({}_RR)K = K$ *for any simple submodule K of M.*

(3) Soc$({}_RR)$Soc$(M) =$ Soc(M).

(4) *If L is an essential maximal left ideal of R, then $r_M(L) = 0$.*

(5) *If L is a maximal left ideal of R, then $r_M(L) = eM$ with $e^2 = e \in R$.*

Then $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Rightarrow (5)$. If M is faithful, then (5) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) We claim that Soc$({}_RR)K \neq 0$ for any simple submodule K of M. If not, then there exists a simple submodule Rx of M such that Soc$({}_RR)Rx = 0$. Since M is a PS module, we have $R = l_R(x) \oplus I$ with I a left ideal of R, and so $Rx = Ix$. But $I \cong R/l_R(x)$ is simple. Thus $Ix = 0$, and hence $Rx = 0$, a contradiction. Thus Soc$({}_RR)K = K$ for any simple submodule K of M since Soc$({}_RR)K \subseteq K$.

(2) $\Rightarrow$ (1) Let K be any simple submodule of M. Then there exists a simple left ideal I such that $IK \neq 0$ by (2). So $I \nsubseteq l_R(x)$ for some $0 \neq x \in K$. Hence $R = l_R(x) \oplus I$ since $l_R(x)$ is a maximal left ideal of R. Thus $K = Rx \cong I$ is projective. Therefore M is a PS module.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2) Let Soc$({}_RM) = \oplus K_i$ with each K_i simple. By (3), Soc$({}_RR)(\oplus K_i) = \oplus K_i$, and so $\oplus$(Soc$({}_RR)K_i) = \oplus K_i$. It is easy to check that each Soc$({}_RR)K_i \neq 0$. Thus Soc$({}_RR)K_i = K_i$.

(1) $\Rightarrow$ (4) Let L be an essential maximal left ideal of R. By [20, Proposition 1], $r_M(L) = 0$ or $L = Rf$ with $f^2 = f \in R$. But it is impossible that $L = Rf$. So $r_M(L) = 0$.

(4) $\Rightarrow$ (5) follows from the fact that a maximal left ideal is either essential or a direct summand.

(5) $\Rightarrow$ (1) Let L be a maximal left ideal of R. Then $r_M(L) = eM$ with $e^2 = e \in R$ by (5). If there exists $0 \neq x \in r_M(L)$, we claim that $1 - e \in L$. For if not, $R = L + R(1 - e)$ since L is maximal. So $Rx = 0$, a contradiction. Thus $R(1 - e) \subseteq L$. On the other hand, since M is faithful, $Le = 0$, and so $L = L(1-e) \subseteq R(1-e)$. Therefore $L = R(1-e)$, and hence M is a PS module by [20, Proposition 1]. ∎

Corollary 2.11. *R is a left PS ring if and only if $(\text{Soc}({}_RR))^2 = \text{Soc}({}_RR)$. In this case, M is a PS left R-module if and only if $\text{Soc}({}_RR)M = \text{Soc}(M)$.*

Proof. The first statement is an immediate consequence of Theorem 2.10.

Now let M be a PS left R-module. By Theorem 2.10, we have

$$\text{Soc}(M) = \text{Soc}({}_RR)\text{Soc}(M) \subseteq \text{Soc}({}_RR)M.$$

In addition, $\text{Soc}({}_RR)M \subseteq \text{Soc}(M)$ by [8, Exercise §6.12 (1)]. It follows that $\text{Soc}({}_RR)M = \text{Soc}(M)$.

Conversely, we assume $\text{Soc}({}_RR)M = \text{Soc}(M)$. Since $(\text{Soc}({}_RR))^2 = \text{Soc}({}_RR)$,

$$\text{Soc}({}_RR)\text{Soc}(M) = \text{Soc}({}_RR)(\text{Soc}({}_RR)M) = \text{Soc}({}_RR)M = \text{Soc}(M).$$

Thus M is a PS left R-module by Theorem 2.10. ∎

Proposition 2.12. *Let R be a commutative ring. The following conditions are equivalent for a projective R-module M:*

(1) *M is a PS module.*

(2) *M is an FS module.*

(3) *Every simple submodule of M is injective.*

(4) *Every simple submodule of M is a direct summand of M.*

Proof. It is straightforward by the fact that a simple R-module N is injective if and only if N is flat (see [17, Lemma 2.6]). ∎

3. M-Min-flat and M-Min-injective Modules

To obtain more properties of min-coherent, PS and FS modules, in this section, we introduce and study M-min-flat and M-min-injective modules.

Definition 3.1. Let M be a left R-module. A right R-module N is said to be *M-min-flat* if the sequence $0 \to N \otimes K \to N \otimes M$ is exact for any simple submodule K of M.

A left R-module Q is called *M-min-injective* if every homomorphism from any simple submodule K of M to Q extends to one from M to Q.

Obviously, the concept of M-min-flat (resp. M-min-injective) modules is a generalization of M-flat (resp. M-injective) modules.

The following lemmas are needed in the sequel.

Lemma 3.2. *Let M be a left R-module. Then a right R-module N is M-min-flat if and only if N^+ is M-min-injective.*

Proof. Let K be a simple submodule of M. Then the sequence $0 \to N \otimes K \to N \otimes M$ is exact if and only if the sequence $(N \otimes M)^+ \to (N \otimes K)^+ \to 0$ is exact if and only if the sequence $\text{Hom}(M, N^+) \to \text{Hom}(K, N^+) \to 0$ is exact. So N is M-min-flat if and only if N^+ is M-min-injective. ■

Let $\mathcal{C}$ be a class of modules and N a module. Following [3], a homomorphism $\phi : N \to F$ with $F \in \mathcal{C}$ is called a *$\mathcal{C}$-preenvelope of N* if for any homomorphism f: $N \to F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Dually we have the definition of a $\mathcal{C}$-precover.

Lemma 3.3. *Let M be a left R-module.*

(1) *The class of M-min-injective left R-modules is closed under direct summands, direct sums and direct products.*

(2) *The class of M-min-flat right R-modules is closed under pure submodules, pure quotient modules, direct summands, direct limits and direct sums. Consequently, every right R-module has an M-min-flat precover.*

Proof.

(1) is easy by definition.

(2) We will prove that the class of M-min-flat right R-modules is closed under pure submodules and pure quotient modules. The rest are clear.

Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of right R-modules with B M-min-flat. Then we get the split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By Lemma 3.2, B^+ is M-min-injective. Thus A^+ and C^+ are M-min-injective by (1). So A and C are M-min-flat by Lemma 3.2 again.

By [7, Theorem 2.5], every right R-module has an M-min-flat precover. ■

Lemma 3.4. *Let M be a min-coherent left R-module.*

(1) *The class of M-min-flat right R-modules is closed under direct products.*

(2) *Every right R-module has an M-min-flat preenvelope.*

(3) *If a left R-module N is M-min-injective, then N^+ is M-min-flat.*

Proof. (1) Let $\{N_i\}$ be a family of M-min-flat right R-modules and K be any simple submodule of M. Then we have the following commutative diagram:

$$\begin{array}{ccc} (\prod N_i) \otimes K & \xrightarrow{\gamma} & (\prod N_i) \otimes M \\ \alpha \downarrow & & \downarrow \\ \prod (N_i \otimes K) & \xrightarrow{\beta} & \prod (N_i \otimes M). \end{array}$$

By [4, Theorem 3.2.22], α is an isomorphism since K is finitely presented. Thus γ is monic since β is monic. So $\prod N_i$ is M-min-flat.

(2) The result is a consequence of (1), Lemma 3.3 (2) and [14, Theorem 3.3].

(3) Let K be any simple submodule of M. Then we have the following commutative diagram:

$$\begin{array}{ccc} N^+ \otimes K & \xrightarrow{f} & N^+ \otimes M \\ \sigma_K \downarrow & & \downarrow \sigma_M \\ \mathrm{Hom}(K, N)^+ & \xrightarrow{g} & \mathrm{Hom}(M, N)^+. \end{array}$$

Since K is finitely presented, σ_K is an isomorphism by [15, Lemma 3.60]. Since $\mathrm{Hom}(M, N) \to \mathrm{Hom}(K, N)$ is epic, g is monic. Thus f is a monomorphism, and so (3) follows. ■

Next we characterize min-coherent modules in terms of M-min-flat and M-min-injective modules.

Theorem 3.5. *The following conditions are equivalent for a finitely presented left R-module M:*

(1) *M is a min-coherent module.*

(2) *The class of M-min-flat right R-modules is closed under direct products.*

(3) *Any direct product of copies of R_R is M-min-flat.*

(4) *Every right R-module has an M-min-flat preenvelope.*

(5) *A left R-module N is M-min-injective if and only if N^+ is M-min-flat.*

(6) *The class of M-min-injective left R-modules is closed under direct limits.*

Proof.

(1) $\Rightarrow (2)$ follows from Lemma 3.4 (1). (2) $\Rightarrow (3)$ is trivial.

(3) $\Rightarrow (1)$ Let K be any simple submodule of M. Then we have the following commutative diagram:

$$\begin{array}{ccc} (\prod R_R) \otimes K & \xrightarrow{\gamma} & (\prod R_R) \otimes M \\ \alpha \downarrow & & \downarrow \beta \\ \prod K & \longrightarrow & \prod M. \end{array}$$

Since M is finitely presented, β is an isomorphism by [4, Theorem 3.2.22]. Since γ is a monomorphism by (3), α is monic. But α is also epic by [4, Lemma 3.2.21]. Thus K is finitely presented by [4, Theorem 3.2.22] again. Hence M is min-coherent.

(2) $\Leftrightarrow (4)$ follows from Lemma 3.3 (2) and [14, Theorem 3.3].

(1) $\Rightarrow (5)$ By Lemma 3.4 (3), it is enough to show that N is M-min-injective if N^+ is M-min-flat. Let K be any simple submodule of M. Then we have the following commutative diagram:

$$\begin{array}{ccc} N^+ \otimes K & \xrightarrow{f} & N^+ \otimes M \\ \downarrow{\scriptstyle \sigma_K} & & \downarrow{\scriptstyle \sigma_M} \\ \mathrm{Hom}(K, N)^+ & \xrightarrow{g} & \mathrm{Hom}(M, N)^+. \end{array}$$

Since K and M are finitely presented, σ_K and σ_M are isomorphisms by [15, Lemma 3.60]. Since f is a monomorphism, g is a monomorphism. Thus $\mathrm{Hom}(M, N) \to \mathrm{Hom}(K, N)$ is an epimorphism, and so N is M-min-injective.

(5) $\Rightarrow (6)$ Let K be any simple submodule of M and $\{N_i : i \in J\}$ a family of M-min-injective left R-modules, where J is a directed set. Then by [18, 33.9], we get the pure exact sequence $0 \to A \to \oplus N_i \to \varinjlim N_i \to 0$, which gives rise to the split exact sequence

$$0 \to (\varinjlim N_i)^+ \to (\oplus N_i)^+ \to A^+ \to 0.$$

Since $\oplus N_i$ is M-min-injective, $(\oplus N_i)^+$ is M-min-flat by (5). Hence $(\varinjlim N_i)^+$ is M-min-flat. So $\varinjlim N_i$ is M-min-injective by (5) again.

(6) $\Rightarrow (1)$ Let K be any simple submodule of M and $\{N_i : i \in J\}$ be a family of M-min-injective left R-modules, where J is a directed set. Then $\varinjlim N_i$ is M-min-injective by (6). Thus we have the following commutative diagram:

$$\begin{array}{ccc} \varinjlim \mathrm{Hom}(M, N_i) & \longrightarrow & \varinjlim \mathrm{Hom}(K, N_i) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \mathrm{Hom}(M, \varinjlim N_i) & \xrightarrow{\alpha} & \mathrm{Hom}(K, \varinjlim N_i). \end{array}$$

Since α is epic and β is an isomorphism by [18, 25.4], γ is epic. But γ is also monic by [18, 24.9]. So K is finitely presented by [18, 25.4] again. Thus M is min-coherent. ■

Remark 3.6. The hypothesis "M is finitely presented" in Theorem 3.5 is not superfluous. In fact, M = Soc(R) in Example 2.9 is not a min-coherent R-module.

But every R-module is both M-min-flat and M-min-injective since M is semisimple.

Corollary 3.7. *The following conditions are equivalent for a finitely presented left R-module M:*

(1) *Every simple submodule of M is a direct summand of M.*

(2) *Every left R-module is M-min-injective.*

(3) *Every right R-module is M-min-flat.*

Proof. $(1) \Leftrightarrow (2)$ is easy. (2) $\Rightarrow (3)$ holds by Lemma 3.2.

(3) $\Rightarrow (2)$ Since every right R-module is M-min-flat, the equivalent conditions of Theorem 3.5 are satisfied. Let N be any left R-module. Then N^+ is M-min-flat by (3). So N is M-min-injective by Theorem 3.5 (5). ■

Lemma 3.8. *If M is a finitely presented min-coherent left R-module, then the class of M-min-injective left R-modules is closed under pure submodules and pure quotient modules. As a consequence, every left R-module has an M-min-injective precover.*

Proof. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of left R-modules with B M-min-injective. Then we get the split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By Theorem 3.5, B^+ is M-min-flat. Thus A^+ and C^+ are M-min-flat. So A and C are M-min-injective. By [7, Theorem 2.5] and Lemma 3.3 (1), every left R-module has an M-min-injective precover. ■

Proposition 3.9. *The following conditions are equivalent for a finitely presented min-coherent left R-module M:*

(1) *${}_RR$ is M-min-injective.*

(2) *Every right R-module has a monic M-min-flat preenvelope.*

(3) *Every left R-module has an epic M-min-injective precover.*

Proof. (1) $\Rightarrow$ (2) Let N be any right R-module. Then N has an M-min-flat preenvelope $f : M \to F$ by Lemma 3.4 (2). Since there exists an exact sequence $0 \to M \to \Pi({}_RR)^+$, M embeds in an M-min-flat right R-module by Theorem 3.5. Thus f is monic.

(2) $\Rightarrow$ (1) By (2), the injective right R-module $({}_RR)^+$ is M-min-flat. So ${}_RR$ is M-min-injective by Theorem 3.5.

(1) $\Rightarrow$ (3) Let M be a left R-module, then M has an M-min-injective precover g by Lemma 3.8. On the other hand, there is an exact sequence $\oplus_R R \to M \to 0$. Since $\oplus_R R$ is M-min-injective by (1) and Lemma 3.3 (1), g is an epimorphism.

(3) $\Rightarrow$ (1) Let $f : N \to {}_RR$ be an epic M-min-injective precover. Then ${}_RR$ is isomorphic to a direct summand of N, and so ${}_RR$ is M-min-injective. ■

Finally, we give some new characterizations of FS and PS modules.

Theorem 3.10. *If M is an FS left R-module, then the class of M-min-flat right R-modules is closed under submodules. The converse holds if M is flat.*

Proof. Let A be a submodule of an M-min-flat right R-module B and K be a simple submodule of M. Then we have the following commutative diagram:

$$\begin{array}{ccc} A \otimes K & \xrightarrow{\gamma} & A \otimes M \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ B \otimes K & \xrightarrow{\beta} & B \otimes M. \end{array}$$

Since K is flat and B is M-min-flat, α and β are monomorphisms, and so γ is a monomorphism. Thus A is M-min-flat.

Conversely, assume that every submodule of any M-min-flat right R-module is M-min-flat and M is flat. Let K be a simple submodule of M and I a right ideal of R. Then we have the following commutative diagram:

$$\begin{array}{ccc} I \otimes K & \xrightarrow{f} & R \otimes K \\ {\scriptstyle g}\downarrow & & \downarrow \\ I \otimes M & \xrightarrow{h} & R \otimes M. \end{array}$$

Since I is M-min-flat and M is flat, g and h are monomorphisms. Thus f is monic and so K is flat. ■

Theorem 3.11. *Consider the following conditions for a min-coherent left R-module M:*

(1) *M is a PS module.*

(2) *M is an FS module.*

(3) *Every right R-module has an epic M-min-flat preenvelope.*

Then $(1) \Leftrightarrow (2) \Rightarrow (3)$. If M is flat, then (3) $\Rightarrow$ (2).

Proof. (1) $\Leftrightarrow$ (2) is clear.

$(2) \Rightarrow$ (3) For any right R-module N, there is an M-min-flat preenvelope $f : N \to F$ by Lemma 3.4 (2). Note that $\mathrm{im}(f)$ is M-min-flat by Theorem 3.10, and so $N \to \mathrm{im}(f)$ is an epic M-min-flat preenvelope.

$(3) \Rightarrow$ (2) Let A be any submodule of an M-min-flat right R-module B. Since A has an epic M-min-flat preenvelope by (3), A is M-min-flat. So M is an FS module by Theorem 3.10. ■

Theorem 3.12. *Consider the following conditions for a left R-module M:*

(1) *M is a PS module.*

(2) *The class of M-min-injective left R-modules is closed under quotient modules.*

(3) *Every left R-module has a monic M-min-injective precover.*

Then $(1) \Rightarrow (2) \Leftrightarrow (3)$. If M is projective, then (2) $\Rightarrow$ (1).

Proof. $(1) \Rightarrow (2)$ Let X be any M-min-injective left R-module and N any submodule of X. We will show that X/N is M-min-injective. Let K be a simple submodule of M, $i : K \to M$ the inclusion and $\pi : X \to X/N$ the canonical map. For any $f : K \to X/N$, there exists $g : K \to X$ such that $\pi g = f$ since K is projective by (1). Hence there is $h : M \to X$ such that $hi = g$ since X is M-min-injective. It follows that $(\pi h)i = f$, and so X/N is M-min-injective.

$(2) \Leftrightarrow (3)$ holds by [5, Proposition 4] and Lemma 3.3 (1).

$(2) \Rightarrow$ (1) Let N be a submodule of an injective left R-module E and $\pi : E \to E/N$ the canonical map. Suppose that K is a simple submodule of M, and $f : K \to E/N$ is any homomorphism. Since E/N is M-min-injective by (2), there exists $g : M \to E/N$ such that $f = g\iota$ where $\iota : K \to M$ is the inclusion. Since M is projective, there exists $h : M \to E$ such that $g = \pi h$. Hence $f = (\pi h)\iota = \pi(h\iota)$ and so K is projective by [15, Lemma 4.22]. Thus M is a PS module. ∎

ACKNOWLEDGMENTS

This research was supported by NSFC (No. 11071111), Jiangsu Six Major Talents Peak Project, NSF of Jiangsu Province of China (No. BK2008365), Jiangsu 333 Project, Jiangsu Qinglan Project, and Research Fund of Nanjing Institute of Technology (No. CKJ2009009). The author would like to thank the referee for the very helpful comments and suggestions.

REFERENCES

1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.
2. V. Camillo, Coherence for polynomial rings, *J. Algebra*, **132** (1990), 72-76.
3. E. E. Enochs, Injective and flat covers, envelopes and resolvents, *Israel J. Math.* **39** (1981), 189-209.
4. E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, Berlin-New York, 2000.
5. J. R. García Rozas and B. Torrecillas, Relative injective covers, *Comm. Algebra*, **22** (1994), 2925-2940.

6. S. Glaz, *Commutative Coherent Rings*, Lecture Notes in Math. 1371, Springer-Verlag, 1989.

7. H. Holm and P. Jørgensen, Covers, precovers, and purity, *Illinois J. Math.*, **52** (2008), 691-703.

8. T. Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag, New York-Heidelberg-Berlin, 1999.

9. Z. K. Liu, Rings with flat left socle, *Comm. Algebra*, **23** (1995), 1645-1656.

10. Z. K. Liu, PS modules over rings of generalized power series, *Northeast. Math. J.*, **18(3)** (2002), 254-260.

11. L. X. Mao, Min-flat modules and min-coherent rings, *Comm. Algebra*, **35(2)** (2007), 635-650.

12. L. X. Mao, On mininjective and min-flat modules, *Publ. Math. Debrecen*, **72(3-4)** (2008), 347-358.

13. W. K. Nicholson and J. F. Watters, Rings with projective socle, *Proc. Amer. Math. Soc.*, **102** (1988), 443-450.

14. J. Rada and M. Saorin, Rings characterized by (pre)envelopes and (pre)covers of their modules, *Comm. Algebra*, **26(3)** (1998), 899-912.

15. J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.

16. D. G. Wang, Modules with flat socle and almost excellent extensions, *Acta Math. Vietnam*, **21** (1996), 295-301.

17. R. Ware, Endomorphism rings of projective modules, *Trans. Amer. Math. Soc.*, **155** (1971), 233-256.

18. R. Wisbauer, *Foundations of Module and Ring Theory*, Philadelphia, Gordon and Breach, 1991.

19. Y. F. Xiao, Rings with flat socles, *Proc. Amer. Math. Soc.*, **123** (1995), 2391-2395.

20. W. M. Xue, Modules with projective socle, *Riv. Mat. Univ.*, **5** (1992), 311-315.

21. X. X. Zhang and J. L. Chen, A note on relative flatness and coherence, *Comm. Algebra*, **35** (2007), 3321-3330.

Lixin Mao
Institute of Mathematics
Nanjing Institute of Technology
Nanjing 211167
P. R. China
E-mail: maolx2@hotmail.com