CUBIC FORMAL POWER SERIES IN CHARACTERISTIC 2 WITH UNBOUNDED PARTIAL QUOTIENTS

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Abstract. There is a theory of continued fractions for formal power series in x^{-1} with coefficients in a field $\mathbb{F}_q$. This theory bears a close analogy with classical continued fractions for real numbers with formal power series playing the role of real numbers and the sum of the terms of non-negative degree in x playing the role of the integral part. We give a family of cubic power series over $\mathbb{F}_2$ with unbounded partial quotients. To be more precise, let $f \in \mathbb{F}_2((x^{-1}))$ such that f is not polynomial but f^3 is polynomial with degree $d \in 3\mathbb{N}$, we prove that the continued fraction expansion of f is unbounded.

1. INTRODUCTION

In [2], Khintchine conjectured that if x is a real algebraic number of degree > 2 then x has a continued fraction expansion whose the sequence of partial quotients is unbounded. Note that the answer to this conjecture is far from being tractable. However, more things are known in the case of algebraic formal series over a finite field $\mathbb{F}_k$ of characteristic p.

In 1976, Baum and Sweet gave, in [1], the first example of an algebraic formal series of degree 3 on $\mathbb{F}_2((x^{-1}))$ whose partial quotients have only a finite number of values as well as examples whose partial quotients have infinite values. In [4], Mills and Robbins continued the work of Baum and Sweet and they gave an example of formal series algebraic over $\mathbb{F}_3((x^{-1}))$ whose the sequence of partial quotients is unbounded and given explicitly. Moreover, Robbins gave a family of cubic formal power series with bounded partial quotients [5].

In this paper, we give a family of cubic formal power series with unbounded partial quotients. Namely, we prove that, if f^3 is polynomial with degree $d \in 3\mathbb{N}$, then the continued fraction expansion of f is unbounded.

2. Field of Formal Series

Let $\mathbb{F}_k$ be a finite field with k elements. We denote by $\mathbb{F}_k[x]$ the ring of polynomials with coefficients in $\mathbb{F}_k$ and $\mathbb{F}_k(x)$ its fractions field. Let $\mathbb{F}_k((x^{-1}))$ be the field of formal power series:

$$\mathbb{F}_k((x^{-1})) = \left\{ f = \sum_{i \geq n_0} f_i x^{-i} \quad : \quad n_0 \in \mathbb{Z} \quad \text{and} \quad f_i \in \mathbb{F}_k \right\}.$$

Let $f = \sum f_i x^{-i}$ be any formal power series, we define its polynomial part, denoted $[f]$, by $[f] := \sum_{i \leq 0} f_i x^{-i}$. If $f \neq 0$, then the degree of f is $\gamma(f) = \inf\{i : f_i \neq 0\}$ and $\gamma(0) = +\infty$. Thus, we define $|f| = k^{-\gamma(f)}$. Note that $|.|$ is a non archimedean absolute value over $\mathbb{F}_k((x^{-1}))$.

Now, similarly to the classical continued fractions for real numbers, we define the continued fractions for power series. Let us consider the set $M_k = \{f \in \mathbb{F}_k((x^{-1})) : |f| < 1\}$ and consider the transformation

$$T : M_k \to M_k, \, f \mapsto \frac{1}{f} - \left[\frac{1}{f}\right].$$

For any $f \in \mathbb{F}_k((x^{-1}))$ we define a polynomial sequence $(a_n)_{n \geq 0}$ by

$$a_0 = [f] \quad \text{and, for } n \geq 1, \quad a_n = \left[\frac{1}{T^{n-1}(f - [f])}\right].$$

We easily check that

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

This expression is called the continued fraction expansion of f and it will be simply denoted by $f = [a_0, a_1, a_2, \ldots]$. The sequence $(a_i)_{i \geq 0}$ is called the sequence of partial quotients of f. If it is bounded, we say that f admits bounded continued fraction expansion.

Besides, we define two sequences of polynomials (p_n) and (q_n) by

$$p_0 = a_0, \quad q_0 = 1, \quad p_1 = a_0 a_1 + 1, \quad q_1 = a_1$$

and for any $n \geq 2$
$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

By a straightforward computation, we can see that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \quad \text{and} \quad \frac{p_n}{q_n} = [a_0, a_1, a_2, \ldots, a_n].$$

The sequence $\left(\dfrac{p_n}{q_n}\right)$ is called the sequence of convergent of f.

We will need the following two lemmas given by Baum and Sweet in [1]:

Lemma 2.1. *Consider* $p, q \in \mathbb{F}_2[x]$ *with* $\gcd(p, q) = 1$.

a. *If* $|qf - p| = 2^{-(\gamma(q)+d)}$, $d > 0$, *then, there exists* $n \in \mathbb{N}$ *such that*

$$q = q_n, \quad p = p_n \quad and \quad \gamma(a_{n+1}) = d.$$

b. *If* $|qf - p| = \dfrac{1}{|q|}$, *then, there exists* $n \in \mathbb{N}$ *such that*

$$p = p_n + p_{n-1} \quad and \quad q = q_n + q_{n-1}.$$

Lemma 2.2. *Let* $g_1, g_2, h_1, h_2 \in \mathbb{F}_2[x]$ *with* $g_1 h_2 + g_2 h_1 \neq 0$. *Then, any* $f \in \mathbb{F}_2((x^{-1}))$ *has bounded partial quotients if and only if* $\dfrac{g_1 f + h_1}{g_2 f + h_2}$ *has bounded partial quotients.*

In 1976, Baum and Sweet gave in [1] some examples of formal series whose the sequence of partial quotients is unbounded. They have proved that the formal series $\left(\dfrac{P}{P+1}\right)^{\frac{1}{2^n-1}}$ admits unbounded partial quotients which are given explicitly .

3. Results

Before giving the mains theorems, we introduce the following two lemmas.

Lemma 3.1. *Let* $P \in \mathbb{F}_2[x]$ *such that the degree of* P *is a multiple of 3, then there is a unique* $f \in \mathbb{F}_2((x^{-1}))$ *such that* $f^3 = P$.

Proof. The uniqueness of f is obvious. Let $P = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ in which n is a multiple of 3. We seek $f = \sum_{i \geq n_0} f_i x^{-i}$ in $\mathbb{F}_2((x^{-1}))$ such that $f^3 = P$ and $f_{n_0} \neq 0$. It is sufficient to determine the f_i step by step. The equality $f^3 = P$ yields first $n = -3n_0$. Moreover, by identification, we get

$$f_{n_0} = 1, \quad f_{n_0+1} = a_{n-1}, \quad f_{n_0+2} = a_{n-2} - a_{n-1}^2, \quad f_{n_0+3} = a_{n-3} - a_{n-1}^3, \ldots \quad \blacksquare$$

Lemma 3.2. *Let* $P \in \mathbb{F}_2[x]$ *such that* $\gamma(P) \in 3\mathbb{N}$. *Then the polynomial* P *is uniquely expressible as* $P = S^3 + T$, *where* $S, T \in \mathbb{F}_2[x]$ *with* $\gamma(T) < 2\gamma(S)$.

Proof. Write $P = x^{3m} + a_{3m-1}x^{3m-1} + \cdots + a_{2m}x^{2m} + a_{2m-1}x^{2m-1} + \cdots + a_0$. From the equality $P = S^3 + T$, together with the condition $\gamma(T) < 2\gamma(S)$, we deduce that $\gamma(S) = m$. Moreover, $S^3 = x^{3m} + a_{3m-1}x^{3m-1} + \cdots + a_{2m}x^{2m} + H$ where $\gamma(H) < 2k$. Thus, we write $S = \sum_{i=0}^{m} b_i x^i \in \mathbb{F}_2[x]$, then, as in the previous lemma, we compute recursively the coefficients $b_m, b_{m-1}, \ldots$ and then we get $P = S^3 + T$ with $\gamma(T) < 2\gamma(S)$. $\blacksquare$

Theorem 3.3. *Let $P = S^3 + T$ with $S, T \in \mathbb{F}_2[x]$, such that T divides P and $\gamma(T) < 2\gamma(S)$. Then the formal power series g such that $g^3 = P$ admits unbounded partial quotients.*

Proof. We follows [1] fairly closely. Consider $f = \prod_{i \geq 0} \left(1 + R^{-1}\right)^{4^i}$ where $R = T^{-1}P$. It is clear that $R \in \mathbb{F}_2[x]$ and $|R| > 1$. We have

$$f^3 = \prod_{i \geq 0} \left(1 + R^{-1}\right)^{3 \times 4^i} = \lim_{m \to +\infty} \left(1 + R^{-1}\right)^{3 \times \sum_{i=0}^{m-1} 4^i} = \lim_{m \to +\infty} \left(1 + R^{-1}\right)^{4^m - 1}$$

$$= \left(1 + R^{-1}\right)^{-1} \lim_{m \to +\infty} \left(1 + R^{-1}\right)^{4^m}.$$

Since $\mathbb{F}_2((x^{-1}))$ is of characteristic 2, we have then $f^3 = (1 + R^{-1})^{-1}$ $\left(1 + \lim_{m \to +\infty} R^{-4^m}\right)$. So, $f^3 = (1 + R^{-1})^{-1} = \dfrac{1}{1 + R^{-1}} = \dfrac{P}{P + T}$, then $f = \left(\dfrac{P}{P + T}\right)^{\frac{1}{3}} \in \mathbb{F}_2((x^{-1}))$.

Let $d_m = \dfrac{(4^m - 1)}{3} = \displaystyle\sum_{i=0}^{m-1} 4^i$. Then

$$(1 + R)^{d_m} = (1 + R)^{\sum_{i=0}^{m-1} 4^i} = \prod_{i=0}^{m-1} (1 + R)^{4^i}$$

$$= \prod_{i=0}^{m-1} R^{4^i} \left(1 + R^{-1}\right)^{4^i} = R^{d_m} \prod_{i=0}^{m-1} \left(1 + R^{-1}\right)^{4^i}.$$

Thus, we get

$$R^{d_m} f = R^{d_m} \prod_{i \geq 0} \left(1 + R^{-1}\right)^{4^i} = R^{d_m} \prod_{0 \leq i < m} \left(1 + R^{-1}\right)^{4^i} \prod_{i \geq m} \left(1 + R^{-1}\right)^{4^i}$$

$$= (1 + R)^{d_m} \prod_{i \geq m} \left(1 + R^{-1}\right)^{4^i} = (1 + R)^{d_m} + (1 + R)^{d_m} (R^{-1})^{4^m} + K.$$

Therefore,

$$R^{d_m} f - (1 + R)^{d_m} = (1 + R)^{d_m} (R^{-1})^{4^m} + K.$$

Moreover, we have $|K| < |(1 + R)^{d_m} (R^{-1})^{4^m}| = |R^{d_m} (R^{-1})^{4^m}|$ since $|(1 + R)^{d_m}| = |R^{d_m}|$. So, we get

$$(3.1) \qquad |R^{d_m} f - (1 + R)^{d_m}| = |R^{-(4^m - d_m)}|$$

and then we have

$$\left| R^{d_m} f - (1+R)^{d_m} \right| = 2^{-(\gamma(R)d_m + \gamma(R)4^m - 2\gamma(R)d_m)}.$$

We have $\gcd(R^{d_m}, (1+R)^{d_m}) = 1$, moreover, $4^m \gamma(R) - 2\gamma(R)d_m > 0$, then according to Lemma 2.1, there exists i such that $\gamma(a_{i+1}) = 4^m \gamma(R) - 2\gamma(R)d_m$. Obviously $\gamma(a_{i+1})$ is unbounded, so, we conclude that the formal series $f = \left(\dfrac{P}{P+T} \right)^{\frac{1}{3}} = \dfrac{1}{S}(S^3 + T)^{\frac{1}{3}}$ admits unbounded partial quotients, but $S \in \mathbb{F}_2[x]$, then, by Lemma 2.2, $(S^3 + T)^{\frac{1}{3}}$ admits unbounded partial quotients. ∎

Theorem 3.4. *Let S, $T \in \mathbb{F}_2[x]$ such that $\gamma(T) < \gamma(S)$. Then the formal series g such that $g^3 = S^3 + T$ admits unbounded partial quotients.*

Proof. Let $P = S^3 + T$. We begin the proof as in Theorem 3.3 until equality (3.1). Besides, multiplying (3.1) by T^{d_m}, we obtain

$$\left| P^{d_m} f - (T+P)^{d_m} \right| = \left| T^{d_m} (T^{-1}P)^{-(4^m - d_m)} \right| = 2^{-(\beta d_m + (\beta 4^m - 2\beta d_m - \alpha 4^m))}.$$

where $\beta = \gamma(P)$ and $\alpha = \gamma(T)$. We have to distinguish two cases:

(i) $\gcd(P, T) = 1$: since $\deg \gamma(T) < \frac{1}{3}\gamma(P)$ then $\beta 4^m - 2\beta d_m - \alpha 4^m > 0$. Moreover, $\gcd(P^{d_m}, (T+P)^{d_m}) = 1$, then by Lemma 2.1, there exists i such that $\gamma(a_{i+1}) = \beta 4^m - 2\beta d_m - \alpha 4^m$. It is clear that $\gamma(a_{i+1})$ is unbounded.

(ii) $\gcd(P, T) = h \neq 1$: there exists P' and $T' \in \mathbb{F}_2[x]$ such that $P = hP'$ and $T = hT'$ with $\gcd(P', T') = 1$. It is easy to verify that $\gamma(T') < \frac{1}{3}\gamma(P')$, and then the result is obvious by the first case.

So, in these two cases, $\left(\dfrac{P}{P+T} \right)^{\frac{1}{3}}$ admits unbounded partial quotients, then, as before, $(S^3 + T)^{\frac{1}{3}}$ admits unbounded partial quotients . ∎

Example 1. Let $f^3 = S^3 + 1$ with $\gamma(S) > 0$, then, by Theorem 3.3, the partial quotients of f are unbounded.

Example 2. Let $f^3 = S^3 + x$ with $\gamma(S) > 1$, then the partial quotients of f are unbounded, by Theorem 3.3 if $S(0) = 0$ and by Theorem 3.4 if $S(0) \neq 0$.

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