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INFINITELY MANY SOLUTIONS FOR A CLASS OF DEGENERATE ANISOTROPIC ELLIPTIC PROBLEMS WITH VARIABLE EXPONENT

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Abstract. We study the nonlinear degenerate problem $-\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right) = f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right)$ is a $\overrightarrow{p}(\cdot)$ - Laplace type operator and the nonlinearity f is $(P_+^+ - 1)$ - superlinear at infinity (with $\overrightarrow{p}(x) = (p_1(x), p_2(x), ...p_N(x))$ and $P_+^+ = \max_{i\in\{1,\dots,N\}} \{\sup_{x\in\Omega} p_i(x)\}$). By means of the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz, we establish the existence of a sequence of weak solutions in appropriate anisotropic variable exponent Sobolev spaces.

1. Introduction and Statement of the Main Result

Our study is conducted in the framework of the anisotropic variable exponent Lebesgue-Sobolev spaces. In the domain of PDEs, characterized by Brézis and Browder [5] as "the major bridge between central issues of applied mathematics and physical sciences on the one hand and the central development of mathematical ideas in active areas of pure mathematics on the other", the theory of anisotropic variable exponent Lebesgue-Sobolev spaces is a bridge itself. Indeed, it is a bridge between the anisotropic Sobolev spaces theory developed by [31, 34, 35, 42, 43] and the variable exponent Sobolev spaces theory developed by [8, 9, 10, 11, 12, 21, 28, 29, 30, 38]. This way, under our dazzled eyes, a delta is born, with new forms of life, or, more exactly, since we refer to a delta of mathematics, with new articles [2, 3, 4, 13, 19, 20, 25, 26, 27]. This state of fact is no surprise, since we know that there are some materials with inhomogeneities for the study of which we can not use the classical Lebesgue-Sobolev spaces L^p and $W^{1,p}$ and we should let the exponent p to vary instead (the need for the variable exponent spaces is confirmed by the large scale of applications in elastic mechanics [45], in the mathematical modeling

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of non-Newtonian fluids [7, 15, 32, 36, 37, 39, 40, 41, 44] and in image restoration [6]). But what happens when we want to consider materials with inhomogeneities that have different behavior on different space directions? Well, in this case we should work on the anisotropic variable exponent Lebesgue-Sobolev spaces $L^{\overrightarrow{p}(\cdot)}$ and $W^{1,\overrightarrow{p}(\cdot)}$, where $\overrightarrow{p}$ verifies the following condition:

(p) $\overrightarrow{p}(x) = (p_1(x), p_2(x), ...p_N(x))$ *and* p_i, $i \in \{1, ..., N\}$, *are continuous functions such that* $1 < p_i(x) < N$ *and* $\sum_{i=1}^N 1/\inf_x p_i(x) > 1$ *for all* x.

In the context of these spaces that will be carefully described in the next section, we are interested in discussing the following problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary and a_i, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Let us denote by A_i, $F : \Omega \times \mathbb{R} \to \mathbb{R}$

$$A_i(x,s) = \int_0^s a_i(x,t)dt \quad \text{for all } i \in \{1, ..., N\},$$

respectively

$$F(x,s) = \int_0^s f(x,t)dt.$$

We set $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1\}$ and we denote, for any $p \in C_+(\overline{\Omega})$,

$$p^+ = \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\Omega} p(x).$$

We denote by $\overrightarrow{P}_+$, $\overrightarrow{P}_- \in \mathbb{R}^N$ the vectors

$$\overrightarrow{P}_+ = (p_1^+, ..., p_N^+), \quad \overrightarrow{P}_- = (p_1^-, ..., p_N^-),$$

and by P_+^+, P_-^+, $P_-^- \in \mathbb{R}^+$ the following:

$$P_+^+ = \max\{p_1^+, ..., p_N^+\}, \quad P_-^+ = \max\{p_1^-, ..., p_N^-\}, \quad P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

We define $P_-^\star \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^\star = \frac{N}{\displaystyle\sum_{i=1}^N 1/p_i^- - 1}, \qquad P_{-,\infty} = \max\{P_-^+, P_-^\star\}\,.$$

The goal of this paper is to prove the following theorem.

Theorem 1. *Suppose that $\overrightarrow{p}$ verifies (p) and, for all $i \in \{1, ..., N\}$, the functions $A_i,\ a_i,\ f$ fulfil the conditions:*

$(A1)$ *A_i is even in s, that is, $A_i(x, -s) = A_i(x, s)$ for all $x \in \Omega$;*

$(A2)$ *there exists a positive constant $c_{1,i}$ such that a_i satisfies the growth condition*

$$|a_i(x, s)| \leq c_{1,i}(1 + |s|^{p_i(x)-1}),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

$(A3)$ *a_i is strictly monotone, that is,*

$$(a_i(x, s) - a_i(x, t))(s - t) > 0,$$

for all $x \in \Omega$ and $s,\ t \in \mathbb{R}$ with $s \neq t$;

$(A4)$ *the following inequalities hold:*

$$|s|^{p_i(x)} \leq a_i(x, s)s \leq p_i(x)\ A_i(x, s),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

$(f1)$ *f is odd in s, that is, $f(x, -s) = -f(x, s)$ for all $x \in \Omega$;*

$(f2)$ *there exist a positive constant c_2 and $q \in C(\overline{\Omega})$ with $1 < P_-^- < P_+^+ < q^- < q^+ < P_-^\star$, such that f satisfies the growth condition*

$$|f(x, s)| \leq c_2|s|^{q(x)-1},$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

$(f3)$ *f verifies the Ambrosetti-Rabinowitz type condition: there exists a constant $\mu > P_+^+$ such that for every $x \in \Omega$*

$$0 < \mu F(x, s) \leq sf(x, s), \quad \forall s > 0.$$

Then problem (1) admits an unbounded sequence of weak solutions.

Remark 1. Since f is odd in its second variable s, we obtain that F is even in s and the relation described by *(f3)* remains valid for all $s \in \mathbb{R} \setminus \{0\}$. Moreover, by rewriting condition *(f3)*, we can obtain the existence of a positive constant c_3 such that

$$F(x, s) \geq c_3\,|s|^\mu, \quad \forall x \in \Omega,\ \forall s \in \mathbb{R}$$

and we can deduce that f is $(P_+^+ - 1)$ - superlinear at infinity:

$$|f(x, s)| \geq c_4\,|s|^{\mu-1} \quad \forall x \in \Omega,\ \forall s \in \mathbb{R},$$

where c_4 is a positive constant.

As for the conditions imposed on A_i and a_i, obviously they are not randomly chosen. In fact, there are already studies where we can find almost identical conditions and, to give some examples, we indicate [20, 25], or [22, 24, 28] if we are referring to problems of the type

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = f(x,u) & \text{for} \quad x \in \Omega, \\ u = 0 & \text{for} \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}$ verifies conditions resembling to (A1)-(A4). The preference for conditions (A1)-(A4) may be explained by giving two examples of well known operators that satisfy them:

(1) when choosing $a_i(x,s) = |s|^{p_i(x)-2}s$ for all $i \in \{1, ..., N\}$, we have $A_i(x,s) = \frac{1}{p_i(x)}|s|^{p_i(x)}$ for all $i \in \{1, ..., N\}$, and we obtain the anisotropic variable exponent Laplace operator

$$\sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \, \partial_{x_i} u \right);$$

(2) when choosing $a_i(x,s) = (1 + |s|^2)^{(p_i(x)-2)/2}s$ for all $i \in \{1, ..., N\}$, we have $A_i(x,s) = \frac{1}{p_i(x)}[(1 + |s|^2)^{p_i(x)/2} - 1]$ for all $i \in \{1, ..., N\}$, and we obtain the anisotropic variable mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i} \left[\left(1 + |\partial_{x_i} u|^2\right)^{(p_i(x)-2)/2} \partial_{x_i} u \right].$$

In the light of the above said, we point out that our problem is closely related to the problem discussed in [4],

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \, \partial_{x_i} u \right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions (f1)-(f3) and p_i are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$ for all $x \in \overline{\Omega}$ and $i \in \{1, ..., N\}$. Their main theorem also asserts the existence of an unbounded sequence of weak solutions. It is clear that our work extends this result since we can consider (2) to be a particular case of problem (1).

2. Abstract Framework

In this section we recall the definition and some important properties of the Lebesgue-Sobolev spaces mentioned above.

Everywhere below we consider $p,\ p_i \in C_+(\overline{\Omega})$ to be logarithmic Hölder continuous. The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u : u \text{ is a measurable real–valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}$$

endowed with the Luxemburg norm

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0 \ : \quad \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\,dx \le 1\right\}.$$

Notice that for p constant this norm becomes the norm

$$|u|_p = \left(\int_\Omega |u|^p\right)^{1/p},$$

that is, the norm of the classical Lebesgue space L^p.

The space $\left(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)}\right)$ has many important qualities. We remind that it is a separable and reflexive Banach space ([21, Theorem 2.5, Corollary 2.7]) and the inclusion between spaces generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous ([21, Theorem 2.8]). In addition, the following Hölder-type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \le 2|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{3}$$

holds true for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ ([21, Theorem 2.1]), where we denoted by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$ ([21, Corollary 2.7]).

Also, the function $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx,$$

which is called the $p(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space, plays a key role in handling these generalized Lebesgue spaces. We present some of its properties (see again [21]): if $u \in L^{p(\cdot)}(\Omega)$, $(u_n)_n \subset L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$, then,

$$|u|_{p(\cdot)} < 1 \ (= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1 \ (= 1; > 1) \tag{4}$$

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|^{p^-}_{p(\cdot)} \le \rho_{p(\cdot)}(u) \le |u|^{p^+}_{p(\cdot)} \tag{5}$$

$$(6) \qquad |u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}$$

$$(7) \qquad |u|_{p(\cdot)} \to 0 \ (\to \infty) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) \to 0 \ (\to \infty)$$

$$(8) \qquad \lim_{n\to\infty} |u_n - u|_{p(\cdot)} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0.$$

Let us pass now to the definition of the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$,

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \, \partial_{x_i} u \in L^{p(\cdot)}(\Omega), \ i \in \{1, 2, ...N\} \right\}$$

endowed with the norm

$$(9) \qquad \|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}.$$

$\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|\right)$ is a separable and reflexive Banach space. $W_0^{1,p(\cdot)}(\Omega)$, the Sobolev space with zero boundary values defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|$, occupies an important place in the theory of variable exponent spaces (see [16, 17]). Note that the norms

$$\|u\|_{1,p(\cdot)} = |\nabla u|_{p(\cdot)},$$

and

$$\|u\|_{p(\cdot)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p(\cdot)}$$

are equivalent to (9) in $W_0^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ is also a separable and reflexive Banach space. Moreover, if $s \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, where $p^\star(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^\star(x) = \infty$ if $p(x) \ge N$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous.

Finally we can present the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, where $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$ is the vectorial function

$$\overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot)).$$

The space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}.$$

The space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ allows the adequate treatment of the existence of the weak solutions for problem (1) and can be considered a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$. On the other hand, playing the previously announced role of "bridge", $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ can be considered a natural generalization of the classical anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1, ..., p_N)$. $W_0^{1,\overrightarrow{p}}(\Omega)$ endowed with the norm

$$\|u\|_{1,\overrightarrow{p}} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i}$$

is a reflexive Banach space for any $\overrightarrow{p} \in \mathbb{R}^N$ with $p_i > 1$ for all $i \in \{1, ..., N\}$. This result can be easily extended to $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, see [27]. Another extension was made in what concerns the embedding between $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega)$ [27, Theorem 1]: if $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\overrightarrow{p}$ verifies (p) and $q \in C(\overline{\Omega})$ verifies $1 < q(x) < P_{-,\infty}$ for all $x \in \overline{\Omega}$, then the embedding

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is continuous and compact.

3. Auxiliary Results

We denote $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ by E and we underline the fact that we work under the conditions of Theorem 1. We base the proof of Theorem 1 on the critical point theory, that is, we associate to our problem a functional energy whose critical points represent the weak solutions of the problem.

Let us start by giving the definition of the weak solution for problem (1).

Definition 1. By a weak solution to problem (1) we understand a function $u \in E$ such that

$$\int_\Omega \left[\sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} \varphi - f(x,u)\varphi \right] dx = 0,$$

for all $\varphi \in E$.

The energy functional corresponding to problem (1) is defined as $I : E \to \mathbb{R}$,

$$I(u) = \int_\Omega \sum_{i=1}^{N} A_i\left(x, \partial_{x_i} u\right) dx - \int_\Omega F(x,u) dx.$$

For all $i \in \{1, 2, ...N\}$, we denote by $J,\ J_i : E \to \mathbb{R}$ the functionals

$$J(u) = \int_\Omega \sum_{i=1}^{N} A_i\left(x, \partial_{x_i} u\right) dx \quad \text{and} \quad J_i(u) = \int_\Omega A_i\left(x, \partial_{x_i} u\right) dx.$$

We recall the following results concerning the functionals J_i.

Lemma 1. ([20, Lemma 3.4]). *For* $i \in \{1, 2, ...N\}$,

(i) *the functional* J_i *is well-defined on* E*;*

(ii) *the functional* J_i *is of class* $C^1(E, \mathbb{R})$ *and*

$$\langle J_i'(u), \varphi \rangle = \int_\Omega a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi dx,$$

for all $u, \varphi \in E$.

A simple calculus leads us to the fact that I is well-defined on E and $I \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle I'(u), \varphi \rangle = \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi dx - \int_\Omega f(x, u) \varphi dx,$$

for all $u, \varphi \in E$. It is easy to see that the critical points of I are weak solutions to (1). Therefore we are preoccupied with the existence of critical points. A major help is provided by the mini-max principles, see for example [1, 33]. Here we focus on the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz:

Theorem 2. ([18, Theorem 11.5]). *Let* X *be a real infinite dimensional Banach space and* $\Phi \in C^1(X; \mathbb{R})$ *a functional satisfying the Palais-Smale condition (that is, any sequence* $(u_n)_n \subset X$ *such that* $(\Phi(u_n))_n$ *is bounded and* $\Phi'(u_n) \to 0$ *admits a convergent subsequence). Assume that* Φ *satisfies:*

(i) $\Phi(0) = 0$ *and there are constants* $\rho, r > 0$ *such that*

$$\Phi_{|\partial B_\rho} \geq r,$$

(ii) Φ *is even, and*

(iii) *for all finite dimensional subspaces* $\widetilde{X} \subset X$ *there exists* $R = R(\widetilde{X}) > 0$ *such that*

$$\Phi(u) \leq 0 \text{ for } u \in \widetilde{X} \setminus B_R(\widetilde{X}).$$

Then Φ *possesses an unbounded sequence of critical values characterized by a mini-max argument.*

To adapt the usual variational methods described by [14, 23] so that we can work on the anisotropic variable exponent Sobolev spaces is not an easy task. Especially when we think at the fact that we inherited the variable exponent from the variable exponent spaces and, in a "world" of partial differential equations, to depend on x may be viewed as a serious "crime". In addition, by the legacy received from the anisotropic spaces, we are dealing with more than just one variable exponent since $\overrightarrow{p}(\cdot)$ is a vector having continuous functions as components. Therefore we must transform our techniques in such manner that we can succeed to overcome all the difficulties and to verify the conditions of Theorem 2. In order to do so, we need the following result, too.

Lemma 2. *The operator J' is of type (S_+) on E, that is, if $(u_n)_n \subset E$ is weakly convergent to $u \in E$ and*

$$\limsup_{n\to\infty} \langle J'(u_n), u_n - u\rangle \leq 0, \tag{10}$$

then $(u_n)_n$ converges strongly to u in E.

Proof. The idea of the proof is the same as in [24, Theorem 4.1] because our lemma extends this theorem from the case of the $p(\cdot)$ - Laplace type operators to the case of the $\overrightarrow{p}(\cdot)$ - Laplace type operators. Therefore we follow the reasoning from [24] and we use Vitali's convergence theorem in order to show that

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\, dx = 0. \tag{11}$$

Consequently, we divide our proof into two parts.

Claim 1. The sequence $\left(\sum_{i=1}^N |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\right)_n$ is uniformly integrable in Ω, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if H is a measurable subset of Ω with $\mathrm{meas(H)} \leq \delta$, where $\mathrm{meas(H)}$ denotes the Lebesgue measure of H, then

$$\int_H \sum_{i=1}^N |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\, dx \leq \varepsilon \quad \forall n \in \mathbb{N}.$$

Since for all $n \in \mathbb{N}$ and for all $x \in \Omega$ there exists $c_5 > 0$ such that

$$\sum_{i=1}^N |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)} \leq c_5 \left(\sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} + \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} \right)$$

and $|\partial_{x_i} u|^{p_i(x)} \in L^1(\Omega)$, if we prove that the sequence $\left(\sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\right)_n$ is uniformly integrable in Ω, then we have the uniform integrability of $\left(\sum_{i=1}^N |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\right)_n$. Let us show that $\left(\sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\right)_n$ is uniformly integrable.

We know that $(u_n)_n$ is weakly convergent to u and we deduce that

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u) \left(\partial_{x_i} u_n - \partial_{x_i} u\right) dx = 0.$$

From this, (10) and (A3) we get

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^N \left[a_i(x, \partial_{x_i} u_n) - a_i(x, \partial_{x_i} u)\right] \left(\partial_{x_i} u_n - \partial_{x_i} u\right) dx = 0. \tag{12}$$

The above relation assures us that for any $\varepsilon > 0$ and any measurable subset H of Ω there exists $N \in \mathbb{N}$ such that

$$\int_H \sum_{i=1}^{N} \left[a_i(x, \partial_{x_i} u_n) - a_i(x, \partial_{x_i} u)\right] \left(\partial_{x_i} u_n - \partial_{x_i} u\right) \, dx \leq \frac{\varepsilon}{6} \quad \text{for all } n \geq N.$$

We fix $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that if H is a measurable subset of Ω with $\mathrm{meas(H)} \leq \delta_1$,

$$\int_H \sum_{i=1}^{N} \left[a_i(x, \partial_{x_i} u_n) - a_i(x, \partial_{x_i} u)\right] \left(\partial_{x_i} u_n - \partial_{x_i} u\right) \, dx \leq \frac{\varepsilon}{6} \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

Using the first inequality of (A4) we obtain that

$$\int_H \sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n) \partial_{x_i} u_n \, dx \geq \int_H \sum_{i=1}^{N} |\partial_{x_i} u_n|^{p_i(x)} \, dx. \tag{14}$$

By a variant of Young's inequality we have that, given $\tau \in (0,1)$, there exists $C(\tau) > 0$ depending on τ and $p_i(\cdot)$, but not on $x \in \overline{\Omega}$, such that for all $a, b \in \mathbb{R}$ and $x \in \overline{\Omega}$,

$$ab \leq \tau |a|^{p_i'(x)} + C(\tau) |b|^{p_i(x)} \tag{15}$$

(see ([24, relation 3.14]). This inequality yields

$$\begin{aligned} &\int_H \sum_{i=1}^{N} a_i(x, \partial_{x_i} u) \partial_{x_i} u_n \, dx \\ &\leq \frac{1}{3} \int_H \sum_{i=1}^{N} |\partial_{x_i} u_n|^{p_i(x)} \, dx + C\left(\frac{1}{3}\right) \int_H \sum_{i=1}^{N} |a_i(x, \partial_{x_i} u)|^{p_i'(x)} \, dx. \end{aligned} \tag{16}$$

By (A2),

$$\begin{aligned} &\int_H \sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n) \partial_{x_i} u \, dx \\ &\leq \overline{C}_1 \int_H \sum_{i=1}^{N} |\partial_{x_i} u| \, dx + \overline{C}_1 \int_H \sum_{i=1}^{N} |\partial_{x_i} u_n|^{p_i(x)-1} |\partial_{x_i} u| \, dx, \end{aligned}$$

where $C_1 = \max\{c_{1,i} : \quad i \in \{1, 2, ...N\}\}$ and $\overline{C}_1 = \max\{C_1, 1\}$. Relying again on (15), we arrive at

$$\begin{aligned} &\int_H \sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n) \partial_{x_i} u \, dx \\ &\leq \overline{C}_1 \int_H \sum_{i=1}^{N} |\partial_{x_i} u| dx + \overline{C}_1 C\left(\frac{1}{3\overline{C}_1}\right) \int_H \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)} dx \\ &+ \frac{1}{3} \int_H \sum_{i=1}^{N} |\partial_{x_i} u_n|^{p_i(x)} dx. \end{aligned} \tag{17}$$

Putting together (13), (14), (16) and (17) we obtain

$$\begin{aligned}
&\frac{1}{3}\int_H \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\, dx \\
\leq\ & \frac{\varepsilon}{6} + C\left(\frac{1}{3}\right)\int_H \sum_{i=1}^N |a_i(x,\partial_{x_i}u)|^{p_i'(x)}\, dx \\
&+\overline{C}_1 \int_H \sum_{i=1}^N |\partial_{x_i}u| dx + \overline{C}_1 C\left(\frac{1}{3\overline{C}_1}\right)\int_H \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)} dx \\
&+\int_H \sum_{i=1}^N |a_i(x,\partial_{x_i}u)||\partial_{x_i}u|\, dx.
\end{aligned}$$

Taking into account (A2), functions $|a_i(x,\partial_{x_i}u)|^{p_i'(x)}$, $|\partial_{x_i}u|$, $|\partial_{x_i}u|^{p_i(x)}$ and $|a_i(x, \partial_{x_i}u)||\partial_{x_i}u|$ belong to $L^1(\Omega)$. Hence there exists $0 < \delta \leq \delta_1$ such that if $\mathrm{meas(H)} \leq \delta$ then

$$\begin{aligned}
\frac{\varepsilon}{6} \quad \geq \quad & C\left(\frac{1}{3}\right)\int_H \sum_{i=1}^N |a_i(x,\partial_{x_i}u)|^{p_i'(x)}\, dx + \\
&+\overline{C}_1 \int_H \sum_{i=1}^N |\partial_{x_i}u| dx + \overline{C}_1 C\left(\frac{1}{3\overline{C}_1}\right)\int_H \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)} dx + \\
&+\int_H \sum_{i=1}^N |a_i(x,\partial_{x_i}u)||\partial_{x_i}u|\, dx.
\end{aligned}$$

The combination of the previous two inequalities conducts us to

$$\frac{1}{3}\int_H \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\, dx \leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6}$$

and we can conclude that that the sequence $\left(\sum_{i=1}^N |\partial_{x_i}u_n|^{p_i(x)}\right)_n$ is uniformly integrable in Ω, thus the proof of the first claim is complete.

Claim 2. The sequence $\left(\sum_{i=1}^N |\partial_{x_i}u_n - \partial_{x_i}u|^{p_i(x)}\right)_n$ converges in measure to 0 on Ω.

In order to prove the second claim, we need to show that

$$\lim_{n\to\infty} |\partial_{x_i}u_n - \partial_{x_i}u| = 0 \quad \text{for a.e. } x \in \Omega. \tag{18}$$

Relation (12) provides the existence of a subset U of Ω with $\mathrm{meas(U)} = 0$ such that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\left[a_i(x,\partial_{x_i}u_n)-a_i(x,\partial_{x_i}u)\right](\partial_{x_i}u_n-\partial_{x_i}u)=0 \quad \forall x\in\Omega\setminus U,$$

therefore, due to (A3), for all $i\in\{1,...,N\}$,

$$\lim_{n\to\infty}\left[a_i(x,\partial_{x_i}u_n)-a_i(x,\partial_{x_i}u)\right](\partial_{x_i}u_n-\partial_{x_i}u)=0 \quad \forall x\in\Omega\setminus U. \tag{19}$$

For $i\in\{1,...,N\}$ and $x\in\Omega\setminus U$, we deduce by (19) that there exists $M>0$ such that, for all $n\in\mathbb{N}$,

$$\begin{aligned}&a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n\\ &\le M+|a_i(x,\partial_{x_i}u_n)||\partial_{x_i}u|+|a_i(x,\partial_{x_i}u)||\partial_{x_i}u_n|+|a_i(x,\partial_{x_i}u)||\partial_{x_i}u|.\end{aligned}$$

Using (A2) and (A4) in the above inequality we produce

$$\begin{aligned}&|\partial_{x_i}u_n|^{p_i(x)}\\ &\le M+c_{1,i}(1+|\partial_{x_i}u_n|^{p_i(x)-1})|\partial_{x_i}u|+|a_i(x,\partial_{x_i}u)||\partial_{x_i}u_n|+|a_i(x,\partial_{x_i}u)||\partial_{x_i}u|\end{aligned}$$

from where we obtain that the sequence $(\partial_{x_i}u_n)_n$ is bounded. Passing to a subsequence, there exists $\xi=\xi(x)$ in $\mathbb{R}$ such that

$$\partial_{x_i}u_{n_k}\to\xi \quad \text{when} \quad k\to\infty.$$

Moreover, a_i is a Carathéodory function, so

$$a_i(x,\partial_{x_i}u_{n_k})\to a_i(x,\xi) \quad \text{when} \quad k\to\infty.$$

Then, replacing the sequence $(\partial_{x_i}u_n)_n$ by its subsequence $(\partial_{x_i}u_{n_k})_k$ in (19) and passing to the limit, we come to

$$\left[a_i(x,\xi)-a_i(x,\partial_{x_i}u)\right](\xi-\partial_{x_i}u)=0.$$

By (A3) and the uniqueness of the limit we deduce that

$$\partial_{x_i}u_{n_k}\to\partial_{x_i}u \quad \text{when} \quad k\to\infty.$$

Since the above arguments are valid for any subsequence of $(u_n)_n$ we obtain that

$$\partial_{x_i}u_n\to\partial_{x_i}u \quad \text{when} \quad n\to\infty,$$

hence (18) holds and the proof of the second claim is complete.

Combining the statements of the two claims with Vitali's convergence theorem we establish that (11) takes place and, using (8), we get the strong convergence of $(u_n)_n$ to u in E ■

The proof of Theorem 1 will follow the steps indicated by Theorem 2 and we rely on Lemma 2 to show that I satisfies the Palais-Smale condition. Furthermore, note that $I(0)=0$ and the fact that A_i and F are even in the second variable implies that I is even. We are now ready to prove our main theorem, under the reservation that the calculus techniques are not completely new and some of the arguments used are similar to some arguments used by [4]. However, for the completeness of the proof, we must include them in our work.

4. Proof of the Main Result

Keeping in mind the statement of Theorem 2 and the above comments, we arrange the proof into three parts, namely into three claims.

Claim 1. The energy functional I satisfies condition Palais-Smale.

Let $(u_n)_n \subset E$ be a sequence such that

$$|I(u_n)| < K, \quad \forall n \geq 1, \tag{20}$$

where K is a positive constant, and

$$I'(u_n) \to 0 \quad \text{when } n \to \infty. \tag{21}$$

To show that $(u_n)_n$ is bounded, we argue by contradiction and we assume that, passing eventually to a subsequence still denoted by $(u_n)_n$,

$$\|u_n\|_{\vec{p}(\cdot)} \to \infty \quad \text{when } n \to \infty. \tag{22}$$

By relations (20), (21), (22) we have

$$\begin{aligned} 1 + K + \|u_n\|_{\vec{p}(\cdot)} &\geq I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\ &\geq \sum_{i=1}^{N} \int_\Omega \left[A_i(x, \partial_{x_i} u_n) - \frac{1}{\mu} a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n\right] dx \\ &\quad - \int_\Omega \left[F(x, u_n) - \frac{1}{\mu} u_n f(x, u_n)\right] dx, \end{aligned}$$

for sufficiently large n, where μ is the constant from (f3). Using (f3) we get

$$1 + K + \|u_n\|_{\vec{p}(\cdot)} \geq \sum_{i=1}^{N} \int_\Omega \left[A_i(x, \partial_{x_i} u_n) - \frac{1}{\mu} a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n\right] dx. \tag{23}$$

From (A4),

$$a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n \leq p_i(x)\, A_i(x, \partial_{x_i} u_n),$$

for all $x \in \Omega$ and all $i \in \{1, 2, \ldots N\}$, which implies

$$-\frac{1}{\mu} a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n \geq -\frac{P_+^+}{\mu}\, A_i(x, \partial_{x_i} u_n),$$

for all $x \in \Omega$ and all $i \in \{1, 2, \ldots N\}$. Introducing the previous inequality into relation (23) we obtain

$$1 + K + \|u_n\|_{\vec{p}(\cdot)} \geq \left(1 - \frac{P_+^+}{\mu}\right) \sum_{i=1}^{N} \int_\Omega A_i(x, \partial_{x_i} u_n) dx.$$

From (A4) we also have

$$A_i(x, \partial_{x_i} u_n) \geq \frac{1}{P_+^+} |\partial_{x_i} u_n|^{p_i(x)}, \tag{24}$$

for all $x \in \Omega$ and all $i \in \{1, 2, ...N\}$, thus

$$1 + K + \|u_n\|_{\vec{p}(\cdot)} \geq \left(\frac{1}{P_+^+} - \frac{1}{\mu} \right) \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)} \, dx. \tag{25}$$

For every n, let us denote by $\mathcal{I}_{n_1}$, $\mathcal{I}_{n_2}$ the indices sets

$$\mathcal{I}_{n_1} = \{ i \in \{1, 2, ...N\} : \quad |\partial_{x_i} u_n|_{p_i(\cdot)} \leq 1 \}$$

and

$$\mathcal{I}_{n_2} = \{ i \in \{1, 2, ...N\} : \quad |\partial_{x_i} u_n|_{p_i(\cdot)} > 1 \}.$$

Using (4), (5), (6) and (25) we infer

$$\begin{aligned} 1 + K + \|u_n\|_{\vec{p}(\cdot)} &\geq \left(\frac{1}{P_+^+} - \frac{1}{\mu} \right) \left(\sum_{i \in \mathcal{I}_{n_1}} |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_+^+} + \sum_{i \in \mathcal{I}_{n_2}} |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} \right) \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\mu} \right) \left[\sum_{i=1}^{N} |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} - \sum_{i \in \mathcal{I}_{n_1}} |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} \right] \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\mu} \right) \left(\sum_{i=1}^{N} |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} - N \right). \end{aligned}$$

By the generalized mean inequality or the Jensen inequality applied to the convex function $a : \mathbb{R}^+ \to \mathbb{R}^+$, $a(t) = t^{P_-^-}$, $P_-^- > 1$ we get

$$1 + K + \|u_n\|_{\vec{p}(\cdot)} \geq \left(\frac{1}{P_+^+} - \frac{1}{\mu} \right) \left(\frac{\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- - 1}} - N \right).$$

Dividing by $\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}$ in the above relation and passing to the limit as $n \to \infty$ we obtain a contradiction, hence $(u_n)_n$ is bounded in E. The fact that E is reflexive yields the existence of a $u_0 \in E$ such that, up to a subsequence, $(u_n)_n$ converges weakly to u_0 in E. It remains to show that $(u_n)_n$ converges strongly to u_0 in E.

Since $q^+ < P_-^\star = P_{-,\infty}$, the embedding $E \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact, which implies that (u_n) converges strongly to u_0 in $L^{q(\cdot)}(\Omega)$. By (f2) and the Hölder-type inequality (3),

$$\left| \int_\Omega f(x, u_n)(u_n - u_0) dx \right| \leq 2c_2 \left| |u_n|^{q(x)-1} \right|_{\frac{q(\cdot)}{q(\cdot)-1}} |u_n - u_0|_{q(\cdot)}. \tag{26}$$

By (26), (7) and the strong convergence of $(u_n)_n$ to u_0 in $L^{q(\cdot)}(\Omega)$, we deduce

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u_0)dx=0. \tag{27}$$

By (21) we deduce

$$\lim_{n\to\infty} < I'(u_n), u_n-u_0 > = 0,$$

that is

$$\begin{aligned}\lim_{n\to\infty}\Bigg(&\int_\Omega\sum_{i=1}^N a_i(x,\partial_{x_i}u_n)(\partial_{x_i}u_n-\partial_{x_i}u_0)dx\\ &-\int_\Omega f(x,u_n)(u_n-u_0)dx\Bigg)=0.\end{aligned} \tag{28}$$

Joining together (27) and (28), we find that

$$\lim_{n\to\infty}\int_\Omega\sum_{i=1}^N a_i(x,\partial_{x_i}u_n)(\partial_{x_i}u_n-\partial_{x_i}u_0)dx=0.$$

The statement of Lemma 2 completes the proof of the first claim.

Claim 2. There exist ρ, $r>0$ such that $I(u)\geq r>0$, for any $u\in E$ with $\|u\|_{\vec{p}(\cdot)}=\rho$.

Note that, by (24),

$$I(u)\geq\frac{1}{P_+^+}\int_\Omega\sum_{i=1}^N|\partial_{x_i}u|^{p_i(x)}dx-\int_\Omega F(x,u)dx,\quad\text{for all } u\in E. \tag{29}$$

For $\rho<1$ we consider $u\in E$ such that $\|u\|_{\vec{p}(\cdot)}=\rho<1$. Thus $|\partial_{x_i}u|_{p_i(\cdot)}<1$ and, by (6),

$$\int_\Omega\sum_{i=1}^N|\partial_{x_i}u|^{p_i(x)}dx\geq\sum_{i=1}^N|\partial_{x_i}u|_{p_i(\cdot)}^{P_+^+}, \tag{30}$$

for all $u\in E$ with $\|u\|_{\vec{p}(\cdot)}<1$.

Again, by the generalized mean inequality or the Jensen inequality applied to the convex function $b:\mathbb{R}^+\to\mathbb{R}^+$, $b(t)=t^{P_+^+}$, $P_+^+>1$, we come to

$$\sum_{i=1}^N|\partial_{x_i}u|_{p_i(\cdot)}^{P_+^+}\geq N\left(\frac{\sum\limits_{i=1}^N|\partial_{x_i}u|_{p_i(\cdot)}}{N}\right)^{P_+^+}. \tag{31}$$

By (30) and (31) there exists $k_0>0$ such that

$$\frac{1}{P_+^+}\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}dx \geq k_0 \, \|u\|_{\vec{p}(\cdot)}^{P_+^+}, \tag{32}$$

for all $u \in E$ with $\|u\|_{\vec{p}(\cdot)} < 1$.

Keeping in mind Remark 1 and using (f2) we obtain,

$$F(x,s) \leq c_2 \int_0^{|s|} |t|^{q(x)-1}dt \leq \frac{c_2}{q^-}|s|^{q(x)} \leq \frac{c_2}{q^-}\left(|s|^{q^-} + |s|^{q^+}\right),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$. Thus

$$\int_\Omega F(x,u)dx \leq \frac{c_2}{q^-}\left(|u|_{q^-}^{q^-} + |u|_{q^+}^{q^+}\right) \qquad \text{for all } u \in E. \tag{33}$$

Since

$$E \hookrightarrow L^{q^-}(\Omega), \quad E \hookrightarrow L^{q^+}(\Omega)$$

continuously we have that there exists a positive constant k_1 such that, using (33),

$$\int_\Omega F(x,u)dx \leq k_1 \|u\|_{\vec{p}(\cdot)}^{q^-} \qquad \text{for all } u \in E \text{ with } \|u\|_{\vec{p}(\cdot)} < 1.$$

Combining the above relation with (32) and (29) we have

$$I(u) \geq k_0 \, \|u\|_{\vec{p}(\cdot)}^{P_+^+} - k_1 \, \|u\|_{\vec{p}(\cdot)}^{q^-} \qquad \text{for all } u \in E \text{ with } \|u\|_{\vec{p}(\cdot)} < 1,$$

where $k_0 = \frac{1}{P_+^+ N^{P_+^+ - 1}}$. Therefore,

$$I(u) \geq \|u\|_{\vec{p}(\cdot)}^{P_+^+}\left(k_0 - k_1 \, \|u\|_{\vec{p}(\cdot)}^{q^- - P_+^+}\right) \qquad \text{for all } u \in E \text{ with } \|u\|_{\vec{p}(\cdot)} < 1.$$

We denote by $g : [0,1] \to \mathbb{R}$ the function defined by

$$g(t) = k_0 - k_1 \, t^{q^- - P_+^+}$$

and we point out the fact that g is positive in a neighborhood of the origin. Since we can choose $0 < \rho < 1$ sufficiently small, the proof of our second claim is complete.

Claim 3.

For any finite dimensional subspace $\widetilde{E} \subset E$ there exists $R = R(\widetilde{E}) > 0$ such that

$$I(u) \leq 0 \text{ for all } u \in \widetilde{E} \setminus B_R(\widetilde{E}).$$

By conditions (A1) and (A2),

$$0 \leq A_i(x,s) \leq c_{1,i} \int_0^{|s|}\left(1 + |t|^{p_i(x)-1}\right)dt = c_{1,i}\left(|s| + \frac{|s|^{p_i(x)}}{p_i(x)}\right) \text{ for all } x \in \Omega, \; s \in \mathbb{R},$$

and we obtain

$$(34)\quad 0 \le J(v) \le C_1 \int_\Omega \sum_{i=1}^N |\partial_{x_i} v|\,dx + \frac{C_1}{P_-^-}\int_\Omega \sum_{i=1}^N |\partial_{x_i} v|^{p_i(x)}\,dx \quad \text{for all } v \in E.$$

Let $\widetilde{E} \subset E$ be a finite dimensional subspace, $u \in \widetilde{E} \setminus \{0\}$ and $t > 1$. Then, by (34),

$$J(tu) \le C_1 \int_\Omega \sum_{i=1}^N |\partial_{x_i}(tu)|\,dx + \frac{C_1}{P_-^-}\int_\Omega \sum_{i=1}^N |\partial_{x_i}(tu)|^{p_i(x)}\,dx$$

and by Remark 1 we infer that

$$I(tu) \le C_1\, t \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|\,dx + \frac{C_1\, t^{P_+^+}}{P_-^-}\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,dx - c_3\, t^\mu \int_\Omega |u|^\mu\,dx \to\ -\infty \quad \text{as } t \to \infty,$$

since $\mu > P_+^+ > 1$.

Notice that, for all $R > 0$,

$$\sup_{\|u\|_{\vec{p}(\cdot)}=R,\, u\in\widetilde{E}} I(u) = \sup_{\|tu\|_{\vec{p}(\cdot)}=R,\, tu\in\widetilde{E}} I(tu) = \sup_{\|tu\|_{\vec{p}(\cdot)}=R,\, u\in\widetilde{E}} I(tu)$$

and combining the above two relations we get

$$\sup_{\|u\|_{\vec{p}(\cdot)}=R,\, u\in\widetilde{E}} I(u) \to -\infty \quad \text{as } R \to \infty.$$

Therefore we can choose $R_0 > 0$ sufficiently large such that $\forall R \ge R_0$ and $\forall u \in \widetilde{E}$ with $\|u\|_{\vec{p}(\cdot)} = R$ we have $I(u) \le 0$. Thus

$$I(u) \le 0 \text{ for all } u \in \widetilde{E} \setminus B_{R_0}$$

and the proof of our final claim is complete.

Finally, taking into account the three claims and using the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz we deduce the existence of an unbounded sequence of weak solutions to problem (1). ∎

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REFERENCES

1. A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory, *J. Funct. Anal.*, **14** (1973), 349-381.

2. S. Antontsev and S. Shmarev, Vanishing solutions of anisotropic parabolic equations with variable nonlinearity, *J. Math. Anal. Appl.*, **361** (2010), 371-391.

3. M.-M. Boureanu, Existence of solutions for anisotropic quasilinear elliptic equations with variable exponent, *Advances in Pure and Applied Mathematics*, **1** (2010), 387-411.

4. M.-M. Boureanu, P. Pucci and V. Rădulescu, Multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent, *Complex Variables and Elliptic Equations*, 2010, 1-13, i First.

5. H. Brézis and F. Browder, Partial Differential Equations in the 20th Century, *Advances in Mathematics*, **135** (1998), 76-144.

6. Y. Chen, S. Levine and R. Rao, Functionals with $p(x)$-growth in image processing, *Duquesne University, Department of Mathematics and Computer Science Technical Report 2004-01*.

7. L. Diening, *Theoretical and Numerical Results for Electrorheological Fluids*, Ph.D. thesis, University of Frieburg, Germany, 2002.

8. D. E. Edmunds, J. Lang and A. Nekvinda, On $L^{p(x)}$ norms, *Proc. Roy. Soc. London Ser. A*, **455** (1999), 219-225.

9. D. E. Edmunds and J. Rákosnîk, Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. Roy. Soc. London Ser. A*, **437** (1992), 229-236.

10. D. E. Edmunds and J. Rákosnîk, Sobolev embedding with variable exponent, *Studia Math.*, **143** (2000), 267-293.

11. X. Fan, Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.*, **312** (2005), 464-477.

12. X. Fan, Q. Zhang and D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, *J. Math. Anal. Appl.*, **302** (2005), 306-317.

13. I. Fragalà, F. Gazzola and B. Kawohl, Existence and nonexistence results for anisotropic quasilinear equations, *Ann. Inst. H. Poincaré, Analyse Non Linéaire*, **21** (2004), 715-734.

14. M. Ghergu and V. Rădulescu, *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Its Applications, **37**, Oxford University Press, 2008.

15. T. C. Halsey, Electrorheological fluids, *Science*, **258** (1992), 761-766.

16. P. Harjulehto, Variable exponent Sobolev spaces with zero boundary values, *Math. Bohem.*, **132(2)** (2007), 125-136.

17. P. Hästö, On the density of continuous functions in variable exponent Sobolev spaces, *Rev. Mat. Iberoamericana*, **23** (2007), 74-82.

18. Y. Jabri, *The Mountain Pass Theorem. Variants, Generalizations and some Applications,* Cambridge University Press, 2003.

19. C. Ji, An eigenvalue of an anisotropic quasilinear elliptic equation with variable exponent and Neumann boundary condition, *Nonlinear Analysis T.M.A.*, **71** (2009), 4507-4514.

20. B. Kone, S. Ouaro and S. Traore, Weak solutions for anisotropic nonlinear elliptic equations with variable exponents, *Electronic Journal of Differential Equations,* **2009** (2009), No. 144, 1-11.

21. O. Kováčik and J. Rákosnîk, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.*, **41** (1991), 592-618.

22. A. Kristály, H. Lisei and C. Varga, Multiple solutions for p-Laplacian type equations, *Nonlinear Anal. TMA*, **68** (2008), 1375-1381.

23. A. Kristály, V. Rădulescu and C. Varga, *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, **136**, Cambridge University Press, Cambridge, 2010.

24. V. K. Le, On a sub-supersolution method for variational inequalities with Leray-Lions operators in variable exponent spaces, *Nonlinear Anal.*, **71** (2009), 3305-3321.

25. M. Mihăilescu and G. Moroşanu, Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions, *Applicable Analysis*, **89** (2010), 257-271.

26. M. Mihăailescu, P. Pucci and V. Rădulescu, Nonhomogeneous boundary value problems in anisotropic Sobolev spaces, *C. R. Acad. Sci. Paris, Ser. I*, **345** (2007), 561-566.

27. M. Mihăilescu, P. Pucci and V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.*, **340** (2008), 687-698.

28. M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London Ser. A*, **462** (2006), 2625-2641.

29. M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.*, **135** (2007), 2929-2937.

30. J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983.

31. S. M. Nikol'skii, On imbedding, continuation and approximation theorems for differentiable functions of several variables, *Russian Math. Surv.*, **16** (1961), 55-104.

32. C. Pfeiffer, C. Mavroidis, Y. Bar-Cohen and B. Dolgin, Electrorheological fluid based force feedback device, *Proc. 1999 SPIE Telemanipulator and Telepresence Technologies VI Conf.* (Boston, MA), **3840** (1999), 88-99.

33. P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. in Math., **65**, Amer. Math. Soc., 1986.

34. J. Rákosnîk, Some remarks to anisotropic Sobolev spaces I, *Beiträge zur Analysis*, **13** (1979), 55-68.

35. J. Rákosnîk, Some remarks to anisotropic Sobolev spaces II, *Beiträge zur Analysis*, **15** (1981), 127-140.

36. K. R. Rajagopal and M. Růžička, Mathematical modelling of electrorheological fluids, *Continuum Mech. Thermodyn.*, **13** (2001), 59-78.

37. M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2002.

38. S. Samko and B. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, *J. Math. Anal. Appl.*, **310** (2005), 229-246.

39. M. Seed, G. S. Hobson, R. C. Tozer and A. J. Simmonds, Voltage-controlled Electrorheological brake, *Proc. IASTED Int. Symp. Measurement, Sig. Proc. and Control: Paper No.* 105-092-1, Taormina, Italy: ACTA Press, 1986.

40. A. J. Simmonds, Electro-rheological valves in a hydraulic circuit, *IEE Proceedings-D*, **138(4)** (1991), 400-404.

41. R. Stanway, J. L. Sproston and A. K. El-Wahed, Applications of electro-rheological fluids in vibration control: a survey, *Smart Mater. Struct.*, **5** (1996), 464-482.

42. M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, *Ricerche Mat.*, **18** (1969), 3-24.

43. L. Ven'-tuan, On embedding theorems for spaces of functions with partial derivatives of various degree of summability, *Vestnik Leningrad Univ.*, **16** (1961), 23-37.

44. W. Winslow, Induced fibration of suspensions, *J. Appl. Phys.*, **20** (1949), 1137-1140.

45. V. V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR Izv.*, **29** (1987), 33-66.

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