TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 5, pp. 2265-2289, October 2011
This paper is available online at http://tjm.math.ntu.edu.tw/index.php/TJM

CANONICAL COORDINATES AND PRINCIPAL DIRECTIONS FOR SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

Franki Dillen, Marian Ioan Munteanu and Ana-Irina Nistor

Abstract. In this paper we characterize and classify surfaces in $\mathbb{H}^2 \times \mathbb{R}$ which have a canonical principal direction. Here $\mathbb{H}^2$ denotes the hyperbolic plane. We study some geometric properties such as minimality and flatness. Several examples are given to complete the study.

1. Introduction

The geometry of surfaces in spaces of dimension 3, especially of the form $\mathbb{M}^2 \times \mathbb{R}$, has been quite developed in recent years. The most interesting situations occur when $\mathbb{M}^2$ has constant Gaussian curvature, since in these cases a lot of classification results are obtained. One of the first properties, minimality, was studied by H. Rosenberg and W. H. Meeks III in [13, 14, 19]. Inspired by these papers, a generalization for arbitrary dimension, namely for $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, is given in [2, 3, 12, 23]. In particular, in [23] the author independently proves a higher dimensional version of Theorem 5 in the present paper. Some other recent results involving minimality and curvature properties for surfaces immersed in ambient spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ can be found in [1, 9, 10, 15. 16, 20-22].

Another problem that is studied in several recent papers consists of characterizing and classifying constant angle surfaces in 3-dimensional spaces $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, or the Heisenberg group Nil_3. A constant angle surface is an orientable surface whose unit normal makes a constant angle, denoted by θ, with a fixed direction. See for instance [5-8, 11, 17]. When the ambient space is of the form $\mathbb{M}^2 \times \mathbb{R}$, a favored direction is $\mathbb{R}$. It is known that for a constant angle surface in $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or in $\mathbb{H}^2 \times \mathbb{R}$, the projection of $\frac{\partial}{\partial t}$ (where t is the global parameter on $\mathbb{R}$) onto the tangent plane of the immersed surface, denoted by T, is a principal direction with

Received January 27, 2010, accepted June 21, 2010.
Communicated by Bang-Yen Chen.
2010 *Mathematics Subject Classification*: 53B25.
Key words and phrases: Hyperbolic plane, Minimal surfaces, Principal direction.
This research was supported by Research Grant G.0432.07 of the Research Foundation-Flanders (FWO). The second author was supported by Grant PN-II ID 398/2007-2010 (Romania).

corresponding principal curvature identically zero. The main topic of the present work is to investigate surfaces in $\mathbb{H}^2 \times \mathbb{R}$ for which T is a principal direction. The study of these surfaces was motivated by the results obtained in [4] for the ambient space $\mathbb{S}^2 \times \mathbb{R}$.

The structure of this paper is the following. After we recall the basic notions in *Preliminaries*, we introduce appropriate coordinates which turn out to be useful in the study of flatness and minimality. We also define canonical coordinates on a surface for which T is a principal direction. The main result of the paper is presented in the last section, in particular we formulate the following classification Theorem 4 (See also Theorem 5):

A surface M in $\mathbb{H}^2 \times \mathbb{R}$ has T as principal direction if and only if the immersion $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is given by

$$F(x, y) = (A(y) \sinh \phi(x) + B(y) \cosh \phi(x),\ \chi(x)),$$

where $(\phi,\ \chi)$ is a regular curve in $\mathbb{R}^2$, A is a curve in $\mathbb{S}^2_1 \subset \mathbb{R}^3_1$, B is a curve in $\mathbb{H}^2 \subset \mathbb{R}^3_1$ orthogonal to A such that the two speeds A' and respectively B' are parallel. Here $\mathbb{S}^2_1$ denotes the de Sitter space.

As a consequence, we recover the classification of all constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$, obtained in [7]. Finally, we give some other theorems under extra assumptions of minimality or flatness. We construct many suggestive examples to illustrate our study.

2. Preliminaries

Let us fix the notations used in this paper. By $\widetilde{M} = \mathbb{H}^2 \times \mathbb{R}$ we denote the ambient space given as the Riemannian product of the hyperbolic space endowed with the metric g_H, namely $(\mathbb{H}^2(-1), g_H)$, and the one dimensional Euclidean space endowed with the usual metric. The metric on the ambient space is given by $\widetilde{g} = g_H + dt^2$, where t is the global coordinate on $\mathbb{R}$. Then $\partial_t := \frac{\partial}{\partial_t}$ denotes an unit vector field in the tangent bundle $T(\widetilde{M})$ that is tangent to the $\mathbb{R}$-direction. We denote by $\widetilde{R}$ either the curvature tensor $\widetilde{R}(X, Y) = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}$, or the Riemann-Christoffel tensor on $\widetilde{M}$ defined by $\widetilde{R}(W, Z, X, Y) = \widetilde{g}(W, \widetilde{R}(X, Y)Z)$. One has

$$\widetilde{R}(X, Y, Z, W) = -g_H(X_H, W_H) g_H(Y_H, Z_H) + g_H(X_H, Z_H) g_H(Y_H, W_H),$$

for any $X, Y, Z, W \in T(\widetilde{M})$ and X_H denotes the projection of X to the tangent space of $\mathbb{H}^2$.

In the sequel we study some important properties of submanifolds $(M, g) \hookrightarrow (\widetilde{M}, \widetilde{g})$ isometrically immersed in $\widetilde{M}$. Roughly speaking, the metric g on M represents the restriction of $\widetilde{g}$ to M.

For an isometric immersion $F : M \to \widetilde{M}$ we recall the classical Gauss and Weingarten formulas:

(G) $\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$

(W) $\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$

where X, Y are tangent to M, $\widetilde{\nabla}$ and ∇ denote the Levi-Civita connections on $\widetilde{M}$ respectively on M, and N denotes any vector field normal to M. Moreover, h is a symmetric (1, 2)-tensor field called the second fundamental form of the submanifold M, A_N is a symmetric (1, 1)-tensor field called the shape operator associated to N and $\nabla^{\perp}$ denotes the connection in the normal bundle.

In particular, let us consider an oriented surface M in $\widetilde{M}$. If ξ is the unit normal to M associated with the shape operator A, the following property holds:

$$\widetilde{g}(h(X, Y), \xi) = g(X, AY),$$

for any vector fields X, Y tangent to M. Taking into account these considerations, since ∂_t is unitary, it can be decomposed as

$$\partial_t = T + \cos\theta\xi, \tag{1}$$

where T is the projection of ∂_t on $T(M)$ and θ is the *angle function* depending on the point of the surface and supposed to take values in the interval $[0, \pi]$.

Denoting by R the curvature tensor on M, after straightforward computations we are able to write the fundamental equations of Gauss and Codazzi

(E.G.) $R(X, Y) = AX \wedge AY - X \wedge Y + T^{\flat} \otimes (X \wedge Y)(T) - (X \wedge Y)(T)^{\flat} \otimes T,$

(E.C.) $(\nabla_X A)\, Y - (\nabla_Y A)\, X = \cos\theta\ (g(X, T)Y - g(Y, T)X),$

where $(X \wedge Y)(Z) := g(X, Z)Y - g(Y, Z)X$, $\flat$ denotes the musical isomorphism *flat* and $(\omega \otimes X)(Y) = \omega(Y)X$, for $\omega \in \Lambda^1(M)$ and for all $X, Y, Z \in T(M)$.

Computing the Gaussian curvature K, from the equation of Gauss it follows

$$K = \det A - \cos^2\theta. \tag{2}$$

Now, taking into account the decomposition of any vector field $X \in T(M)$ as $X = X_H + g(X, T)\partial_t$, the expression of ∂_t from (1) and the formulas **(G)** and **(W)**, the following proposition holds as in [7].

Proposition A. *Let X be an arbitrary tangent vector to M. Then we have*

$$\nabla_X T = \cos\theta AX, \tag{3}$$

$$X(\cos\theta) = -g(AX, T). \tag{4}$$

The set of equations **(E.G.)**, **(E.C.)**, (3) and (4) are called the *compatibility conditions*. The following result was given in [2]:

Theorem B. *Let M be a simply connected Riemannian surface endowed with the metric g and its corresponding Levi-Civita connection ∇. Let A be a field of symmetric operators $A_p : T_p(M) \to T_p(M)$ and T a vector field on M with $\|T\|^2 = \sin^2\theta$, where θ is a smooth function defined on M. Assume that (g, A, T, θ) satisfies the compatibility conditions for $\mathbb{H}^2 \times \mathbb{R}$. Then there exists an isometric immersion $F : M \to \mathbb{H}^2 \times \mathbb{R}$ such that the shape operator with respect to the unit normal ξ is given by A and $\partial_t = F_*T + \cos\theta\xi$. Moreover, the immersion is unique up to isometries of $\mathbb{H}^2 \times \mathbb{R}$ preserving the orientation of both $\mathbb{H}^2$ and $\mathbb{R}$.*

3. Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

In this section we suppose that the angle function θ is different from 0 and $\frac{\pi}{2}$.

Proposition 1. *If θ is never 0 or $\frac{\pi}{2}$, then we can choose local coordinates (x, y) on the surface M isometrically immersed in $\widetilde{M}$ with ∂_x in the direction of T such that the metric on M has the following form*

$$g = \frac{1}{\sin^2\theta}dx^2 + \beta^2(x,y)dy^2. \tag{5}$$

In the basis $\{\partial_x,\ \partial_y\}$ the shape operator A can be expressed as

$$A = \begin{pmatrix} \theta_x \sin\theta & \theta_y \sin\theta \\ \dfrac{\theta_y}{\sin\theta\beta^2} & \dfrac{\sin^2\theta\beta_x}{\cos\theta\beta} \end{pmatrix}, \tag{6}$$

and the functions θ and β are related by the PDE

$$\frac{\sin^2\theta}{\cos\theta}\frac{\beta_{xx}}{\beta} + \frac{\sin\theta\theta_x}{\cos^2\theta}\frac{\beta_x}{\beta} + \frac{\theta_y}{\sin\theta}\frac{\beta_y}{\beta^3} + \left(2\frac{\cos\theta\theta_y^2}{\sin^2\theta} - \frac{\theta_{yy}}{\sin\theta}\right)\frac{1}{\beta^2} - \cos\theta = 0. \tag{7}$$

Proof. Here, and for the rest of the paper, we denote, for the sake of simplicity, $\frac{\partial}{\partial x} = \partial_x$ and $\frac{\partial f}{\partial x} = f_x$ for any function f.

From the general theory, choosing an arbitrary point $p \in M$ such that the angle function $\theta(p) \neq 0$ and $\theta(p) \neq \frac{\pi}{2}$, we can take local orthogonal coordinates (x, y) such that ∂_x is in the direction of T and the metric g on M has the form

$$g = \alpha^2(x,y)dx^2 + \beta^2(x,y)dy^2, \tag{8}$$

for certain functions α and β on M.

The Levi-Civita connection for this metric is given by the following expressions

$$\nabla_{\partial_x}\partial_x = \frac{\alpha_x}{\alpha}\partial_x - \frac{\alpha\alpha_y}{\beta^2}\partial_y, \quad \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = \frac{\alpha_y}{\alpha}\partial_x + \frac{\beta_x}{\beta}\partial_y,$$

$$\nabla_{\partial_y}\partial_y = -\frac{\beta\beta_x}{\alpha^2}\partial_x + \frac{\beta_y}{\beta}\partial_y.$$

In order to determine the shape operator A, we use formula (3) for $X = \partial_x$ and respectively for $X = \partial_y$. Since $T = \frac{\sin\theta}{\alpha}\partial_x$ we get

$$A\partial_x = \frac{\theta_x}{\alpha}\partial_x - \tan\theta\frac{\alpha_y}{\beta^2}\partial_y, \tag{9}$$

$$A\partial_y = \frac{\theta_y}{\alpha}\partial_x + \tan\theta\frac{\beta_x}{\alpha\beta}\partial_y. \tag{10}$$

On the other hand, since A is symmetric, i.e. $g(A\partial_y, \partial_x) = g(\partial_y, A\partial_x)$, we obtain

$$\tan\theta\alpha_y + \alpha\theta_y = 0. \tag{11}$$

It follows that the shape operator is given by $A = \begin{pmatrix} \frac{\theta_x}{\alpha} & \frac{\theta_y}{\alpha} \\ \frac{\alpha\theta_y}{\beta^2} & \frac{\tan\theta\beta_x}{\alpha\beta} \end{pmatrix}$.

After an integration in (11) one obtains $\alpha = \dfrac{\phi(x)}{\sin\theta}$, where ϕ is a function depending on x. Changing the x-coordinate we may assume that $\alpha = \dfrac{1}{\sin\theta}$ and substituting it in (8) we get (5). Moreover, replacing α in the previous expression of A we obtain that the shape operator is given exactly by formula (6). In order to find a relation between β and θ, we substitute in the Codazzi equation **(E.C.)** $X = \partial_x,\ Y = \partial_y$, $T = \sin^2\theta\partial_x$ and we get

$$\nabla_{\partial_x}(A\partial_y) - \nabla_{\partial_y}(A\partial_x) - \cos\theta\partial_y = 0.$$

After straightforward calculations, the PDE (7) is obtained, concluding the proof. ∎

Remark 1. Every two functions θ and β defined on a smooth simply connected surface M, related by (7), determine an isometric immersion of M into $\mathbb{H}^2 \times \mathbb{R}$ such that the shape operator is given by (6).

Proof. Construct the metric g as in (5) and define the field of operators A such that its matrix in the local basis $\{\partial_x, \partial_y\}$ is given by (6). It is easy to notice that all A_p are symmetric ($p \in M$). Take $T = \sin^2\theta\ \partial_x$. A straightforward computation shows that all compatibility conditions are satisfied. Applying Theorem B, we get the conclusion. ∎

Once the background of the study of a surface M in $\mathbb{H}^2 \times \mathbb{R}$ is established, we are interested to find some particular classes of surfaces involving the minimality and flatness conditions. Following the same idea as in the general case but under the restriction imposed by the minimality condition, one gets the following proposition.

Proposition 2. *If M is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\theta \neq 0, \frac{\pi}{2}$, then we can choose local coordinates (x, y) such that ∂_x is in the direction of T, the metric is given by*

$$g = \frac{1}{\sin^2 \theta}\,(dx^2 + dy^2), \tag{12}$$

and the shape operator w.r.t. the basis $\{\partial_x, \partial_y\}$ has the expression

$$A = \sin\theta \begin{pmatrix} \theta_x & \theta_y \\ \theta_y & -\theta_x \end{pmatrix}. \tag{13}$$

Moreover, denoting $\Delta = \sin^2\theta(\partial_x^2 + \partial_y^2)$ the Laplacian of the surface M, the angle function θ satisfies the PDE

$$\Delta \ln\left(\tan\left(\frac{\theta}{2}\right)\right) = -\cos\theta. \tag{14}$$

Proof. Using Proposition 1 and in particular (6), the minimality condition $H = 0$ becomes $\cos\theta\theta_x\beta + \sin\theta\beta_x = 0$. Integrating once w.r.t. x one gets $\beta(x, y) = \dfrac{\psi(y)}{\sin\theta}$ for some function ψ on M depending only on y. Making a change of the y-coordinate one can assume $\beta(x, y) = \dfrac{1}{\sin\theta}$. Substituting it in (5) and (6) the relations (12) and (13) are proved. Note that (x, y) are *isothermal coordinates*.

Computing the partial derivatives of β and replacing them in (7), an equivalent condition is obtained

$$\cos\theta(\theta_x^2 + \theta_y^2 - 1) - \sin\theta(\theta_{xx} + \theta_{yy}) = 0. \tag{15}$$

A straightforward computation shows that this is equivalent with (14). ∎

Remark 2. Every smooth function θ defined on a smooth simply connected surface M satisfying the elliptic equation (14), gives rise to an isometric minimal immersion of M into $\mathbb{H}^2 \times \mathbb{R}$ such that the shape operator is given by (13).

Example 1. In order to give an example of angle function θ that corresponds to a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$, it is necessary to solve first the equation (15). Let us look for θ such that $\theta_x = k\theta_y$, for some non-vanishing real constant k. Taking the derivatives with respect to x, respectively y, we get $\theta_{xx} = k\theta_{xy}$ and $\theta_{xy} = k\theta_{yy}$. Substituting in (15) the expressions $\theta_x^2 + \theta_y^2 = (k^2 + 1)\theta_y^2$ and $\theta_{xx} + \theta_{yy} = \frac{k^2+1}{k}\theta_{xy}$ and integrating w.r.t. x we get that $\frac{(k^2+1)\theta_y^2 - 1}{\sin^2\theta} = c(y)$. It can be proved that $c(y)$ is constant, hence $\frac{\theta_y}{\sqrt{1 + c\sin^2\theta}} = \pm\frac{1}{\sqrt{k^2+1}}$ and $\frac{\theta_x}{\sqrt{1 + c\sin^2\theta}} = \pm\frac{k}{\sqrt{k^2+1}}$. Denoting $F(\theta\,|-c) = \displaystyle\int_0^\theta \frac{1}{\sqrt{1 + c\sin^2\theta(t)}}\,dt$ the elliptic integral of first kind, its differential is $dF(\theta\,|-c) = \pm\, d\left(\frac{kx+y}{\sqrt{k^2+1}}\right)$. It follows that $\theta = am\left(\pm\frac{kx+y}{\sqrt{k^2+1}} \;\middle|\; -c\right)$, where am denotes the *Jacobi amplitude*, namely the inverse of the elliptic function $F(\theta \mid -c)$.

If we look at the surfaces M in $\mathbb{H}^2 \times \mathbb{R}$ being at the same time minimal and flat, we obtain the following classification theorem.

Theorem 1. *The only surfaces in* $\mathbb{H}^2 \times \mathbb{R}$ *which are both flat and minimal are given by* $f \times \mathbb{R}$, *where* f *is a geodesic line in* $\mathbb{H}^2$.

Proof. If $\theta \neq 0$ or $\frac{\pi}{2}$, then the metric takes the form (12) and the shape operator is given by (13). Asking for the Gaussian curvature to vanish identically, we obtain from (2) and using (13) that θ fulfills $\theta_x^2 + \theta_y^2 = -\cot^2\theta$. Hence θ is constant different from 0 or $\frac{\pi}{2}$, which is a contradiction. A surface with $\theta = 0$ is not flat, hence $\theta = \frac{\pi}{2}$ and the surface is of the form $f \times \mathbb{R}$ with f a geodesic line in $\mathbb{H}^2$. ∎

4. Surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with a Canonical Principal Direction

The problem of studying surfaces for which T is a principal direction arises from the previous papers on constant angle surfaces. We recall [7] where it is proved that for a constant angle surface M in $\mathbb{H}^2 \times \mathbb{R}$, i.e. $\theta \in [0, \pi]$ is constant, T is a principal direction of the surface with corresponding principal curvature 0. One natural generalization consists of the case when T is assumed to be a principal direction with arbitrary principal curvature. At this point we denominate T as a *canonical principal direction*.

Let us say few words about the ambient space. There are many models describing the hyperbolic plane and we will use the hyperboloid model, also known as the *Minkowski model*. Recall some basic facts about this model which will be used in the sequel. We denote $\mathbb{R}^3_1$ the Minkowski 3-space endowed with the *Lorentzian metric*

$$\langle\ ,\ \rangle = dx_1^2 + dx_2^2 - dx_3^2.$$

The space $\mathbb{H}^2$ can be modeled by the upper sheet of the two-sheeted hyperboloid, namely,

$$\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid\ x_1^2 + x_2^2 - x_3^2 = -1,\ x_3 > 0\}.$$

The *Lorentzian cross-product* for vectors $u, v \in \mathbb{R}^3_1$ is defined as $\boxtimes : \mathbb{R}^3_1 \times \mathbb{R}^3_1 \to \mathbb{R}^3_1$, $\big((u_1, u_2, u_3),\ (v_1, v_2, v_3)\big) \mapsto (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_2v_1 - u_1v_2)$.

Concerning the curves in Minkowski 3-space, we recall that a regular curve $\gamma : I \to \mathbb{R}^3_1$ is called *spacelike* if $\langle\dot\gamma, \dot\gamma\rangle > 0$ everywhere, *timelike* if $\langle\dot\gamma, \dot\gamma\rangle < 0$ in any point, and respectively, *lightlike* if $\langle\dot\gamma, \dot\gamma\rangle = 0$ everywhere.

In order to study under which conditions T is a canonical principal direction, we regard the surface M as a surface immersed in $\mathbb{R}^3_1 \times \mathbb{R}$ (also denoted $\mathbb{R}^4_1$) having codimension 2.

The metric on the ambient space is given by $\widetilde{g} = dx_1^2 + dx_2^2 - dx_3^2 + dt^2$.

At this point, let us consider the surface M given by the isometric immersion $F : M \to \mathbb{R}^3_1 \times \mathbb{R}$, $F = (F_1,\ F_2,\ F_3,\ F_4)$. Denote by $\widetilde{\xi} = (F_1,\ F_2,\ F_3,\ 0)$ the normal to $\mathbb{H}^2 \times \mathbb{R}$ in the points of M and by $\xi = (\xi_1,\ \xi_2,\ \xi_3,\ \cos\theta)$ the normal to M in $\mathbb{H}^2 \times \mathbb{R}$. (See [7] for details.) Moreover, from now on, by $D^\perp$ we mean the normal connection of M in $\mathbb{R}^4_1$ and by $R^\perp$ the normal curvature

$$R^\perp(X,\ Y) = [D^\perp_X,\ D^\perp_Y] - D^\perp_{[X,\ Y]}\ ,\ \text{for all } X, Y \text{ tangent to } M.$$

The first result in this section is the characterization theorem of surfaces M having T as principal direction in terms of the normal curvature.

Theorem 2. (Characterization Theorem). *Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$ such that $\theta \neq 0$. T is a principal direction if and only if M is normally flat in $\mathbb{R}^3_1 \times \mathbb{R}$.*

Proof. With the above considerations, for any X tangent to M, we compute $D^\perp_X \tilde{\xi} = -\cos\theta g(X,T)\xi$ which implies $D^\perp_X \xi = \cos\theta g(X,T)\tilde{\xi}$.

Since Proposition 1 holds, the metric g is given by (5), and using the previous expressions, one has

$$R^\perp(\partial_x,\ \partial_y)\xi = \sin\theta\theta_y\tilde{\xi} \qquad \text{and} \qquad R^\perp(\partial_x,\ \partial_y)\tilde{\xi} = -\sin\theta\theta_y\xi.$$

Taking into account that ξ and $\tilde{\xi}$ are unitary and $\sin\theta$ cannot vanish, we get that M is normally flat if and only if $\theta_y = 0$. On the other hand, T is a canonical principal direction if and only if $\theta_y = 0$. This follows from expression (6) of the Weingarten operator A. Hence, we get the conclusion. ∎

An analogue result with Proposition 1, formulated for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ having T as principal direction, is the following

Proposition 3. *Let M be isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$ such that $\theta \neq 0, \frac{\pi}{2}$ with T a principal direction. Then, we can choose local coordinates (x, y) such that ∂_x is in the direction of T, the metric is given by*

$$g = dx^2 + \beta^2(x, y)dy^2, \tag{16}$$

and the shape operator w.r.t. $\{\partial_x,\ \partial_y\}$ can be written as

$$A = \begin{pmatrix} \theta_x & 0 \\ 0 & \tan\theta\frac{\beta_x}{\beta} \end{pmatrix}. \tag{17}$$

Moreover, the functions θ and β are related by the PDE

$$\beta_{xx} + \tan\theta\theta_x\beta_x - \beta\cos^2\theta = 0, \tag{18}$$

and $\theta_y = 0$.

Proof. Looking back at the proof of Proposition 1, if T is a principal direction, then one may choose local coordinates (x, y) such that $g = \alpha^2(x, y)dx^2 + \beta^2(x, y)dy^2$ and $\theta_y = 0$, i.e. the angle function $\theta(x, y)$ depends only of x. This means that we can do a change of the x-coordinate such that the metric is given now by (16). Moreover, following the same line of proof as in Proposition 1 for $\alpha = 1$ and $\theta_y = 0$, we obtain that the shape operator is given by (17). Finally, the equation of Codazzi yields (18). ∎

Remark 3. For every two functions θ and β defined on a smooth simply connected surface M such that $\beta_{xx} + \tan\theta\theta_x\beta_x - \beta\cos^2\theta = 0$ and $\theta_y = 0$ for certain coordinates (x, y), we can construct an isometric immersion $F : M \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ with the shape operator (17) and such that it has a canonical principal direction.

Remark 4. Let M be an isometrically immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ such that T is a principal direction. Coordinates (x, y) on M such that the hypotheses of Proposition 3 are fulfilled, i.e. ∂_x is collinear with T and the metric g has the form $g = dx^2 + \beta^2(x, y)dy^2$, will be called *canonical coordinates*. Of course, they are not unique. More precisely, if (x, y) and $(\overline{x}, \overline{y})$ are both canonical coordinates, then they are related by $\overline{x} = \pm\, x + c$ and $\overline{y} = \overline{y}(y)$, where $c \in \mathbb{R}$.

We are interested in solving equation (18) in order to find explicit parametrizations of surfaces M in $\mathbb{H}^2 \times \mathbb{R}$ with T as principal direction. We first give some auxiliary results.

Lemma 1. *The solution of the PDE in* $\mathbf{f} = \mathbf{f}(x, y)$

$$\frac{\mathbf{f}_x^2}{\cos^2\theta(x)} - \mathbf{f}^2 = \tilde{\mu}(y),$$

with θ a function of one variable x and $\tilde{\mu}$ a function of y that does not change sign is given by

$$\mathbf{f} = \mu(y)\sinh(\phi(x) + \psi(y)) \quad \text{when} \quad \tilde{\mu}(y) = \mu^2(y), \tag{19}$$

$$\mathbf{f} = \mu(y)\cosh(\phi(x) + \psi(y)) \quad \text{when} \quad \tilde{\mu}(y) = -\mu^2(y), \tag{20}$$

$$\mathbf{f} = \psi(y)e^{\pm\phi(x)} \quad \text{when} \quad \tilde{\mu}(y) = 0, \tag{21}$$

where ψ is a function depending only on y and ϕ depends only on x and denotes a primitive of $\cos\theta$.

Lemma 2. *The ODE in* $\mathbf{f} = \mathbf{f}(x)$

$$\mathbf{f}'' + \tan\theta\ \theta'\mathbf{f}' - \cos^2\theta\mathbf{f} = 0,$$

with $\theta = \theta(x)$ *has solution*

$$\mathbf{f} = c_1 \sinh\phi(x) + c_2 \cosh\phi(x),$$

where ϕ *is a primitive of* $\cos\theta$ *and* $c_1,\ c_2$ *are real constants.*

We can now state the next theorem.

Theorem 3. *If* $F : M \to \mathbb{H}^2 \times \mathbb{R}$ *is an isometric immersion with* $\theta \neq 0,\ \frac{\pi}{2}$*, then* T *is a principal direction if and only if* F *is given by*

$$F(x,\ y) = (F_1(x,\ y),\ F_2(x,\ y),\ F_3(x,\ y),\ F_4(x)),$$

with $F_j(x,y) = A_j(y)\sinh\phi(x) + B_j(y)\cosh\phi(x)$*, for* $j = 1,2,3$ *and* $F_4(x) = \displaystyle\int_0^x \sin\theta(\tau)d\tau$*, where* $\phi'(x) = \cos\theta$*. The six functions* A_j *and* B_j *are found in one of the following cases.*

- **Case 1**.
$$\begin{aligned} A_j(y) &= \int_0^y H_j(\tau)\cosh\psi(\tau)d\tau + c_{1j}, \\ B_j(y) &= \int_0^y H_j(\tau)\sinh\psi(\tau)d\tau + c_{2j}, \\ H_j'(y) &= B_j(y)\sinh\psi(y) - A_j(y)\cosh\psi(y); \end{aligned}$$

- **Case 2**.
$$\begin{aligned} A_j(y) &= \int_0^y H_j(\tau)\sinh\psi(\tau)d\tau + c_{1j}, \\ B_j(y) &= \int_0^y H_j(\tau)\cosh\psi(\tau)d\tau + c_{2j}, \\ H_j'(y) &= -A_j(y)\sinh\psi(y) + B_j(y)\cosh\psi(y); \end{aligned}$$

- **Case 3**.
$$\begin{aligned} A_j(y) &= \pm\int_0^y H_j(\tau)d\tau + c_{1j}, \\ B_j(y) &= \int_0^y H_j(\tau)d\tau + c_{2j}, \\ H_j'(y) &= c_{2j} \mp c_{1j}; \end{aligned}$$

where $H = (H_1, H_2, H_3)$ *is a curve on the de Sitter space* $\mathbb{S}_1^2$*,* ψ *is a smooth function on* M *and* $c_1 = (c_{11}, c_{12}, c_{13})$*,* $c_2 = (c_{21}, c_{22}, c_{23})$ *are constant vectors.*

Proof. We choose canonical coordinates (x, y) as in Proposition 3.

First, let us determine the 4^{th} component of F. Taking the derivatives w.r.t. x and respectively w.r.t. y, one has $(F_4)_x = \widetilde{g}(F_x, \partial_t) = \sin\theta$, $(F_4)_y = 0$. It follows that F_4 depends only on x and it can be expressed, after a translation of M along the t-axis, as $F_4(x) = \int_0^x \sin\theta(\tau)d\tau$.

We point our attention now on the first three components. We are able to write the Levi-Civita connection of the metric given by (16)

$$\nabla_{\partial_x}\partial_x = 0, \tag{22}$$

$$\nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = \frac{\beta_x}{\beta}\partial_y, \tag{23}$$

$$\nabla_{\partial_y}\partial_y = -\beta\beta_x\partial_x + \frac{\beta_y}{\beta}\partial_y. \tag{24}$$

Recall that M is a codimension 2 surface in $\mathbb{R}^4_1$. Computing explicitly the two normals, we have

$$\xi = (\xi_1,\ \xi_2,\ \xi_3,\ \cos\theta),\ \text{where } \xi_j = -\tan\theta(F_j)_x,\ j = \overline{1,3}, \tag{25}$$

$$\tilde{\xi} = (F_1,\ F_2,\ F_3,\ 0), \tag{26}$$

with corresponding shape operators A given by (17) and

$$\widetilde{A} = \begin{pmatrix} -\cos^2\theta & 0 \\ 0 & -1 \end{pmatrix}. \tag{27}$$

Cf. also [7].

Using the Gauss formula **(G)** in combination with (22)-(26) one gets

$$(F_j)_{xx} = \cos^2\theta F_j - \tan\theta\theta_x(F_j)_x,\ j = \overline{1,3}, \tag{28}$$

$$(F_j)_{xy} = \frac{\beta_x}{\beta}(F_j)_y,\ j = \overline{1,3}, \tag{29}$$

$$(F_j)_{yy} = \beta^2 F_j - \frac{1}{\cos^2\theta}\beta\beta_x(F_j)_x + \frac{\beta_y}{\beta}(F_j)_y,\ j = \overline{1,3}. \tag{30}$$

We remark that (28) does not depend on β.

Our aim is to determine the function β. A first integration in (18) leads to $\frac{\beta_x^2}{\cos^2\theta} - \beta^2 = \tilde{k}(y)$ where $\tilde{k}$ is an arbitrary function on M depending only on y. Solving this PDE according to Lemma 1 we distinguish the following cases:

Case 1. If $\tilde{k}(y) = \mu^2(y)$, the solution has the form (19).

Changing the y-coordinate, we consider $\beta = \sinh(\phi(x) + \psi(y))$, with $\phi'(x) = \cos\theta$ and hence the metric has the form

$$g = dx^2 + \sinh^2(\phi(x) + \psi(y))dy^2. \tag{31}$$

From (29) and (30) and taking into account (31) we get

$$(F_j)_{xy} = \coth(\phi(x)+\psi(y))\cos\theta (F_j)_y, \tag{32}$$

and

$$\begin{aligned}(F_j)_{yy} = \sinh^2(\phi(x)+\psi(y))F_j - \frac{1}{2\cos\theta}\sinh 2(\phi(x)+\psi(y))(F_j)_x + \\ + \coth(\phi(x)+\psi(y))\psi'(y)(F_j)_y.\end{aligned} \tag{33}$$

After two consecutive integrations in (32) w.r.t. x and w.r.t. y we have

$$F_j(x,\ y) = \int_0^y H_j(\tau)\sinh(\phi(x)+\psi(\tau))d\tau + I_j(x).$$

In addition, since F_j fulfills (28), each I_j satisfies the second oder ordiuary differential equation $I_j'' + \tan\theta\theta_x I_j' - \cos^2\theta I_j = 0$ which, by Lemma 2, has the general solution $I_j = c_{1j}\sinh\phi(x) + c_{2j}\cosh\phi(x)$. Hence, we find

$$\begin{aligned}F_j(x,\ y) = \int_0^y H_j(\tau)\sinh(\phi(x)+\psi(\tau))d\tau + \\ + c_{1j}\sinh\phi(x) + c_{2j}\cosh\phi(x).\end{aligned} \tag{34}$$

We still have to use the condition (33). This yields

$$\begin{aligned}H_j'(y) = -\cosh(\phi(x)+\psi(y))\int_0^y H_j(\tau)\cosh(\phi(x)+\psi(\tau))d\tau + \\ + \sinh(\phi(x)+\psi(y))\int_0^y H_j(\tau)\sinh(\phi(x)+\psi(\tau))d\tau + \\ + c_{2j}\sinh\psi(y) - c_{1j}\cosh\psi(y).\end{aligned} \tag{35}$$

We remark that apparently the right hand in (35) depends both on x and y, while the left one only depends on y. Further-on we will see that everything depends only of y. If we define

$$\begin{aligned}A_j(y) = \int_0^y H_j(\tau)\cosh\psi(\tau)d\tau + c_{1j}, \\ B_j(y) = \int_0^y H_j(\tau)\sinh\psi(\tau)d\tau + c_{2j},\end{aligned} \tag{36}$$

then we can rewrite (34) and (35) in a simpler form

$$F_j(x,\ y) = A_j(y) \sinh \phi(x) + B_j(y) \cosh \phi(x), \tag{37}$$

and

$$H_j'(y) = B_j(y) \sinh \psi(y) - A_j(y) \cosh \psi(y).$$

We will keep in mind that A_j and B_j for $j = \overline{1,3}$ depend on y but, for simplicity, we drop the "y" in writing.

Let us consider $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$. Then we have

$$\sum_{j=1}^{3} \epsilon_j F_j^2 = -1, \tag{38}$$

$$\sum_{j=1}^{3} \epsilon_j (F_j)_x^2 = \cos^2 \theta(x), \tag{39}$$

$$\sum_{j=1}^{3} \epsilon_j (F_j)_x (F_j)_y = 0, \tag{40}$$

$$\sum_{j=1}^{3} \epsilon_j (F_j)_y^2 = \sinh^2(\phi(x) + \psi(y)). \tag{41}$$

Combining in a proper manner (37) with (38)-(40), denoting $A = (A_1,\ A_2,\ A_3)$ and $B = (B_1,\ B_2,\ B_3)$ we get the following relations written in terms of the Lorentzian scalar product

$$\langle A,\ A\rangle = 1, \langle B,\ B\rangle = -1, \langle A,\ B\rangle = 0, \left\langle A',\ B\right\rangle = \left\langle A,\ B'\right\rangle = 0. \tag{42}$$

As a consequence one has

$$\left\langle A,\ A'\right\rangle = 0,\ \left\langle B,\ B'\right\rangle = 0,\ \langle A,\ H\rangle = 0,\ \langle B,\ H\rangle = 0.$$

Finally, developing (41) one obtains

$$\langle H,\ H\rangle = \left\langle A',\ A'\right\rangle - \left\langle B',\ B'\right\rangle = 1.$$

Moreover, $\langle H',\ H'\rangle = 1$. We conclude that H is a unit speed spacelike curve on the Lorentzian unit sphere, known as the de Sitter space $\mathbb{S}_1^2$.

Case 2. If $\tilde{k}(y) = -\mu^2(y)$, the solution has the form (20).

Again, changing the y-coordinate we consider $\beta = \cosh(\phi(x) + \psi(y))$, where $\phi'(x) = \cos\theta$ and the metric in this case is given by

$$g = dx^2 + \cosh^2(\phi(x) + \psi(y))dy^2.$$

In a similar way as in **Case 1**, by straightforward computations one gets

$$F_j(x,\ y) = \int_0^y H_j(\tau)\cosh(\phi(x)+\psi(\tau))d\tau + c_{1j}\sinh\phi(x) + c_{2j}\cosh\phi(x),$$

with

$$\begin{aligned} H_j'(y) &= -\sinh(\phi(x)+\psi(y))\int_0^y H_j(\tau)\sinh(\phi(x)+\psi(\tau))d\tau \\ &\quad + \cosh(\phi(x)+\psi(y))\int_0^y H_j(\tau)\cosh(\phi(x)+\psi(\tau))d\tau \\ &\quad + c_{2j}\cosh\psi(y) - c_{1j}\sinh\psi(y). \end{aligned}$$

As in the previous case, we define the following quantities

$$A_j(y) = \int_0^y H_j(\tau)\sinh\psi(\tau)d\tau + c_{1j} \quad \text{and}$$

$$B_j(y) = \int_0^y H_j(\tau)\cosh\psi(\tau)d\tau + c_{2j}.$$

Then we have

$$F_j(x,\ y) = A_j(y)\sinh\phi(x) + B_j(y)\cosh\phi(x),$$

$$H_j'(y) = -A_j(y)\sinh\psi(y) + B_j(y)\cosh\psi(y).$$

Formulas (38)-(40) together with

$$\sum_{j=1}^{3}\epsilon_j(F_j)_y^2 = \cosh^2(\phi(x)+\psi(y)),$$

imply that the expressions (42) again hold, and that

$$\langle H,\ H\rangle = -\left\langle A',\ A'\right\rangle + \left\langle B',\ B'\right\rangle = 1.$$

In this case we find $\langle H',\ H'\rangle = -1$, hence H is a unit speed timelike curve in the de Sitter space $\mathbb{S}_1^2$.

Case 3. $\tilde{k}(y) = 0$. The solution for β is given by (21).

After a change of the y-coordinate, we have locally $\beta = e^{\pm\phi(x)}$, where $\phi'(x) = \cos\theta$, and the metric has the form

$$g = dx^2 + e^{\pm 2\phi(x)}dy^2.$$

By straightforward computations we find in this case

$$F_j(x,y) = A_j(y)\sinh\phi(x) + B_j(y)\cosh\phi(x),$$

where we denoted

$$A_j = \pm \int_0^y H_j(\tau)d\tau + c_{1j} \quad \text{and} \quad B_j = \int_0^y H_j(\tau)d\tau + c_{2j}, \tag{43}$$

with H_j satisfying

$$H_j'(y) = c_{2j} \mp c_{1j}. \tag{44}$$

Applying the same technique as in previous cases, we get

$$\langle A,\ A\rangle = 1,\ \langle B,\ B\rangle = -1,\ \langle A,\ B\rangle = 0,\ \left\langle A',\ B\right\rangle = 0,\ \left\langle A,\ B'\right\rangle = 0.$$

Moreover, $\langle A,\ H\rangle = 0,\ \langle B,\ H\rangle = 0$, since $A' = \pm H$, $B' = H$.

Remark that H is unitary, i.e. $\langle H,\ H\rangle = 1$, with $H' \neq 0$.

Denoting by $c_1 = (c_{11},\ c_{12},\ c_{13})$, $c_2 = (c_{21},\ c_{22},\ c_{23})$, from (44) we compute $H = (c_2 \mp c_1)y + c_3$ where $c_3 = (c_{31}, c_{32}, c_{33})$ is a constant vector. Plugging this value in (43) we get

$$A = \frac{\pm c_2 - c_1}{2}y^2 \pm c_3 y + c_1 \qquad \text{and} \qquad B = \frac{c_2 \mp c_1}{2}y^2 + c_3 y + c_2.$$

We conclude with the following relations satisfied by c_1, c_2 and c_3

$$\begin{aligned} &\langle c_1,\ c_1\rangle = 1,\ \langle c_2,\ c_2\rangle = -1,\ \langle c_3,\ c_3\rangle = 1,\\ &\langle c_1,\ c_2\rangle = 0,\ \langle c_1,\ c_3\rangle = 0, \quad \langle c_2,\ c_3\rangle = 0. \end{aligned} \tag{45}$$

From (44) and (45) it follows that $\langle H',\ H'\rangle = 0$. Hence, in this case, H is a lightlike curve in the de Sitter space $\mathbb{S}^2_1$.

Conversely, in each of the three cases we prove that the corresponding surface has T as principal direction. Since the idea of the proof is the same in all cases, we sketch the proof only in the first case.

We prove that the surface parametrized by

$$F(x,y) = \left(F_1(x,y),\ F_2(x,y),\ F_3(x,y),\ \int_0^x \sin\theta(\tau)d\tau\right),$$

where $F_j(x,y) = A_j(y)\sinh\phi(x) + B_j(y)\cosh\phi(x)$ with A_j and B_j given by (36) has T as principal direction.

Since $\widetilde{g}(F_x, F_x) = 1$, $\widetilde{g}(F_x, F_y) = 0$, $\widetilde{g}(F_y, F_y) = \sinh^2(\phi(x) + \psi(y))$, it follows that the metric g can be written in form (16). Computing the shape operator (e.g. from (9) and (10)) and using its symmetry we get $\theta_y = 0$ and (17). It is easy to prove that $\widetilde{g}(F_x, T) = \sin\theta$ and $\widetilde{g}(F_y, T) = 0$ concluding that T is a principal direction. ∎

Remark 5. In order to obtain a unified description, we note that in all cases F is given by

$$F(x,y)=\left(A(y)\sinh\phi(x)+B(y)\cosh\phi(x),\ \int_0^x \sin\theta(\tau)d\tau\right),$$

where A is a curve in $\mathbb{S}^2_1$ and B is a curve in $\mathbb{H}^2$ orthogonal to A such that the two speeds A' and B' are parallel. Denoting by H the unit vector of their common direction, one has $H=A\boxtimes B$ and moreover

- H is a spacelike curve in the first case,
- H is a timelike curve in the second case,
- H is a lightlike curve in the last case.

Theorem 4. (Classification theorem). *If $F:M\to\mathbb{H}^2\times\mathbb{R}$ is an isometric immersion with angle function $\theta\neq 0,\ \frac{\pi}{2}$, then T is a principal direction if and only if F is locally given by*

$$F(x,y)=(A(y)\sinh\phi(x)+B(y)\cosh\phi(x),\chi(x)),$$

where $A(y)$ is a curve in $\mathbb{S}^2_1$, $B(y)$ is a curve in $\mathbb{H}^2$, such that $\langle A,B\rangle=0$, $A'||B'$ and where $(\phi(x),\chi(x))$ is a regular curve in $\mathbb{R}^2$. The angle function θ of M depends only on x and coincides with the angle function of the curve (ϕ,χ). In particular, we can arc length reparametrize (ϕ,χ); then (x,y) are canonical coordinates and $\theta'(x)=\kappa(x)$, the curvature of (ϕ,χ).

Remark 6. Since ϕ is determined up to constants ($\phi'(x)=\cos\theta$) we check what happens if we put $\widetilde{\phi}(x)=\phi(x)-\phi_0$ for certain $\phi_0\in\mathbb{R}$. We immediately obtain that

$$F(x,y)=\left(\widetilde{A}(y)\sinh\tilde{\phi}(x)+\widetilde{B}(y)\cosh\tilde{\phi}(x),\ \chi(x)\right),$$

where

$$\widetilde{A}(y)=A(y)\cosh\phi_0+B(y)\sinh\phi_0,\quad \widetilde{B}(y)=A(y)\sinh\phi_0+B(y)\cosh\phi_0.$$

The new curves $\widetilde{A}$ and $\widetilde{B}$ satisfy the same conditions as the initial curves A and B.

While Theorem 4 provides an elegant classification, a combination with Theorem 3 allows to produce some explicit examples, for instance if we put $\psi=0$ or $\psi(y)=y$.

Example 2. In **Case 1** of Theorem 3 with $\psi=0$, we get

$$A_j(y)=\int_0^y H_j(\tau)d\tau+c_{1j},\ B_j(y)=c_{2j},\ H_j'(y)=-\int_0^y H_j(\tau)d\tau-c_{1j}.$$

It follows that $H(y) = l\cos y + m\sin y$, where $l, m = -c_1 \in \mathbb{R}^3$ are constant vectors. Since H is a unit vector, we find $\langle l,l\rangle = 1$, $\langle m,m\rangle = 1$, $\langle l,m\rangle = 0$. We can choose $l = (1,0,0)$ and $m = (0,-1,0)$. Then $A(y) = (\sin y, \cos y, 0)$ and we consider $B(y) = (0,0,1)$.

The parametrization F in this case is obtained by taking the curve

$$(\sinh\phi(x), \cosh\phi(x),\ \chi(x)),$$

in $\mathbb{H}^1\times\mathbb{R}\subset\mathbb{H}^2\times\mathbb{R}$ and rotating it in an appropriate way (in the (x_1x_2)-plane of $\mathbb{R}^3_1\times\mathbb{R}$).

For example, if $\theta(x) = x$, which yields $\kappa = 1$ in the classification theorem, a nice parametrization arises, namely

$$F(x,y) = \big(\sin y\sinh(\sin x),\ \cos y\sinh(\sin x),\ \cosh(\sin x),\ 1-\cos x\big).$$

Example 3. In **Case 2** of Theorem 3, when $\psi = 0$, we get $B(y) = (\sinh y, 0, \cosh y)$ and we can take $A(y) = (0,1,0)$. For instance, if $\theta(x) = \arccos(x)$ the surface is given by

$$F(x,y) = \left(\sinh y\cosh\frac{x^2}{2},\ \sinh\frac{x^2}{2},\ \cosh y\cosh\frac{x^2}{2},\ \frac{1}{2}\big(x\sqrt{1-x^2}+\arcsin x\big)\right).$$

Example 4. For **Case 1** with $\psi(y) = y$, we find

$$A(y) = \Big(y\sinh y-\cosh y,\ -\frac{y^2}{2}\sinh y + y\cosh y,\ \frac{y^2}{2}\sinh y - y\cosh y+\sinh y\Big),$$

$$B(y) = \Big(y\cosh y-\sinh y,\ -\frac{y^2}{2}\cosh y + y\sinh y,\ \frac{y^2}{2}\cosh y - y\sinh y+\cosh y\Big).$$

Example 5. For the second case of Theorem 3 with $\psi(y) = y$ we find

$$\begin{aligned} A(y) &= \Big(\sinh(y\sqrt{2})\sinh y - \frac{1}{\sqrt{2}}\cosh(y\sqrt{2})\cosh y,\ \frac{1}{\sqrt{2}}\cosh y,\\ &\qquad \cosh(y\sqrt{2})\sinh y - \frac{1}{\sqrt{2}}\sinh(y\sqrt{2})\cosh y\Big),\\ B(y) &= \Big(\sinh(y\sqrt{2})\cosh y - \frac{1}{\sqrt{2}}\cosh(y\sqrt{2})\sinh y,\ \frac{1}{\sqrt{2}}\sinh y,\\ &\qquad \cosh(y\sqrt{2})\cosh y - \frac{1}{\sqrt{2}}\sinh(y\sqrt{2})\sinh y\Big). \end{aligned}$$

Example 6. Concerning the last case in Theorem 3 with $\psi(y) = y$, we can choose for example $c_1 = (0,1,0)$, $c_2 = (0,0,1)$ and $c_3 = (1,0,0)$. Then, $A(y) = \Big(y,\ 1-\frac{y^2}{2},\ \frac{y^2}{2}\Big)$ and $B(y) = \Big(y,\ -\frac{y^2}{2},\ 1+\frac{y^2}{2}\Big)$.

We reformulate the classification Theorem 4 using just one curve as follows.

Theorem 5. *Let $F : M \to \mathbb{H}^2 \times \mathbb{R}$ be an isometric immersion with $\theta \neq 0, \frac{\pi}{2}$. Then M has T as a principal direction if and only if F is given by*

$$F(x, y) = \Big(f(y) \cosh \phi(x) + N_f(y) \sinh \phi(x), \chi(x)\Big), \tag{46}$$

where $f(y)$ is a regular curve in $\mathbb{H}^2$ and $N_f(y) = \frac{f(y) \boxtimes f'(y)}{\sqrt{\langle f'(y), f'(y)\rangle}}$ represents the normal of f. Moreover, (ϕ, χ) is a regular curve in $\mathbb{R}^2$ and the angle function θ of this curve is the same as the angle function of the surface parametrized by F.

Proof. In the classification Theorem 4, if B is not a constant (timelike) vector, rename it by f and hence $f \in \mathbb{H}^2$ with $\langle f', f'\rangle > 0$. The curve A lies on $\mathbb{S}^2_1$ and it is orthogonal to B. Moreover A' and B' are parallel. This implies that A can be identified by $A = \pm \frac{1}{\sqrt{\langle f'(y), f'(y)\rangle}} f(y) \boxtimes f'(y)$ and hence parametrization (46) is obtained.

Suppose B is a constant vector. Adding a constant to ϕ as in Remark 6, we change B to $\widetilde{B}$ such that $\widetilde{B}'$ is different from 0 and we can proceed as in the previous case.

The converse part follows from direct computations. ■

Remark 7. If in the previous theorem the angle function θ is constant, we recover Theorem 3.2 on the classification of constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$ in [7]. See also [12] for an alternative proof.

Another classical problem is the minimality of surfaces. In the following we will investigate this property for surfaces having T as a principal direction.

We give first an auxiliary result useful in the proof of the theorem.

Lemma 3. *The solution of the ODE*

$$\mathbf{f}'' - 2 \cot \mathbf{f} \; \mathbf{f}'^2 + \cos \mathbf{f} \sin \mathbf{f} = 0,$$

is given by $\mathbf{f} = \arctan\left(\frac{1}{a(x)}\right)$ where $a(x) = c_1 \cosh x + c_2 \sinh x$, $(c_1, c_2 \in \mathbb{R})$ never vanishes.

We state now the following classification theorem.

Theorem 6. *Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then M is minimal with T as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by one of the next cases*

$$F(x,\ y) = \left(\frac{b(x)}{\sqrt{1+c_1^2-c_2^2}}\ ,\ \frac{\sqrt{a^2(x)+1}}{\sqrt{1+c_1^2-c_2^2}}\ \sinh y, \frac{\sqrt{a^2(x)+1}}{\sqrt{1+c_1^2-c_2^2}}\ \cosh y, \chi(x)\right), \tag{47a}$$

(47b) $$F(x,\ y)=\left(\frac{\sqrt{a^2(x)+1}}{\sqrt{c_2^2-c_1^2-1}}\cos y,\ \frac{\sqrt{a^2(x)+1}}{\sqrt{c_2^2-c_1^2-1}}\sin y,\ \frac{b(x)}{\sqrt{c_2^2-c_1^2-1}},\chi(x)\right),$$

(47c) $$F(x,\ y)=\left(b(x)\ y,\ \frac{b(x)}{2}\ (1-y^2)-\frac{1}{2b(x)},\ \frac{b(x)}{2}\ (1+y^2)+\frac{1}{2b(x)},\ \chi(x)\right),$$

where

(48) $$\chi(x) = \int_0^x \frac{1}{\sqrt{a^2(\tau)+1}}\ d\tau\ ,$$

with $a(x) = c_1 \cosh x + c_2 \sinh x$*,* $b(x) = a'(x)$ *and* c_1, c_2 *are constants such that all quantities involved in the previous expressions are well defined.*

Remark 8. In all three cases, the curve $(\phi(x), \chi(x))$ is determined up to some real constants c_1 and c_2 by the angle function $\theta = \arctan\left(\frac{1}{a(x)}\right)$, where $a(x) = c_1 \cosh x + c_2 \sinh x$ with $c_1^2 + c_2^2 \neq 0$. In particular, in each case of the previous theorem we have

(49a) $$\sinh\phi(x) = \frac{b(x)}{\sqrt{1+c_1^2-c_2^2}}\ ,\ \cosh\phi(x) = \frac{\sqrt{a^2(x)+1}}{\sqrt{1+c_1^2-c_2^2}}\ ,$$
$$A(y) = (1,\ 0,\ 0) \text{ and } B(y) = (0,\ \sinh y,\ \cosh y);$$

(49b) $$\sinh\phi(x) = \frac{\sqrt{a^2(x)+1}}{\sqrt{c_2^2-c_1^2-1}}\ ,\ \cosh\phi(x) = \frac{b(x)}{\sqrt{c_2^2-c_1^2-1}},$$
$$A(y) = (\cos y,\ \sin y,\ 0) \text{ and } B(y) = (0,\ 0,\ 1);$$

(49c) $$\phi(x) = \pm \ln b(x)\ ,$$
$$A(y) = \left(y,\ 1-\tfrac{y^2}{2},\ \tfrac{y^2}{2}\right) \text{ and } B(y) = \left(y,\ -\tfrac{y^2}{2},\ 1+\tfrac{y^2}{2}\right).$$

Proof of Theorem 6. We choose canonical coordinates x and y as in Proposition 3. Starting with the classification Theorem 4, the isometric immersion is given by
$$F(x,y) = (A(y)\sinh\phi(x) + B(y)\cosh\phi(x), \chi(x)),$$
where $A(y)$ is a curve in $\mathbb{S}_1^2$, $B(y)$ is a curve in $\mathbb{H}^2$, such that $\langle A, B\rangle = 0$ and $A'||B'$. Recall that $\phi'(x) = \cos\theta(x)$ and $\chi'(x) = \sin\theta(x)$. The metric of this surface is obtained from (16), where

(50) $$\beta^2(x,y) = \frac{1}{2}\big(\langle A',A'\rangle + \langle B',B'\rangle\big)\cosh 2\phi(x) + \\ + \langle A',B'\rangle \sinh 2\phi(x) + \frac{1}{2}\big(-\langle A',A'\rangle + \langle B',B'\rangle\big).$$

The minimality condition yields $\theta_x + \tan\theta\dfrac{\beta_x}{\beta} = 0$. Integrating it once we obtain $\beta = \frac{1}{\sin\theta}$ after a change of y-coordinate. Recall that β and θ are also related by the general equation (18). Substituting here the expression of β, we get that the angle function θ satisfies

$$\theta_{xx} - 2\cot\theta\theta_x^2 + \cos\theta\sin\theta = 0,$$

which has the general solution given by Lemma 3, namely

$$\theta = \arctan\left(\frac{1}{a(x)}\right) \text{ where } a(x) = c_1\cosh x + c_2\sinh x \ (c_1, c_2 \in \mathbb{R}).$$

Next, we immediately compute $\sin\theta = \frac{1}{\sqrt{a^2(x)+1}}$. Hence, the fourth component of the parametrization F, i.e. $\chi(x)$, is given by (48) and $\beta = \sqrt{a^2(x)+1}$. Moreover, $\cos\theta = \frac{b'(x)}{\sqrt{b^2(x)+c_1^2-c_2^2+1}}$.

In order to determine explicitly $\phi(x) = \int\cos\theta(x)\ dx$ we distinguish the following cases.

Case 1. If $c_1^2 - c_2^2 + 1 > 0$, then $\phi(x) = \operatorname{arcsinh}\left(\frac{b(x)}{\sqrt{c_1^2-c_2^2+1}}\right)$, yielding that $\cosh\phi(x)$ and $\sinh\phi(x)$ are obtained as in (49a). Combining (50) with $\beta = \sqrt{c_1^2 - c_2^2 + 1}\cosh\phi$, one gets that the curves A and B must satisfy the following conditions

$$\langle A', A'\rangle = 0,\ \langle B', B'\rangle = c_1^2 - c_2^2 + 1,\ \langle A', B'\rangle = 0.$$

At this point we conclude that A is a constant curve on $\mathbb{S}_1^2$. Hence, up to some constants and after a change of $y-$coordinate $y\sqrt{1 + c_1^2 - c_2^2} \equiv y$, one can choose them as in (49a) obtaining parametrization (47a).

Case 2. If $c_1^2 - c_2^2 + 1 < 0$, then $\phi(x) = \operatorname{arccosh}\left(\frac{b(x)}{\sqrt{c_2^2-c_1^2-1}}\right)$. Following the same idea as in the previous case we get similar conditions that must be satisfied by the curves A and B:

$$\langle A', A'\rangle = c_2^2 - c_1^2 - 1,\ \langle B', B'\rangle = 0,\ \langle A', B'\rangle = 0.$$

Hence, B is a constant curve on $\mathbb{H}^2$. As in Case 1, curves A and B can be taken as in (49b). With these considerations (47b) is obtained.

Case 3. If $c_1^2 - c_2^2 + 1 = 0$, $\phi(x) = \pm\ln b(x)$. In this case the curves A and B must satisfy

$$\langle A', A'\rangle = 1,\ \langle B', B'\rangle = 1,\ \langle A', B'\rangle = \pm 1,$$

and up to some constants, can be chosen as in (49c) obtaining parametrization (47c).

Conversely, it can be proved that parametrizations (47) determine a minimal surface having T as principal direction. ■

Remark 9. When one of the two constants vanishes, the 4^{th} component of the parametrization, in the previous theorem, can be rewritten using elliptic functions as follows:

$$\chi(x) = \tfrac{1}{\sqrt{c^2+1}}\, F\left(\arccos \tfrac{1}{\cosh x} \;\middle|\; \tfrac{1}{1+c^2}\right) \text{ if } c_1 = c \text{ and } c_2 = 0,$$
$$\chi(x) = \; F\left(\arccos \tfrac{1}{\cosh x} \;\middle|\; 1 - c^2\right) \text{ if } c_1 = 0 \text{ and } c_2 = c,$$

where $F\,(z \mid m) = \displaystyle\int_0^z \frac{dt}{\sqrt{1 - m\sin^2 t}}$ is the elliptic integral of the first kind.

Concerning the flatness property for surfaces having T as a principal direction, we give the following classification theorem.

Theorem 7. *Let M be a surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\theta \neq 0, \frac{\pi}{2}$. Then M is flat with T a principal direction if and only if the immersion F is, up to isometries of the ambient space, given by*

$$F(x,\ y) = \left(\frac{x}{\sqrt{c+1}}\cos y,\ \frac{x}{\sqrt{c+1}}\sin y,\ \frac{\sqrt{x^2+c+1}}{\sqrt{c+1}},\ \chi(x)\right), \tag{51a}$$

$$F(x,\ y) = \left(\frac{\sqrt{x^2+c+1}}{\sqrt{-c-1}},\ \frac{x}{\sqrt{-c-1}}\sinh y,\ \frac{x}{\sqrt{-c-1}}\cosh y,\ \chi(x)\right), \tag{51b}$$

$$F(x,\ y) = \left(xy,\ \frac{x}{2}(1-y^2) - \frac{1}{2x},\ \frac{x}{2}(1+y^2) + \frac{1}{2x},\ \chi(x)\right), \tag{51c}$$

where

$$\chi(x) = \int^x \frac{\sqrt{\tau^2+c}}{\sqrt{\tau^2+c+1}}\, d\tau,\ c \in \mathbb{R}. \tag{52}$$

Proof. We use canonical coordinates and start with the parametrization given in Theorem 4. Under the flatness condition we obtain that $\tan\theta\theta_x\beta_x - \cos^2\theta\beta = 0$. Combining it with the PDE (18) we obtain $\beta = \mathfrak{a}(y)x + \mathfrak{b}(y)$ where $\mathfrak{a}$ and $\mathfrak{b}$ are smooth functions on M depending only on y with $\mathfrak{a}$ nowhere vanishing, and such that

$$\frac{\mathfrak{b}(y)}{\mathfrak{a}(y)} = \frac{\tan\theta\theta_x - x\cos^2\theta}{\cos^2\theta}\ . \tag{53}$$

Since the right hand side of (53) depends only on x, it follows that both sides are equal to the same constant, say $c_0 \in \mathbb{R}$. Thus

$$\beta = \mathfrak{a}(y)(x + c_0), \tag{54}$$

(55) $$\cos^2\theta(x+c_0)-\tan\theta\theta_x=0.$$

One can change the $(x,y)-$coordinates in (54) such that $\beta=x$, but with no effect on the other formulas in Proposition 3. Then from (55) one finds

(56) $$\theta=\arctan(\sqrt{x^2+c}),\ c\in\mathbb{R}.$$

By direct computations, $\sin\theta=\frac{\sqrt{x^2+c}}{\sqrt{x^2+c+1}}$ and $\cos\theta=\frac{1}{\sqrt{x^2+c+1}}$. Hence, the last component of the parametrization is given by (52). Let us distinguish the following cases for the real constant c.

Case 1. $c\geq 0$. The solution (56) for θ is well defined for $x\in(0,\ +\infty)$. As $c\geq 0$, one gets that $c+1>0$ and so, $\phi(x)=\operatorname{arcsinh}\left(\frac{x}{\sqrt{c+1}}\right)$. This yields $\cosh\phi(x)=\frac{\sqrt{x^2+c+1}}{\sqrt{c+1}}$. Combining both $\beta=\sqrt{c+1}\ \sinh\phi$ and (50) it follows that the curves A and B must satisfy

$$\langle A',A'\rangle=c+1,\ \langle B',B'\rangle=0,\ \langle A',B'\rangle=0.$$

Therefore, B is a constant curve in $\mathbb{H}^2$. Up to some constants and after a change of the $y-$coordinate $y\sqrt{c+1}\equiv y$, one can choose $A=(\cos y,\ \sin y,\ 0)$ and $B=(0,\ 0,\ 1)$ obtaining (51a).

Case 2. $c<0$.

Case 2.a $c\in(-1,\ 0)$. The angle function (56) is well defined for $x\in(\sqrt{-c},\ +\infty)$. We get again $c+1>0$ and the rest of the computations are the same as in **Case 1** leading also in this case to the parametrization (51a).

Case 2.b $c<-1$. Using similar arguments as in the previous cases the domain of x is $(\sqrt{-c},\ +\infty)$. Moreover, $\phi(x)=\operatorname{arccosh}\left(\frac{x}{\sqrt{-c-1}}\right)$. The curves A and B fulfill

$$\langle A',A'\rangle=0,\ \langle B',B'\rangle=-c-1,\ \langle A',B'\rangle=0.$$

Hence A is a constant curve in $\mathbb{S}^2_1$, and consequently, one gets the parametrization (51b).

Case 2.c $c=-1$. From (56) one obtains that $x\in(1,\ +\infty)$. Finally, $\phi(x)=\pm\ln(x)$ and $\beta=e^{\pm\phi(x)}$. Hence, the curves A and B satisfy

$$\langle A',A'\rangle=1,\ \langle B',B'\rangle=1,\ \langle A',B'\rangle=\pm 1.$$

Using the same technique, the parametrization (51c) is obtained.

Conversely, it is not difficult to prove that the parametrizations (51) indeed give a flat surface in $\mathbb{H}^2\times\mathbb{R}$ with T as a principal direction. ∎

At the end of this paper, we give some results concerning constant mean curvature surfaces with canonical principal directions.

Proposition 4. Let M be a surface in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$ and T as principal direction. If M has constant principal curvatures, then θ is constant and M is given by Example 6 with constant θ.

Proof. We look for such surfaces using canonical coordinates. As a consequence of (17) we have

$$\theta_x = \kappa_1 \text{ and } \tan\theta \, \frac{\beta_x}{\beta} = \kappa_2, \text{ where } \kappa_1, \kappa_2 \in \mathbb{R}.$$

If $\kappa_1 = 0$ then M is a constant angle surface belonging to Case 3 of Theorem 3. The result follows also from [7, Remark 3.5].
If $\kappa_1 \neq 0$ then $\theta = \kappa_1 x + c$, where $c \in \mathbb{R}$. Therefore, $\beta = \psi(y)[\sin\theta]^{\frac{\kappa_2}{\kappa_1}}$, where ψ is a smooth function on M. Moreover, θ and β fulfill also (18) which can be rewritten, if $\kappa_2 \neq 0$ and $\kappa_1 \neq \kappa_2$, as

$$\cos^2\theta[\sin\theta]^{\frac{\kappa_2}{\kappa_1}-2}\Big((\kappa_2 - \kappa_1)\kappa_2 - \sin^2\theta\Big) = 0,$$

yielding a contradiction with θ non constant ($\kappa_1 \neq 0$).
If $\kappa_2 = 0$ or $\kappa_1 = \kappa_2$ then $\beta_x = \kappa_2\psi(y)\cos\theta$ which implies $\beta\cos^2\theta = 0$, i.e. $\theta = \frac{\pi}{2}$. ∎

Yet, there exist non-minimal CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ having T as principal direction.

Example 7. We give an example of such a surface.

Looking at the expression (17) of the Weingarten operator A, we may write $\partial_x\big(\beta\sin\theta\big) = 2H\beta\cos\theta$, where $H \neq 0$ is the mean curvature of the surface M. We try to find θ and β such that $\beta\cos\theta$ is constant. Integrating the previous equation and taking into account that θ depends only on x and $\beta\cos\theta$ is a constant it follows, after a translation in parameter x, that $\theta = \arctan(2Hx)$. Moreover, after a homothetic transformation of the y-parameter, $\beta = \sqrt{1+4H^2x^2}$. As the compatibility equation (18) must be identically satisfied by θ and β, we get $H^2 = \frac{1}{4}$. Hence, $\theta = \arctan(x)$ and $\beta = \sqrt{x^2+1}$. Using Theorem 4, we find, $\phi(x) = \operatorname{arcsinh}(x)$ and $\chi(x) = \sqrt{1+x^2}$. Combining the expression of $\beta = \cosh\phi(x)$ with (50) we get that the curves A and B satisfy $\langle A', A'\rangle = 0$, $\langle A', B'\rangle = 0$, $\langle B', B'\rangle = 1$. Let us choose $A(y) = (1,0,0)$ and $B = (0, \cosh y, \sinh y)$. We conclude that

$$F(x,y) = \left(x, \sqrt{1+x^2}\ \sinh y, \sqrt{1+x^2}\ \cosh y, \sqrt{1+x^2}\ \right).$$

Acknowledgments

The authors would like to thank Ruy Tojeiro for valuable discussions leading to a simplification of Theorem 5.

References

1. J. A. Aledo, J. M. Espinar and J. A. Gálvez, Complete surfaces of constant curvature in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, *Calc. Var. and Partial Differential Equations*, **29(3)** (2007), 347-363.

2. B. Daniel, Isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and applications to minimal surfaces, *Trans. Amer. Math. Soc.*, **361(12)** (2009), 6255-6282.

3. F. Dillen, J. Fastenakels and J. Van der Veken, Rotation hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, *Note Mat.*, **29(1)** (2008), 41-54.

4. F. Dillen, J. Fastenakels and J. Van der Veken, Surfaces in $\mathbb{S}^2 \times \mathbb{R}$ with a canonical principal direction, *Ann. Global Anal. Geom.*, **35(4)** (2009), 381-396.

5. F. Dillen, J. Fastenakels, J. Van der Veken and L. Vrancken, Constant Angle Surfaces in $\mathbb{S}^2 \times \mathbb{R}$, *Monatsh. Math.*, **152(2)** (2007), 89-96.

6. F. Dillen and M. I. Munteanu, Surfaces in $\mathbb{H}^+ \times \mathbb{R}$, *Proceedings of the conference Pure and Applied Differential Geometry, PADGE*, Brussels 2007, pp. 185-193.

7. F. Dillen and M. I. Munteanu, Constant Angle Surfaces in $\mathbb{H}^2 \times \mathbb{R}$, *Bull. Braz. Math. Soc.*, **40(1)** (2009), 85-97.

8. J. Fastenakels, M. I. Munteanu and J. Van der Veken, Constant angle surfaces in the Heisenberg group, *Acta Math. Sin.* $($*Engl. Ser.*$)$, **27(4)** (2011), 747-756.

9. I. Fernández and P. Mira, A characterization of constant mean curvature surfaces in homogeneous 3-manifolds, *Differential Geom. Appl.*, **25(3)** (2007), 281-289.

10. L. Hauswirth, Minimal surfaces of Riemann type in three-dimensional product manifolds, *Pacific J. Math.*, **224(1)** (2006), 91-118.

11. R. López and M. I. Munteanu, Constant angle surfaces in Minkowski space, to appear in *Bull. Belg. Math. Soc.*, 2011.

12. F. Manfio and R. Tojeiro, Hypersurfaces with constant sectional curvature of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, to appear in *Illinois J. Math.*, available on arxiv:0909.2261v1 [math.DG] 2009.

13. W. H. Meeks III and H. Rosenberg, Stable minimal surfaces in $\mathbb{M} \times \mathbb{R}$, *J. Differential Geom.*, **68(3)** (2004), 515-534.

14. W. H. Meeks III and H. Rosenberg, The theory of minimal surfaces in $\mathbb{M} \times \mathbb{R}$, *Comment. Math. Helv.*, **80(4)** (2005), 811-858.

15. F. Mercuri, S. Montaldo and P. Piu, A Weierstrass representation Formula for Minimal Surfaces in $\mathbb{H}_3$ and $\mathbb{H}^2 \times \mathbb{R}$, *Acta Math. Sin.* $($*Engl. Ser.*$)$, **22(6)** (2006), 1603-1612.

16. S. Montaldo and I. I. Onnis, A note on surfaces in $\mathbb{H}^2 \times \mathbb{R}$, *Boll. U.M.I.*, **8(10-B)** (2007), 939-950.

17. M. I. Munteanu and A. I. Nistor, A new approach on constant angle surfaces in $\mathbb{E}^3$, *Turkish J. Math.*, **33(2)** (2009), 169-178.

18. B. Nelli and H. Rosenberg, Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, *Bull. Braz. Math. Soc.*, **33(2)** (2002), 263-292.

19. H. Rosenberg, Minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$, *Illinois J. Math.*, **46(4)** (2002), 1177-1195.

20. R. Sa Earp, Parabolic and hyperbolic screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$, *J. Aust. Math. Soc.*, **85(1)** (2008), 113-143.

21. R. Sa Earp and E. Toubiana, Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, *Illinois J. Math.*, **49(4)** (2005), 1323-1362.

22. R. Souam and E. Toubiana, Totally umbilic surfaces in homogeneous 3-manifolds, *Comment. Math. Helv.*, **84(3)** (2009), 673-704.

23. R. Tojeiro, On a class of hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, *Bull. Braz. Math. Soc.*, **41(2)** (2010), 199-209.

Franki Dillen and Ana-Irina Nistor
Katholieke Universiteit Leuven
Departement Wiskunde
Celestijnenlaan 200 B
B-3001 Leuven
Belgium
E-mail: franki.dillen@wis.kuleuven.be
ana.irina.nistor@gmail.com

Marian Ioan Munteanu
University 'Al. I. Cuza' Iaşi
Faculty of Mathematics
Bd. Carol I, no. 11
700506 Iaşi
Romania
E-mail: marian.ioan.munteanu@gmail.com