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THE FORMATION OF SINGULARITIES IN THE HARMONIC MAP HEAT FLOW WITH BOUNDARY CONDITIONS

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Abstract. Let M be a compact manifold with boundary and N be compact manifold without boundary. Let $u(x,t)$ be a smooth solution of the harmonic heat equation from M to N with Dirichlet or Neumann condition. Suppose that M is strictly convex, we will prove a monotonicity formula for u. Moreover, if T is the blow-up time for u, and $\sup_M |Du|^2(x,t) \leq C/(T-t)$, we prove that a subsequence of the rescaled solutions converges to a homothetically shrinking soliton.

1. Introduction

Let M and N be compact manifolds and let $u(x,t)$ be a smooth solution of the harmonic heat equation

$$u_t = \Delta_M u + \Gamma_N(u)(Du, Du) \qquad \text{in} \quad M \times (0,T). \tag{1.1}$$

Suppose that T is the blow-up time for u, i.e.,

$$\sup_M |Du|(x,t) \to \infty \quad \text{as} \quad t \to T.$$

Let x_0 be a singularity point. We define

$$u_\lambda(x,t) = u\left(\exp_{x_0} \lambda x, T + \lambda^2 t\right). \tag{1.2}$$

When M is a compact manifold without boundary and has dimension n, in [2], Grayson and Hamilton proved that if the singularity forms rapidly, i.e.,

$$\sup_M |Du|^2(x,t) \leq \frac{C}{T-t}, \tag{1.3}$$

there is a sequence λ_i such that on each compact set in $\mathbb{R}^n \times (-\infty, 0)$, the rescaled maps $\{u_{\lambda_i}\}$ converges uniformly to a non-constant map $\bar{u} : \mathbb{R}^n \times (-\infty, 0) \to N$

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and $\bar{u}$ satisfies the harmonic map heat flow on $\mathbb{R}^n$, and is dilation-invariant, i.e., for any $\lambda > 0$, we have

$$\bar{u}(x,t) = \bar{u}(\lambda x, \lambda^2 t). \tag{1.4}$$

We call a solution of the harmonic heat equation (1.1) satisfying the dilation-invariant condition (1.4) a homothetic soliton.

To prove their results, Grayson and Hamilton made use of a monotonicity formula from [4]: Let $u(x,t) : M \times (0,T) \to N$ be a smooth solution to the harmonic map heat flow, and

$$\int_M |Du|^2(x,t)\, dx \le E_0 \quad \text{for} \quad 0 < t < T.$$

If we define

$$Z(t) = (T-t)\int_M |Du|^2 k\, dx,$$

where k is the backward heat kernel on M, then, there are constants $B > 0$ and $C > 0$ such that for any $0 < t < T$,

$$\frac{d}{dt}\left(e^{2C\varphi} Z\right) \le -2e^{2C\varphi}(T-t)\int_M \left|\Delta u + \frac{Du \cdot Dk}{k}\right|^2 k\, dx + 4CE_0 e^{2C\varphi},$$

where

$$\varphi(t) = (T-t)\left(\frac{n}{2} + \log\left(B/(T-t)^{n/2}\right)\right).$$

This involves a nontrivial estimates on the matrix of second derivatives of the heat kernel on a compact manifold M: there are constants B and C depending only on M such that,

$$D_iD_jk - \frac{D_ikD_jk}{k} + \frac{1}{2t}kg_{ij} + Ck\left(1 + \log\left(\frac{Bk}{t^{m/2}}\right)\right)g_{ij} \ge 0.$$

See [3].

Here, we would like to consider the case where M has non-empty boundary and the solution $u(x,t)$ satisfies the Dirichlet boundary condition

$$u(x,t) = h(x) \qquad \text{on} \quad \partial M \times (0,T) \tag{1.5}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \partial M \times (0,T). \tag{1.6}$$

Let x_0 and x be points in M. We denote $r(x_0; x)$ to be the distance between x_0 and x. We define

$$\mathcal{E}(x_0;t) = (T-t)\int_M |Du|^2(x,t) G(x_0,T;x,t)\, dx,$$

where

$$G(y,s;x,t) = \left(\frac{1}{4\pi|s-t|}\right)^{n/2} \exp\left(\frac{r^2(y;x)}{4(t-s)}\right).$$

When $M = \mathbb{R}^n$, the function $G(y,s;x,t)$ is the backward heat kernel. When ∂M is strictly convex and $u(x,t)$ is a smooth solution of the harmonic heat equation and satisfies the Dirichlet boundary condition (1.5), we will prove a monotonicity formula: there is a constant $A > 0$, such that

$$\begin{aligned} &\frac{d}{dt}\left(\exp\left(2|T-t|^{1/2}\right)\mathcal{E}(t) + A|T-t|^{1/2}\right) \\ &\leq -2\exp\left(2|T-t|^{1/2}\right)|T-t|\int_M \left(u_t + \frac{Du\cdot Dr^2}{4(t-T)}\right)^2 G\,dx. \end{aligned} \tag{1.7}$$

Using this formula, we obtain the similar results as in [2]. Let u_λ be the function defined in (1.2). Suppose that (1.3) holds and (x_0,T) is an interior singularity point, then there is a sequence λ_i such that on each compact set in $\mathbb{R}^n\times(-\infty,0)$, $\{u_{\lambda_i}\}$ in converges uniformly to a non-constant map $\bar{u} : \mathbb{R}^n\times(-\infty,0)\to N$ and $\bar{u}$ satisfies the harmonic map heat flow on $\mathbb{R}^n$, and is dilation-invariant. Let $\mathbb{R}^n_+ = \{x\in\mathbb{R}^n : x_n > 0\}$. If (x_0,T) is a boundary singularity point, we show that there is a sequence λ_i such that on each compact set in $\mathbb{R}^n_+\times(-\infty,0)$, $\{u_{\lambda_i}\}$ in converges uniformly to a non-constant map $\bar{u} : \mathbb{R}^n_+\times(-\infty,0)\to N$. Also, the limit function $\bar{u}$ satisfies the harmonic map heat flow on $\mathbb{R}^n_+\times(-\infty,0)$, and is dilation-invariant, and is a constant on the hyperplane $\{(x,t)\in\mathbb{R}^n\times(-\infty,0) : x_n = 0\}$.

It is interesting to know whether boundary singularities exist. This is equivalent to ask whether there is non-constant solution to the harmonic map heat flow on $\mathbb{R}^n_+\times(-\infty,0)$, and is dilation-invariant and is a constant on the hyperplane $\{(x,t)\in\mathbb{R}^n\times(-\infty,0) : x_n = 0\}$. In fact, there are harmonic maps from $B^3(1) = \{x\in\mathbb{R}^3 : |x| < 1\}$ to $S^2 = \{x\in\mathbb{R}^3 : |x| = 1\}$ which is smooth in B^3 and have singularities on the boundary, [6].

Let $u : M\times[0,T)\to N$ be a regular solution of (1.1) with Neumann boundary condition (1.6). Suppose that M is a compact manifold with convex boundary. We prove that similar results are true. Let $\mathcal{E}(x_0;t)$ be the energy function defined in the above, we show that there is a constant $B > 0$ such that

$$\begin{aligned} &\frac{d}{dt}\left(\exp\left(2|T-t|^{1/2}\right)\mathcal{E}(t) + B|T-t|^{1/2}\right) \\ &\leq -2\exp\left(2|T-t|^{1/2}\right)|T-t|\int_M \left(u_t + \frac{Du\cdot Dr^2}{4(t-T)}\right)^2 G\,dx. \end{aligned}$$

Using this monotonicity formula, it is not difficult to see that the small-energy-regularity theory also works and the rescaled solution converges to a homothetically shrinking solition solution.

In a forthcoming paper, we will use similar method to treat the equation

$$u_t = \Delta u + u^p$$

defined on a compact manifold with convex boundary.

2. Monotonicity Formula

Let M be a compact manifold with $C^{2,\alpha}$ boundary and N be a compact manifold. Let $u(x,t)$ be a smooth solution of the harmonic heat equation

$$u_t = \Delta_M u + \Gamma_N(u)(Du, Du) \qquad \text{in} \quad M \times (0,T). \tag{2.1}$$

The term $\Gamma_N(u)(Du, Du)$ is perpendicular to the tangent plane at $u(x)$ and for some constant $C > 0$, depending only on N,

$$|\Gamma_N(u)(Du, Du)| \leq C|Du|^2.$$

We assume that $u(x,t)$ satisfies the Dirichlet boundary condition

$$u(x,t) = h(x) \qquad \text{on} \quad \partial M \times (0,T) \tag{2.2}$$

where h is a function in $C^{2,\alpha}(\bar{M}, N)$. Let x and x_0 be in $\bar{M}$. We denote $r(x_0; x)$ to be the distance between x_0 and x on M. We say ∂M is strictly convex, if there is a constant $\gamma > 0$ so that for any $x_0 \in \bar{M}$,

$$Dr^2 \cdot \nu \geq \gamma r^2 > 0 \qquad \text{on} \quad \partial M, \tag{2.3}$$

where ν is the unit outward normal on ∂M.

Suppose that Ω is a strictly convex domain in $\mathbb{R}^n$ with smooth boundary. There exists $R > 0$ such that for any $x \in \partial\Omega$, there is $y \in \mathbb{R}^n$, Ω is contained in $B(y,R) = \{x : |x-y| < R\}$ and $\partial B(y,R) \cap \partial\Omega = \{x\}$. In that case, if $v(x)$ is the unit outward normal at x, then we have $\nu(x) = (x-y)/|x-y|$. Also, for any $x_0 \in \bar{\Omega}$, we have $r(x, x_0) = |x - x_0|$ and $Dr^2(x, x_0) = 2(x - x_0)$. Thus,

$$Dr^2(x, x_0) \cdot \nu(x) = 2\frac{(x-x_0)\cdot(x-y)}{|x-y|} = \frac{2|x-y|^2 - 2(x_0-y)\cdot(x-y)}{|x-y|}.$$

Since $|x-y| = R$ and $|x_0 - y| \leq R$, we have

$$Dr^2(x, x_0) \cdot \nu(x) \geq \frac{|x-y|^2 - 2(x_0-y)\cdot(x-y) + |x_0-y|^2}{|x-y|} = \frac{r^2(x, x_0)}{R}.$$

Hence, (2.3) is true with $\gamma = 1/R$.

For any $x_0 \in M$, we also define the function

$$G(x_0, T; x, t) = \left(\frac{1}{4\pi|T-t|}\right)^{n/2} \exp\left(\frac{r^2(x_0; x)}{4(t-T)}\right).$$

Suppose that

$$\max_{x\in M}|Du|(x,t)\to\infty \qquad \text{as} \quad t\to T.$$

For any $x_0\in\bar{M}$, let

$$\mathcal{E}(x_0;t)=(T-t)\int_M |Du|^2(x,t)G(x_0,T;x,t)\,dx.$$

Theorem 2.1. *Suppose that ∂M is strictly convex. Let $u(x,t)$ be a smooth solution of the harmonic heat equation with Dirichlet boundary condition, and*

$$\int_M |Du|^2(x,t)\,dx\le E_0 \qquad \text{for} \quad t\in(0,T). \tag{2.4}$$

Then, there is $A>0$, depending only on M, N, h, T and E_0, so that, for all $t\in(0,T)$,

$$\begin{aligned}&\frac{d}{dt}\left(\exp\left(2|T-t|^{1/2}\right)\mathcal{E}(x_0;t)+A|T-t|^{1/2}\right)\\ &\le -2\exp\left(2|T-t|^{1/2}\right)|T-t|\int_M\left(u_t+\frac{Du\cdot Dr^2}{4(t-T)}\right)^2 G(x_0,T;x,t)\,dx.\end{aligned}\tag{2.5}$$

We will need the following propositions. The first one concerns the hessian of the distance function, the second one concerns an integral on the boundary.

Proposition 2.2. *Let $x_0\in\bar{M}$ and $r(x)=\mathrm{dist}(x,x_0)$. There is a constant C depending on M so that*

$$|\Delta r^2-2n|\le Cr^2$$

and

$$\left|D^2(r^2)(X,X)-2|X|^2\right|\le Cr^2|X|^2,$$

where $D^2(f)$ denotes the hessian of a function f and X is any tangent vector on T_xM.

Proposition 2.3. *There is a constant $C>0$, depending on the geometries of ∂M and M only, so that, for any $x_0\in\bar{M}$,*

$$\int_{\partial M} G(x_0,T;x,t)\,d\sigma\le\frac{C}{|t|^{1/2}}.$$

Proof. Since ∂M is $C^{2,\alpha}$ and compact, there is $R>0$ such that for any $a\in M$, and dist$(a,\partial M)<R$, there is $\tilde{a}\in\partial M$ such that dist$(a,\partial M)$=dist$(a,\tilde{a})$. Moreover, we may choose R small enough, such that for each $\tilde{a}\in\partial M$, the set

$$B(\tilde{a},R)=\{x\in\bar{M}:\mathrm{dist}(x,\tilde{a})<R\}$$

can be represented by a chart $(\phi_1,\ldots,\phi_n)$ so that $B(\tilde{a},R)\cap M$ is identified with a region Ω,

$$\Omega \subset \{\phi \in \mathbb{R}^n : |\phi| \le 2R,\ \phi_n > \varphi(\phi_1, ..., \phi_{n-1}\},$$

for some $C^{2,\alpha}$ function φ, $\varphi(0) = 0$. The boundary region $\partial M \cap B(\tilde{a}, R)$ is identified with the graph $\phi_n = \varphi(\phi_1, ..., \phi_n)$ and the point $\tilde{a}$ is identified with the point $0 \in \mathbb{R}^n$. Since ∂M is a compact set, if R is chosen small enough, there is a constant $\delta > 0$, depending only on M, such that if $x, \bar{x} \in B(\tilde{a}, R) \cap M$, and $\phi, \bar{\phi}$ be corresponding points in Ω, we have

$$\delta \mathrm{dist}_M(x, \bar{x}) \le \mathrm{dist}_{\mathbb{R}^n}(\phi, \bar{\phi}) \le \frac{1}{\delta} \mathrm{dist}_M(x, \bar{x}).$$

Furthermore, if we choose R and δ small enough, for $x, \bar{x} \in \partial M \cap B(\tilde{a}, R)$, we also have

$$\delta \mathrm{dist}_{\partial M}(x, \bar{x}) \le \mathrm{dist}_{\mathbb{R}^n}(\phi, \bar{\phi}) \le \frac{1}{\delta} \mathrm{dist}_{\partial M}(x, \bar{x}).$$

Now, let $x_0 \in \bar{M}$ and $\mathrm{dist}(x_0, \partial M) = d < R/2$. We can find $\tilde{x}_0 \in \partial M$ and a chart $(\phi_1, ..., \phi_n)$ described in the above. After a rotation, we may assume that the point $\tilde{x}_0$ is identified with the origin in the chart and the point x_0 is identified with the point $(0, ..., 0, d)$. For any $x \in \partial M \cap B(\tilde{x}_0, R)$, which is identified with a point $\phi \in \partial\Omega$, we have

$$\begin{aligned} \frac{1}{\delta}\ \mathrm{dist}^2_M(x, x_0) &\ge \phi_1^2 + ... + \phi_{n-1}^2 + (\phi_n - d)^2 \ge \phi_1^2 + ... + \phi_{n-1}^2 \\ &\ge \delta \mathrm{dist}^2_{\partial M}(x, \tilde{x}_0) \ge \delta \mathrm{dist}^2_M(x, \tilde{x}_0). \end{aligned}$$

We let $\tilde{r}(x) = \mathrm{dist}^2_{\partial M}(x, \tilde{x}_0)$ for $x \in \partial M$. Then,

$$G(x,t) \le \frac{1}{|t|^{n/2}} \exp\left(\frac{\delta^2 \tilde{r}^2(x)}{4t}\right) \quad \text{when} \quad x \in \partial M \cap B(\tilde{x}_0, R), \quad t < 0,$$

and

$$G(x,t) \le \frac{1}{|t|^{n/2}} \exp\left(\frac{R^2}{4t}\right) \quad \text{when} \quad x \in \partial M - B(\tilde{x}_0, R), \quad t < 0.$$

Thus, when $\mathrm{dist}(x_0, \partial M) \le R/2$, we have

$$\begin{aligned} \int_{\partial M} G\ d\sigma &= \int_{\partial M \cap B(\tilde{x}_0, R)} G\ d\sigma + \int_{\partial M - B(\tilde{x}_0, R)} G\ d\sigma \\ &\le \frac{C_2}{|t|^{1/2}} + \frac{1}{|t|^{n/2}} \exp\left(\frac{R}{4t}\right) \mathrm{vol}(\partial M) \\ &\le \frac{C_3}{|t|^{1/2}}. \end{aligned} \tag{2.6}$$

If $\mathrm{dist}(x_0, \partial M) > R/2$, then

$$\int_{\partial M} G\ d\sigma = \frac{1}{|t|^{n/2}} \int_{\partial M} \exp\left(\frac{r^2}{4t}\right)\ dx \le \frac{1}{|t|^{n/2}} \exp\left(\frac{R^2}{16t}\right) \mathrm{vol}(\partial M). \tag{2.7}$$

From (2.6) and (2.7), there is a constant $C_4 > 0$ so that

$$\int_{\partial M} G \, d\sigma \le \frac{C_4}{|t|^{1/2}}. \tag{2.8}$$

We note that the constant C_4 depends on the geometries of ∂M and M only. ∎

Proof of Theorem 2.1. After a translation in time, we may assume the $u(x,t)$ is defined on $(-T,0)$. Let $x_0 \in \bar{M}$. We will write $r(x) = r(x_0; x) = \operatorname{dist}(x_0, x)$, and

$$\mathcal{E}(t) = \mathcal{E}(x_0; t) = |t| \int_M |Du|^2(x,t) G(x,t)\, dx,$$

where

$$G(x,t) = \left(\frac{1}{4\pi|t|}\right)^{n/2} \exp\left(\frac{r^2(x)}{4t}\right),$$

for $x \in M$ and $t \in (-T, 0)$. By straightforward computations, we have

$$\begin{aligned}
&\mathcal{E}'(t) \\
&= - \int_M |Du|^2(x,t)G(x,t)\, dx + |t| \int_M \left(2Du \cdot Du_t G + |Du|^2 G_t\right) dx \\
&= - \int_M |Du|^2(x,t)G(x,t)\, dx + 2|t| \int_M \left(Du \cdot Du_t + \frac{Du \cdot D^2 u \cdot Dr^2}{4t}\right) G\, dx \\
&\quad + |t| \int_M |Du|^2 (G_t + \Delta G)\, dx + 2|t| \int_{\partial M} |Du|^2 \frac{Dr^2 \cdot \nu}{4t} G\, d\sigma \\
&= - \int_M |Du|^2(x,t)G(x,t)\, dx + 2|t| \int_M Du \cdot D\left(u_t + \frac{Du \cdot Dr^2}{4t}\right) G\, dx \\
&\quad - 2|t| \int_M \frac{Du \cdot D^2 r^2 \cdot Du}{4t} G\, dx + |t| \int_M |Du|^2 (G_t + \Delta G)\, dx \\
&\quad + 2|t| \int_{\partial M} |Du|^2 \frac{Dr^2 \cdot \nu}{4t} G\, d\sigma \\
&= - 2|t| \int_M \left(\Delta u + \frac{Du \cdot Dr^2}{4t}\right)\left(u_t + \frac{Du \cdot Dr^2}{4t}\right) G\, dx \\
&\quad - \int_M |Du|^2(x,t)G(x,t)\, dx - 2|t| \int_M \frac{Du \cdot D^2 r^2 \cdot Du}{4t} G\, dx \\
&\quad + |t| \int_M |Du|^2 (G_t + \Delta G)\, dx + 2|t| \int_{\partial M} |Du|^2 \frac{Dr^2 \cdot \nu}{4t} G\, d\sigma \\
&\quad + 2|t| \int_{\partial M} \frac{\partial u}{\partial \nu}\left(u_t + \frac{Du \cdot Dr^2}{4t}\right) G\, d\sigma.
\end{aligned}$$

By equation (2.1), since the term $\Gamma_N(u)(Du, Du)$ is orthogonal to $T_{u(x)}N$, we have

$$
\begin{aligned}
&\mathcal{E}'(t)\\
&= -2|t|\int_M \left(u_t+\frac{Du\cdot Dr^2}{4t}\right)^2 G\ dx + |t|\int_M |Du|^2(G_t+\Delta G)\ dx\\
&\quad - \int_M |Du|^2(x,t)G(x,t)\ dx - 2|t|\int_M \frac{Du\cdot D^2r^2\cdot Du}{4t}G\ dx\\
&\quad + 2|t|\int_{\partial M}|Du|^2\frac{Dr^2\cdot\nu}{4t}G d\sigma + 2|t|\int_{\partial M}\frac{\partial u}{\partial\nu}\left(u_t+\frac{Du\cdot Dr^2}{4t}\right)G\ d\sigma.
\end{aligned}
\tag{2.9}
$$

Since $u_t=0$ on ∂M, from (2.9), we have

$$
\begin{aligned}
&\mathcal{E}'(t)\\
&= -2|t|\int_M \left(u_t+\frac{Du\cdot Dr^2}{4t}\right)^2 G\ dx + |t|\int_M |Du|^2(G_t+\Delta G)\ dx\\
&\quad - \int_M |Du|^2(x,t)G(x,t)\ dx - 2|t|\int_M \frac{Du\cdot D^2r^2\cdot Du}{4t}G\ dx\\
&\quad + 2|t|\int_{\partial M}|Du|^2\frac{Dr^2\cdot\nu}{4t}G\ d\sigma + 2|t|\int_{\partial M}\frac{\partial u}{\partial\nu}\frac{Du\cdot Dr^2}{4t}G\ d\sigma.
\end{aligned}
\tag{2.10}
$$

On ∂M, we may write

$$
Du=\frac{\partial u}{\partial\nu}+D_Tu \qquad \text{and} \qquad Dr^2 = Dr^2\cdot\nu + D_Tr^2.
$$

Then,

$$
\frac{\partial u}{\partial\nu}(Du\cdot Dr^2) = \frac{\partial u}{\partial\nu}\left(\frac{\partial u}{\partial\nu}(Dr^2\cdot\nu)+D_Tu\cdot D_Tr^2\right).
$$

When $t\in(-T,0)$, this gives

$$
\begin{aligned}
&2|t|\int_{\partial M}|Du|^2\frac{Dr^2\cdot\nu}{4t}G\ d\sigma + 2|t|\int_{\partial M}\frac{\partial u}{\partial\nu}\frac{Du\cdot Dr^2}{4t}G\ d\sigma\\
&= -\frac{1}{2}\int_{\partial M}|D_Tu|^2(Dr^2\cdot\nu)G\ d\sigma - \frac{1}{2}\int_{\partial M}\frac{\partial u}{\partial\nu}\left(D_Tu\cdot D_Tr^2\right)G\ d\sigma\\
&\quad - \int_{\partial M}\left(\frac{\partial u}{\partial\nu}\right)^2(Dr^2\cdot\nu)G\ d\sigma
\end{aligned}
$$

Also, by (2.3), we have

$$
\begin{aligned}
&2|t|\int_{\partial M}|Du|^2\frac{Dr^2\cdot\nu}{4t}G\ d\sigma + 2|t|\int_{\partial M}\frac{\partial u}{\partial\nu}\frac{Du\cdot Dr^2}{4t}G d\sigma\\
&\le -\int_{\partial M}\left(\frac{\partial u}{\partial\nu}\right)^2(Dr^2\cdot\nu)G\ d\sigma + \frac{1}{2}\int_{\partial M}\left|\frac{\partial u}{\partial\nu}\right||D_Tu||D_Tr^2|G\ d\sigma\\
&\le -\gamma\int_{\partial M}\left(\frac{\partial u}{\partial\nu}\right)^2 r^2G\ d\sigma + \int_{\partial M}\left|\frac{\partial u}{\partial\nu}\right||D_Tu|r|D_Tr|G\ d\sigma
\end{aligned}
$$

$$\leq -\gamma\int_{\partial M}\left(\frac{\partial u}{\partial \nu}\right)^2 r^2 G\, d\sigma + \gamma\int_{\partial M}\left|\frac{\partial u}{\partial \nu}\right|^2 r^2 G\, d\sigma + \frac{1}{4\gamma}\int_{\partial M}|D_T u|^2 |D_T r|^2 G\, d\sigma$$
$$\leq \frac{1}{4\gamma}\int_{\partial M} |D_T u|^2 |D_T r|^2 G\, d\sigma$$

Thus, one can see that there is a constant C_1, depending only on h and γ and the geometries of ∂M and M, so that

$$\begin{aligned}(2.11)\qquad & 2|t|\int_{\partial M} |Du|^2 \frac{Dr^2\cdot\nu}{4t} G\, d\sigma + 2|t|\int_{\partial M} \frac{\partial u}{\partial \nu}\frac{Du\cdot Dr^2}{4t} G\, d\sigma \\ & \leq \frac{\max(D_T h)^2}{4\gamma}\int_{\partial M} G\, d\sigma = C_1\int_{\partial M} G\, d\sigma.\end{aligned}$$

By Proposition 2.3, we obtain

$$2|t|\int_{\partial M} |Du|^2 \frac{Dr^2\cdot\nu}{4t} G\, d\sigma + 2|t|\int_{\partial M} \frac{\partial u}{\partial \nu}\frac{Du\cdot Dr^2}{4t} G\, d\sigma \leq \frac{C_5}{|t|^{1/2}}.$$

Then, equation (2.10) becomes

$$\begin{aligned}& \mathcal{E}'(t) \\ (2.12)\qquad & \leq -2|t|\int_M \left(u_t + \frac{Du\cdot Dr^2}{4t}\right)^2 G\, dx + |t|\int_M |Du|^2 (G_t + \Delta G)\, dx \\ & \quad -\int_M |Du|^2(x,t)G(x,t)\, dx - 2|t|\int_M \frac{Du\cdot D^2r^2\cdot Du}{4t} G\, dx + \frac{C_5}{|t|^{1/2}}.\end{aligned}$$

On the other hand, it is easy to compute that

$$G_t + \Delta G = \left(-\frac{n}{2t} + \frac{\Delta r^2}{4t}\right) G.$$

By Proposition 2.2, we have

$$(2.13)\qquad |G_t + \Delta G| \leq C_6 \frac{r^2}{|t|} G$$

and

$$(2.14)\qquad \left|\frac{|Du|^2}{|t|} + \frac{D_i u D_{ij} r^2 D_j u}{2t}\right| \leq C_7 \frac{r^2}{|t|}|Du|^2.$$

Let t be fixed and $\Gamma = \{x \in M : r^2(x) < |t|^{1/2}\}$. Then,

$$\begin{aligned}& \int_M |Du|^2(x,t)\frac{r^2}{|t|} G(x,t)\, dx \\ & = \int_\Gamma |Du|^2(x,t)\frac{r^2}{|t|} G(x,t)\, dx + \int_{M-\Gamma} |Du|^2(x,t)\frac{r^2}{|t|} G(x,t)\, dx \\ & \leq \frac{1}{|t|^{1/2}}\int_M |Du|^2(x,t)G(x,t)\, dx + \int_M |Du|^2 \frac{r^2}{|t|}\frac{1}{|t|^{n/2}}\exp\left(\frac{-1}{4|t|^{1/2}}\right)\, dx\end{aligned}$$

$$\leq \frac{1}{|t|^{1/2}} \int_M |Du|^2(x,t)G(x,t)\ dx + C_8 \exp\left(\frac{-1}{8|t|^{1/2}}\right) \int_M |Du|^2\ dx.$$

Thus, by (2.4), we have

$$\begin{aligned} \int_M |Du|^2(x,t)\frac{r^2}{|t|}G(x,t)\ dx \leq & \frac{1}{|t|^{1/2}} \int_M |Du|^2(x,t)G(x,t)\ dx \\ & + C_9 \exp\left(\frac{-1}{8|t|^{1/2}}\right). \end{aligned} \tag{2.15}$$

Combining (2.12), (2.13), (2.14) and (2.15), we have

$$\begin{aligned} \mathcal{E}'(t) \leq & -2|t| \int_M \left(u_t + \frac{Du \cdot Dr^2}{4t}\right)^2 G\ dx \\ & + \frac{1}{|t|^{1/2}}\mathcal{E}(t) + \frac{C_{10}}{|t|^{1/2}}. \end{aligned}$$

The constant C_{10} depends only on M, N, h and E_0 only. It follows that, for $t \in (-T, 0)$,

$$\begin{aligned} & \frac{d}{dt}\left(\exp\left(2|t|^{1/2}\right)\mathcal{E}(t)\right) \\ \leq & -2\exp\left(2|t|^{1/2}\right)|t| \int_M \left(u_t + \frac{Du \cdot Dr^2}{4t}\right)^2 G\ dx + \frac{C_{10}}{|t|^{1/2}}. \end{aligned}$$

By choosing a constant $A > 0$ large enough, one sees that, for $t \in (-T, 0)$,

$$\begin{aligned} & \frac{d}{dt}\left(\exp\left(2|t|^{1/2}\right)\mathcal{E}(t) + A|t|^{1/2}\right) \\ \leq & -2\exp\left(2|t|^{1/2}\right)|t| \int_M \left(u_t + \frac{Du \cdot Dr^2}{4t}\right)^2 G\ dx. \end{aligned}$$

This completes the proof. ∎

3. Partial Regularity Results

Let $u : M \times [-4R_0^2, 0] \to N$ be a regular solution of (2.1) with Dirichlet boundary condition (2.2). Let $x_0 \in \bar{M}$ be fixed. Let

$$r(x) = \mathrm{dist}_M(x, x_0),$$

$$P(R)(x_0) = \{(x,t) : x \in M,\ r(x) < R,\ t \in (-R^2, 0)\},$$

$$T(R)(x_0) = \{(x,t) : x \in M,\ r(x) < R,\ t \in (-4R^2, -R^2)\}.$$

Lemma 3.1. *Let $u : M \times [-1, 0] \to N$ be a regular solution of (2.1) with Dirichlet boundary condition (2.2). Suppose that for some $A > 0$,*

$$|Du|^2(x,t) \le A \qquad \text{on} \quad P(2R). \tag{3.1}$$

Then, if $x_0 \in \bar{M}$ and $R > 0$ and R is less than the injectivity radius on M, then

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(P(R/8))} \le C\left(A + \|h\|_{C^{2+\alpha}(\partial M)}\right).$$

Proof. We first assume that $\operatorname{dist}(x_0, \partial M) > R/4$. We note that in equation (2.1), we have

$$|\Gamma_N(u)(Du, Du)| \le C|Du|^2.$$

By interior regularity theory, ([5], Chap. IV, Theorem 9.1), for any $q > 1$,

$$\|u\|_{W_q^{2,1}(P(R/2))} \le CA,$$

where for any $Q \subset \mathbb{R}^n \times \mathbb{R}$, and $q > 1$,

$$\|u\|_{W_q^{2,1}(Q)} = \left(\int\int_Q \left(|u_t|^q + |D^2u|^q + |Du|^q + |u|^q\right) dx\ dt\right)^{1/q}.$$

We choose $q > (n+2)/(1-\alpha)$. Then, by the Sobolev inequality, Lemma 3.3, Chapter II, [5], $Du \in C^{\alpha,\alpha/2}(P(R/4))$ and

$$\|Du\|_{C^{\alpha,\alpha/2}(P(R/4))} \le C\|u\|_{W_q^{2,1}(P(R/2))} \le CA.$$

It follows from the parabolic Schauder's estimates that

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(P(R/8))} \le CA.$$

Suppose that $x_0 \in \partial M$. For any $q > 1$, by the boundary regularity theory, we have

$$\|u\|_{W_q^{2,1}(P(2R))} \le C\left(A + \|h\|_{C^2(\partial M)}\right).$$

We choose $q > (n+2)/(1-\alpha)$. Then, by the Sobolev inequality, Lemma 3.3, Chapter II, [5], $Du \in C^{\alpha,\alpha/2}(P(R))$ and

$$\|Du\|_{C^{\alpha,\alpha/2}(P(R))} \le C\|u\|_{W_q^{2,1}(P(2R))} \le C\left(A + \|h\|_{C^2(\partial M)}\right).$$

It follows from the parabolic Schauder's estimates that

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(P(R/2))} \le C\left(A + \|h\|_{C^{2+\alpha}(\partial M)}\right).$$

If $x_0 \in M$ and $\operatorname{dist}(x_0, \partial M) \le R/4$, we can choose $x_1 \in \partial M$ such that $P(R/8)(x_0) \subset P(R/2))(x_1)$. Then we obtain Lemma 3.1. ∎

Corollary 3.2. *Let $u : M \times [0,T) \to N$ be a regular solution of (2.1) with Dirichlet boundary condition (2.2). Suppose that for some $C_1 > 0$, we have*

$$\sup_{x\in M} |Du|^2(x,t) \leq \frac{C_1}{T-t}.$$

Then there is a constant $C_2 > 0$ so that

$$\sup_{x\in M} \left(|D^2u|(x,t) + |u_t|(x,t)\right) \leq \frac{C_2}{T-t}.$$

As in the previous section, for any $x_0 \in \bar{M}$, we let $r(x) =$dist(x, x_0) and

$$G(x,t) = \left(\frac{1}{4\pi|t|}\right)^{n/2} \exp\left(\frac{r^2(x)}{4t}\right).$$

In [1], Y. Chen proved that

Lemma 3.3. *Suppose that M is a compact manifold with non-empty boundary. There is a constant $\epsilon_1 > 0$ depending only on M, N and h only, such that for any regular solution $u : M \times [-4R_0^2, 0] \to N$ of (2.1) with Dirichlet boundary condition (2.2) and*

$$\int_M |Du|^2(x,t)\, dx \leq E_0 < \infty, \qquad \text{for} \quad t \in [-4R_0^2, 0),$$

the following is true: If for some $R \in (0, R_0)$ there holds

$$\int_{T(R)} |Du|^2 G\, dx\, dt < \epsilon_1,$$

then there are constants $\delta > 0$, depending on M, N, h, E_0, and R only, and $C > 0$ depending on M, N and h only, so that

$$\sup_{P(\delta R)} |Du|^2 \leq C(\delta R)^{-2}.$$

From Chen's result, we have

Theorem 3.4. *Suppose that M is a compact manifold with strictly convex boundary. There are constants $\epsilon_2 > 0$ and $\beta > 0$, depending only on M, N and h only, such that for any regular solution $u : M \times [-T, 0) \to N$ of (2.1) with Dirichlet boundary condition (2.2) and*

$$\int_M |Du|^2(x,t)\, dx \leq E_0 < \infty, \qquad \text{for} \quad t \in [-T, 0),$$

the following is true: If

$$|t_0| \int_M |Du|^2(x,t_0) G(x,t_0)\, dx < \epsilon_2 \tag{3.2}$$

for some $t_0 \in (-\beta, 0)$*, then there are constants* $\delta > 0$*, depending on* M*,* N*,* E_0*, and* β *only, and* $C > 0$ *depending on* M*,* N *only, so that*

$$\sup_{P(\delta\sqrt{|t_0|})} |Du|^2 \leq \frac{C}{\delta^2 |t_0|}.$$

Proof. Let $t_0 = -4R^2$. If x_0 lies in the interior of M and dist$(x_0, \partial M) > R$, using the monotonicity formula (2.5), we may follow the arguments in [2] to prove the Theorem.

Suppose that dist$(x_0, \partial M) \leq R$. By the monotonicity formula (2.5), if (3.2) holds, there is $C_1 > 0$ so that

$$\begin{aligned}\int_{T(R)} |Du|^2 G\ dx\ dt &\leq \int_{-4R^2}^{-R^2} \int_{r(x)<R} |Du|^2 G\ dx\ dt \\ &\leq \frac{1}{4R^2} \int_{-4R^2}^{-R^2} |t| \int_M |Du|^2 G\ dx\ dt \\ &\leq C_1 \epsilon_2.\end{aligned}$$

If ϵ_2 is chosen small enough, by Lemma 3.3, Theorem 3.4 follows. ∎

Let S be a subset in M. We denote the k-dimensional Hausdorff measure of S by $\mathcal{H}_k(S)$. As in [2], using Theorem 3.4, we can prove that

Theorem 3.5. *Suppose that* M *is a compact manifold with strictly convex boundary. Let* $u : M \times [0, T) \to N$ *be a regular solution of (2.1) with Dirichlet boundary condition (2.2) and*

$$\int_M |Du|^2(x, t)\ dx \leq E_0 < \infty, \qquad \text{for} \quad t \in [0, T).$$

Let n *be the dimension of* M*. Then, there exists a closed set* S *with finite* $n-2$ *dimensional measure such that* $u(x, t)$ *converges smoothly to a limit* $u(x, T)$ *as* $t \to T$ *on compact sets in* $M - S$*. Moreover, there exists a constant* $C > 0$ *depending only on* M*,* N*,* h *and* E_0 *such that if* U *is any relatively open set containing* S*, then*

$$\mathcal{H}_{n-2}(S) \leq C \liminf_{t \to T} \int_U |Du|^2(x, t)\ dx.$$

4. Convergence to the Homothetically Shrinking Solition

Let M be a compact manifold with non-empty $C^{2,\alpha}$, strictly convex boundary. Let $u : M \times [0, T) \to N$ be a regular solution of (2.1) with Dirichlet boundary condition (2.2). We assume that there is a constant $C_1 > 0$ so that

$$\sup_{x \in M} |Du|^2(x, t) \leq \frac{C_1}{T - t}. \tag{4.1}$$

We denote
$$B(R)=\{x\in M:\ \operatorname{dist}(x,x_0)<R\}$$
and
$$P(R)=\{(x,t)\in M\times(0,T):\ \operatorname{dist}(x,x_0)<R,\ t\in(T-R^2,T)\}.$$

Let (x_0,T) be an interior singularity, i.e., $x_0\in M$ and there are sequences $x_n\in M$ and $t_n\in(0,T)$, such that $x_n\to x_0$ and $T_n\to T$ as $n\to\infty$, and
$$\lim_{n\to\infty}|Du|(x_n,t_n)=\infty.$$
We let
$$u_\lambda(x,t)=u\left(\exp_{x_0}\lambda x, T+\lambda^2 t\right).$$
Using almost the same arguments as in [2], we can show that there is a sequence λ_i such that on each compact set in $\mathbb{R}^n\times(-\infty,0)$, $\{u_{\lambda_i}\}$ in C^∞ converges to a non-constant map
$$\bar u:\mathbb{R}^n\times(-\infty,0)\to N$$
and $\bar u$ satisfies the harmonic map heat flow, and is dilation-invariant, i.e., for any $\lambda>0$, we have
$$\bar u(x,t)=\bar u(\lambda x,\lambda^2 t).$$

Now we examine the boundary singularities in greater detail by blowing them up. Let $u:M\times[0,T)\to N$ be a regular solution of (2.1) with Dirichlet boundary condition (2.2). Let $x_0\in\partial M$ and for $\lambda>0$, let
$$u_\lambda(x,t)=u\left(\exp_{x_0}\lambda x, T+\lambda^2 t\right).$$
Let $R>0$ be a number less than the injectivity radius on M. Using a local chart, we can identify the set $\{x\in M:\operatorname{dist}(x,x_0)<R\}$ with
$$\Omega=\{x\in\mathbb{R}^n:|x|<R,\ x_n\ge\phi(x_1,...,x_{n-1}\},$$
where $\phi(x')$ is a $C^{2,\alpha}$ function, $\phi(0)=0$, $D\phi(0)=0$. When $0<\lambda<1$, $u_\lambda(x,t)$ is defined on the set $\Omega_\lambda\times(-T/\lambda,0)$, where
$$\Omega_\lambda=\{(x,t):|x|<R/\lambda,\ \lambda x_n\ge\phi(\lambda x_1,...,\lambda x_{n-1})\}.$$

For each $\lambda>0$, we have
$$|Du_\lambda|^2(x,t)=\lambda^2|Du|^2(\lambda x,T+\lambda^2 t)\le\frac{C_1}{|t|}. \tag{4.2}$$
By Corollary 3.2,
$$\|u_\lambda(x,t)\|_{C^{2+\alpha,1+\alpha/2}(\Omega_\lambda\times(-R/\lambda,0))}\le\frac{C_1}{|t|}.$$
Hence, there is a subsequence $\{u_{\lambda_i}\}$ such that on each compact set in $\mathbb{R}^n_+\times(-\infty,0)$, $\{u_{\lambda_i}\}$ converges in $C^{2+\alpha,1+\alpha/2}$ to a map

$$\bar{u}:\mathbb{R}^n_+\times(-\infty,0)\to N$$

where $\mathbb{R}^n_+=\{x\in\mathbb{R}^n: x_n\geq 0\}$, and $\bar{u}$ satisfies the harmonic map heat flow. Since the function h in (2.2) is $C^{2,\alpha}$, we have $\bar{u}(x)=h(x_0)$ whenever $x_n=0$. We claim that the function $\bar{u}$ satisfies the dilation-invariant condition:

$$\text{for any } \lambda>0,\quad \bar{u}(x,t)=\bar{u}(\lambda x,\lambda^2 t). \tag{4.3}$$

In fact, from the monotonicity formula Theorem 2.1, we have

$$\int_{T-1}^{T}(T-t)\int_M\left(u_t+\frac{Du\cdot Dr^2}{4(t-T)}\right)^2 G\,dx\,dt\leq C<\infty, \tag{4.4}$$

where

$$G(x,t)=\left(\frac{1}{|T-t|}\right)^{n/2}\exp\left(\frac{\mathrm{dist}^2(x,x_0)}{4(t-T)}\right).$$

Then, for any $\epsilon>0$, we can find $\delta>0$ such that

$$\int_{T-\delta}^{T}(T-t)\int_M\left(u_t+\frac{Du\cdot Dr^2}{4(t-T)}\right)^2 G\,dx\,dt\leq\epsilon.$$

Let $R>0$ be a number less than the injectivity radius on M. From (4.4), for any $\lambda>0$,

$$\int_{-\delta/\lambda^2}^{0}|t|\int_{B(R/\lambda)}\left(u_{\lambda t}+\frac{Du_\lambda\cdot Dr^2}{4t}\right)^2 G_\lambda\,dx\,dt\leq\epsilon,$$

where

$$G_\lambda(x,t)=\left(\frac{1}{\pi|t|}\right)^{n/2}\exp\left(\frac{\mathrm{dist}^2_M(\exp_{x_0}(\lambda x),x_0)}{4\lambda^2 t}\right).$$

When $\lambda\to 0$, we have

$$\int_{-\infty}^{0}|t|\int_{\mathbb{R}^n}\left(\bar{u}_t+\frac{D\bar{u}\cdot x}{2t}\right)^2\bar{G}\,dx\,dt\leq\epsilon,$$

where

$$\bar{G}(x,t)=\left(\frac{1}{4\pi|t|}\right)^{n/2}\exp\left(\frac{|x|^2}{4t}\right)$$

is the backward heat heat kernel on $\mathbb{R}^n$. Since ϵ can be any positive number, we have

$$\int_{-\infty}^{0}|t|\int_{\mathbb{R}^n_+}\left(\bar{u}_t+\frac{D\bar{u}_\lambda\cdot x}{2t}\right)^2\bar{G}\,dx\,dt=0.$$

It shows that

$$\bar{u}_t+\frac{D\bar{u}\cdot x}{2t}=0\quad\text{in}\quad\mathbb{R}^n_+\times(-\infty,0),$$

and (4.3) holds.

By (4.2), when $\lambda \to 0$, we have

$$|D\bar{u}|^2(x,t) \le \frac{C_1}{|t|}. \tag{4.5}$$

By the small energy regularity, Theorem 3.4, if $x_0 \in \partial M$ and (x_0, T) is a singular point, then, there is $\beta > 0$ such that for all $T - \beta \le t \le T$, we have

$$|T-t| \int_M |Du|^2(x,t) G(x,t)\, dx > \epsilon.$$

Let $\rho > 0$ be large enough so that

$$\int_{\mathrm{dist}(x,x_0) \ge \rho\sqrt{T-t}} G(x,t)\, dx \le \frac{\epsilon}{2C_1}.$$

Then, for all $T - \beta \le t \le T$, we have

$$|T-t| \int_{\mathrm{dist}(x,x_0) \le \rho\sqrt{T-t}} |Du|^2(x,t) G(x,t)\, dx \ge \epsilon/2.$$

Since u_{λ_i} converges to $\bar{u}$ on compact sets in $\mathbb{R}^n_+ \times (-\infty, 0)$, it is not difficult to see that the same will hold for $\bar{u}$: for $t < 0$,

$$|t| \int_{\{x \in \mathbb{R}^n_+, |x| \le \rho\sqrt{|t|}\}} |D\bar{u}|^2(x,t) \bar{G}(x,t)\, dx \ge \epsilon/2.$$

This implies that $\bar{u}$ is not a constant function.

5. Harmonic Heat Maps with Neumann Boundary Condition

We say ∂M is convex, if for any $a \in \bar{M}$,

$$Dr \cdot \nu \ge 0 \qquad \text{on} \quad \partial M \tag{5.1}$$

where $r(x) = \mathrm{dist}(a, x)$ and ν is the unit outward normal on ∂M.

Suppose that ∂M is convex. Let $u(x,t) : M \times (0,T) \to N$ be a smooth solution of the harmonic heat equation with Neumann boundary condition. Suppose that

$$\max_{x \in M} |Du|(x,t) \to \infty \qquad \text{as} \quad t \to T.$$

As before, for any $x_0 \in \bar{M}$, let

$$\mathcal{E}(x_0; t) = (T-t) \int_M |Du|^2(x,t) G(x_0, T; x, t)\, dx,$$

where

$$G(x_0, T; x, t) = \left(\frac{1}{4\pi|T-t|}\right)^{n/2} \exp\left(\frac{r^2(x_0; x)}{4(t-T)}\right).$$

Theorem 5.1. *Suppose that ∂M is convex. Let $u(x,t) : M \times (0,T) \to N$ be a smooth solution of the harmonic heat equation with Neumann boundary condition,*

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \partial M \times (0,T) \tag{5.2}$$

and

$$\int_M |Du|^2(x,t)\, dx \le E_0 \qquad \text{for} \quad t \in (0,T).$$

Then there is a constant $B > 0$, depending only on M, N, T and E_0 only, so that, for all $t \in (0,T)$,

$$\begin{aligned} &\frac{d}{dt}\left(\exp\left(2|T-t|^{1/2}\right)\mathcal{E}(x_0;t) + B|T-t|^{1/2}\right) \\ &\le -2\exp\left(2|T-t|^{1/2}\right)|T-t| \int_M \left(u_t + \frac{Du\cdot Dr^2}{4(t-T)}\right)^2 G(x_0,T;x,t)\, dx. \end{aligned} \tag{5.3}$$

Proof. After a translation in time, we may assume that u is defined on $M \times [-T,0)$. As in section 2, we will write $r(x) = r(x_0;x) = \operatorname{dist}(x_0,x)$, and

$$\mathcal{E}(t) = \mathcal{E}(x_0;t) = |t| \int_M |Du|^2(x,t)G(x,t)\, dx,$$

where

$$G(x,t) = \left(\frac{1}{4\pi|t|}\right)^{n/2} \exp\left(\frac{r^2(x)}{4t}\right),$$

for $x \in M$ and $t \in (-T,0)$. By (5.1) and (5.2), equation (2.10) becomes

$$\begin{aligned} &\mathcal{E}'(t) \\ &\le -2|t|\int_M \left(u_t + \frac{Du\cdot Dr^2}{4t}\right)^2 G\, dx + |t|\int_M |Du|^2(G_t + \Delta G)\, dx \\ &\quad - \int_M |Du|^2(x,t)G(x,t)\, dx - 2|t|\int_M \frac{Du\cdot D^2r^2\cdot Du}{4t} G\, dx \end{aligned} \tag{5.4}$$

By (2.13) and (2.14), we have

$$\begin{aligned} \mathcal{E}'(t) \le &-2|t|\int_M \left(u_t + \frac{Du\cdot Dr^2}{4t}\right)^2 G\, dx \\ &+ C_3|t|\int_M |Du|^2(x,t)\frac{r^2}{|t|}G(x,t)\, dx. \end{aligned} \tag{5.5}$$

The rest of the proof is the same as the proof of Theorem 2.1. ■

Lemma 5.2. *Let* $u : M \times [-1, 0] \to N$ *be a regular solution of* (*2.1*) *with Neumann boundary condition* (*5.1*). *Suppose that for some* $A > 0$,

(5.6) $$|Du|^2(x,t) \le A \qquad \text{on} \quad P(2R).$$

Then,

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(M\times(-1/8,0))} \le CA.$$

Proof. Suppose that $x_0 \in \partial M$. Let $R > 0$ be a number less than the injectivity radius of M. By choosing a $C^{2,\alpha}$ chart, we may identify a set $\Omega \subset \{x \in M : \mathrm{dist}(x, x_0) < R\}$ with the set

$$D_+(R/2) = \{x \in \mathbb{R}^n : |x| < R/2,\ x_n > 0\}.$$

If R is chosen small enough, the map $(y_1, y_2, ..., y_n)$ is $C^{2,\alpha}$ and its inverse exists and is $C^{2,\alpha}$. In $D_+(R/2)$, u is a solution of an equation of the form:

(5.7) $$u_t = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a^{ij}\frac{\partial u}{\partial x_j}\right) + \Gamma(Du, Du),$$

where a^{ij} and Γ are C^α functions and $\Gamma(Du, Du) \le C|Du|^2$, and

$$\frac{\partial u}{\partial x_n} = 0 \quad \text{whenever} \quad x_n = 0.$$

Let $u(x,t) = u(-x,t)$ when $x_n < 0$. Then, $u(x,t)$ is a solution of (5.7) in $D(R/2) \times (0,T)$, where $D(R/2) = \{x \in \mathbb{R}^n : |x| < R/2\}$. As in section 3, using the regularity theory and Sobolev inequality, we obtain

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(B(x_0,R/8)\times(-R/8,0))} \le CA.$$

If x_0 lies in the interior of M, we argue as in Lemma 3.1. This proves the Lemma. ∎

As in section 3, we have the small-energy-regularity result:

Theorem 5.3. *Suppose that* M *is a compact manifold with convex boundary. There are constants* $\epsilon_4 > 0$ *and* $\beta > 0$, *depending only on* M, N *and* h *only, such that for any regular solution* $u : M \times [-T, 0) \to N$ *of (2.1) with Neumann boundary condition (5.2) and*

$$\int_M |Du|^2(x,t)\, dx \le E_0 < \infty, \qquad \text{for} \quad t \in [-T, 0),$$

the following is true: If

$$|t_0| \int_M |Du|^2(x,t_0)G(x,t_0)\,dx < \epsilon_4$$

for some $t_0 \in (-\beta, 0)$*, then there are constants* $\delta > 0$*, depending on* M*,* N*,* E_0*, and* β *only, and* $C > 0$ *depending on* M*,* N *only, so that*

$$\sup_{P(\delta\sqrt{|t_0|})} |Du|^2 \le \frac{C}{\delta^2 |t_0|}.$$

From Theorem 5.3, we have

Theorem 5.4. *Suppose that* M *is a compact manifold with strictly convex boundary. Let* $u : M \times [0,T) \to N$ *be a regular solution of (2.1) with Neumann boundary condition (5.2) and*

$$\int_M |Du|^2(x,t)\,dx \le E_0 < \infty, \qquad \text{for} \quad t \in [0,T).$$

Let n *be the dimension of* M*. Then, there exists a closed set* S *with finite* $n-2$ *dimensional measure such that* $u(x,t)$ *converges smoothly to a limit* $u(x,T)$ *as* $t \to T$ *on compact sets in* $M - S$*. Moreover, there exists a constant* $C > 0$ *depending only on* M*,* N*,* h *and* E_0 *such that if* U *is any relatively open set containing* S*, then*

$$\mathcal{H}_{n-2}(S) \le C \liminf_{t\to T} \int_M |Du|^2(x,t)\,dx.$$

Now, suppose that

$$\sup_M |Du|^2(x,t) \le \frac{C}{T-t}.$$

As in section 4, we let

$$u_\lambda(x,t) = u\left(\exp_{x_0} \lambda x, T + \lambda^2 t\right).$$

Using the almost the same arguments, we can show that if $x_0 \in M$ is a singular point, there is a sequence λ_i such that on each compact set in $\mathbb{R}^n \times (-\infty, 0)$, $\{u_{\lambda_i}\}$ in $C^{2,\alpha}$ converges to a non-constant map

$$\bar{u} : \mathbb{R}^n \times (-\infty, 0) \to N$$

and $\bar{u}$ satisfies the harmonic map heat flow, and is dilation-invariant, i.e., for any $\lambda > 0$, we have

$$\bar{u}(x,t) = \bar{u}(\lambda x, \lambda^2 t).$$

If $x_0 \in \partial M$ is a singular point, there is a sequence λ_i such that on each compact set in $\mathbb{R}^n_+ \times (-\infty, 0)$, $\{u_{\lambda_i}\}$ in $C^{2,\alpha}$ converges to a non-constant map

$$\bar{u} : \mathbb{R}^n_+ \times (-\infty, 0) \to N$$

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$, and $\bar{u}$ satisfies the harmonic map heat flow, and

$$\frac{\partial \bar{u}}{\partial x_n}(x,t) = 0 \quad \text{whenever} \quad x_n = 0,$$

and is dilation-invariant, i.e., for any $\lambda > 0$, we have

$$\bar{u}(x,t) = \bar{u}(\lambda x, \lambda^2 t).$$

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