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ON ENTIRE SOLUTIONS OF A CERTAIN TYPE OF NONLINEAR DIFFERENTIAL AND DIFFERENCE EQUATIONS

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Abstract. In this paper, we investigate some analogous results on the existence of entire solutions of a certain type of nonlinear differential and differential-difference equations of the following form

$$f^n(z) + P_d(f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z},$$

where $P_d(f)$ is a differential polynomial or differential-difference polynomial in $f(z)$. And we find out its entire solutions or prove that it has no entire solution for some special $P_d(f)$.

1. Introduction and Main Results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the following standard notations in value distribution theory (see[2, 8]).

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \cdots.$$

And we denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o\{T(r,f)\}, \text{ as } r \to \infty,$$

possibly outside of a set E with finite linear measure, not necessarily the same at each occurrence. The order of a meromorphic function $f(z)$ is defined as

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r}.$$

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And the deficiency of a with respect to $f(z)$ is defined by

$$\Theta(a,f)=1-\varlimsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

A differential polynomial in $f(z)$ means that it is a polynomial in $f(z)$ and its derivatives with small functions of $f(z)$ as coefficients. A differential-difference polynomial in $f(z)$ means that it is a polynomial in $f(z)$, its derivatives and its shifts $f(z+c)$ with small functions of $f(z)$ as coefficients. We shall use $P_d(f)$ to denote a differential polynomial in $f(z)$ or a differential-difference polynomial in $f(z)$ with degree d. Furthermore, Nevanlinna's value distribution theory of meromorphic functions plays an important role in studying the growth and existence of meromorphic solutions of the differential or differential-difference equations. For instance, it is shown in [6] that the equation $4f^3(z)+3f''(z)=-\sin 3z$ has exactly three nonconstant entire solutions, namely $f_1(z)=\sin z$, $f_2(z)=\frac{\sqrt{3}}{2}\cos z-\frac{1}{2}\sin z$, $f_3(z)=-\frac{\sqrt{3}}{2}\cos z-\frac{1}{2}\sin z$. And Li-Yang in [6, 4] also considered a general case as follows.

Theorem A. (see [6]). *Let $n\geq 3$ be an integer, $P_{n-3}(f)$ be an algebraic differential polynomial in $f(z)$ of degree $d\leq n-3$, $b(z)$ a meromorphic function and λ, c_1, c_2 three nonzero constants. Then the equation*

$$(*)\qquad f^n(z)+P_{n-3}(f)=b(z)(c_1e^{\lambda z}+c_2e^{-\lambda z})$$

does not have any transcendental entire solution $f(z)$ satisfying that $T(r,b)=S(r,f)$.

Theorem B. (see [4]). *Let $n\geq 4$ be an integer and $P_d(f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d\leq n-3$. If $p_1(z), p_2(z)$ are two nonzero polynomials and α_1, α_2 are two nonzero constants such that $\frac{\alpha_1}{\alpha_2}$ is not rational, then the equation*

$$f^n(z)+P_d(f)=p_1(z)e^{\alpha_1 z}+p_2(z)e^{\alpha_2 z}$$

does not have any transcendental entire solution.

An important question is that the condition that the degree of $P_d(f)$ satisfying $d\leq n-3$ can be weaken? In this paper, we obtained

Theorem 1. *Let $n\geq 3$ be an integer and $P_d(f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d\leq n-2$. If $p_1(z), p_2(z)$ are two nonzero polynomials and α_1, α_2 are two nonzero constants such that $\frac{\alpha_1}{\alpha_2}\neq(\frac{d}{n})^{\pm 1}, 1$. Then any transcendental entire solution of the following equation*

$$(1)\qquad f^n(z)+P_d(f)=p_1(z)e^{\alpha_1 z}+p_2(z)e^{\alpha_2 z}$$

$f(z)$ satisfies that $\Theta(0,f)=0$.

Remark 1. Comparing the proof of Theorem 1 and Theorem B, we can obtain that Theorem B remains valid if the condition "$\frac{\alpha_1}{\alpha_2}$ is not rational" is placed by "$\frac{\alpha_1}{\alpha_2} \neq (\frac{d}{n})^{\pm 1}, 1$" and the latter condition is necessary. In fact, the equation (1) has the entire solution $f(z)=pe^{\frac{\alpha_1}{n}z}$ if $\alpha_1=\alpha_2$, $P_d(f)=0$, $p_1(z)+p_2(z)=p^n(z)$; or the entire solution $f(z)=(p_2(z))^{\frac{1}{n}}e^{\frac{\alpha_1}{d}z}$ if $\frac{\alpha_1}{\alpha_2}=\frac{d}{n}$, $P_d(f)=f^d(z)$, ${p_1}^n(z)={p_2}^d(z)$.

We will give some examples to show that the case that $\Theta(0, f) = 0$ in Theorem 1 does exist.

Example 1. (see [4] Theorem 4). Let a, P_1, P_2, λ be non-zero constants. Then the differential equation

$$f^3(z) + af'' = P_1e^{\lambda z} + P_2e^{-\lambda z}$$

has transcendental entire solutions if and only if the condition $P_1P_2+(a\lambda^2/27)^3 = 0$ holds. Moreover if the condition holds, then the solutions are

$$f(z) = \varrho_j e^{\frac{\lambda z}{3}} - (\frac{a\lambda^2}{27\varrho_j})e^{-\frac{\lambda z}{3}}, (j = 1, 2, 3),$$

where $\varrho_j, (j = 1, 2, 3)$ are the cubic roots of P_1.

Example 2. The differential equation

$$f^4(z) - 64ff'' + 2 = e^z + e^{-z}$$

has a transcendental entire solution

$$f(z) = e^{\frac{z}{4}} + e^{-\frac{z}{4}}.$$

But for some special $P_d(f)$ in Theorem 1, the equation (1) has no entire solution.

Theorem 2. *Let a, P_1, P_2 be non-zero constants. Then the equation*

$$f^3(z) + af'(z) = P_1e^{\lambda z} + P_2e^{-\lambda z} \tag{2}$$

does not have any transcendental entire solution.

Corresponds to the Theorem 1, we also considered the case that the differential polynomial $P_d(f)$ is placed by differential-difference polynomial. And we obtained

Theorem 3. *Let $n \geq 4$ be an integer and $P_d(f)$ denote an algebraic differential-difference polynomial in $f(z)$ of degree $d \leq n-3$. If $p_1(z), p_2(z)$ are two nonzero polynomials and α_1, α_2 are two nonzero constants with $\frac{\alpha_1}{\alpha_2} \neq (\frac{d}{n})^{\pm 1}, 1$, then the equation (1) does not have any transcendental entire solution of finite order.*

Theorem 4. *Let P_1, P_2 and λ be non-zero constants. For the difference equation*

$$f^3(z)+a(z)f(z+1)=P_1e^{\lambda z}+P_2e^{-\lambda z}, \tag{3}$$

where $a(z)$ is a polynomial, we have

(i) *if $a(z)$ is not a constant, then the equation (3) does not have any transcendental entire solution of finite order;*

(ii) *if $a(z)$ is a nonzero constant, then the equation (3) admit transcendental entire solutions of finite order if and only if the condition*

$$e^{\frac{1}{3}\lambda}=\mp 1 \text{ and } P_1P_2=\pm(\frac{a}{3})^3$$

holds, furthermore if the condition above holds, then the transcendental entire solution of finite order of the equation (3) has the form as following

$$f(z)=\varrho_j e^{2k\pi iz}-\frac{a}{3\varrho_j}e^{-2k\pi iz} \text{ or } f(z)=\varrho_j e^{2k\pi iz+\pi iz}+\frac{a}{3\varrho_j}e^{-(2k\pi iz+\pi iz)}.$$

Theorem 2 and Theorem 4 show that there is no causal link between the existences of the solution of a differential equation and the corresponding differential-difference equation.

2. LEMMAS

To prove our results, we need some lemmas.

Lemma 1. (see [3]). *Let $f(z)$ be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$H(z,f)P(z,f)=Q(z,f),$$

where $H(z,f), P(z,f), Q(z,f)$ are difference polynomials in $f(z)$ such that the total degree of $H(z,f)$ in $f(z)$ and its shifts is n, and that the total degree of $Q(z,f)$ is at most n. If $H(z,f)$ just contains one term of maximal total degree, then for any $\varepsilon>0$,

$$m(r,P(z,f))=O(r^{\rho-1+\varepsilon})+S(r,f)$$

holds possibly outside of an exceptional set of finite logarithmic measure.

Remark 2. Particularly, if $H(z,f)=f^n(z)$, then a similar conclusion holds when $P(z,f)$, $Q(z,f)$ are differential-difference polynomials in $f(z)$.

Lemma 2. (see [1]). *Let $f(z)$ be meromorphic and transcendental function in the plane and satisfy*

$$f^n(z)P(f)=Q(f),$$

where $P(f), Q(f)$ are differential polynomials in $f(z)$ with functions of small proximity related to $f(z)$ as the coefficients and the degree of $Q(f)$ is at most n, then

$$m(r,P(f))=S(r,f).$$

Lemma 3. (see [6]). *Suppose that* c *is a non-zero constant and* α *is a nonconstant meromorphic function. Then the equation*

$$f^2(z) + (cf^{(n)}(z))^2 = \alpha$$

has no transcendental meromorphic solution $f(z)$ *satisfying* $T(r, \alpha) = S(r, f)$.

Lemma 4. (see [5]). *Let* m, n *be positive integers satisfying* $\frac{1}{m} + \frac{1}{n} < 1$. *Then there are no transcendental entire solutions* $f(z)$ *and* $g(z)$ *satisfy the equation*

$$a(z)f^n(z) + b(z)g^m(z) = 1$$

with $a(z), b(z)$ *being small functions of* $f(z)$.

Lemma 5. (see [7]). *Let* $f(z)$ *be a nonconstant meromorphic function. Then*

$$m(r, \frac{f'}{f}) = O(\log r), (r \to \infty),$$

if f *is of finite order, and*

$$m(r, \frac{f'}{f}) = O(\log(rT(r, f))), (r \to \infty),$$

possibly outside a set E *of* r *with finite linear measure if* $f(z)$ *is of infinite order.*

Lemma 6. (see [7]). *Suppose that* $f_1(z), f_2(z), \ldots f_n(z), (n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \ldots g_n(z)$ *are entire functions satisfying the following conditions*

(*i*) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.

(*ii*) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$.

(*iii*) *For* $1 \leq j \leq n, 1 \leq h < k \leq n, T(r, f_j) = o\{T(r, e^{g_h - g_k})\}(r \to \infty, r \notin E)$.

Then $f_j(z) \equiv 0, (j = 1, 2, \ldots n)$.

3. The Proofs

3.1. Proof of theorem 1

Let $f(z)$ be a transcendental entire solution of the equation (1) with $\Theta(0, f) > 0$. Then by differentiating both sides of the equation (1), we get

$$nf^{n-1}f' + (P_d(f))' = (p_1\alpha_1 + p_1{}')e^{\alpha_1 z} + (p_2\alpha_2 + p_2{}')e^{\alpha_2 z}. \tag{4}$$

Eliminating $e^{\alpha_1 z}, e^{\alpha_2 z}$ from the equations (1) and (4), we obtain

$$(p_1\alpha_1 + p_1{}')f^n - np_1 f^{n-1}f' + Q_d(f) = \beta e^{\alpha_2 z}, \tag{5}$$

and

$$(p_2\alpha_2 + {p_2}')f^n - np_2 f^{n-1}f' + R_d(f) = -\beta e^{\alpha_1 z}, \tag{6}$$

where

$$\beta = (p_1\alpha_1 + {p_1}')p_2 - (p_2\alpha_2 + {p_2}')p_1, \tag{7}$$

$$Q_d(f) = (p_1\alpha_1 + {p_1}')P_d(f) - p_1(P_d(f))', \tag{8}$$

$$R_d(f) = (p_2\alpha_1 + {p_2}')P_d(f) - p_2(P_d(f))'. \tag{9}$$

By differentiating the equation (5) , we get

$$\begin{aligned}(\beta' + \beta\alpha_2)e^{\alpha_2 z} &= (p_1\alpha_1 + {p_1}')'f^n + np_1\alpha_1 f^{n-1}f' \\ &\quad - n(n-1)p_1 f^{n-2}f'^2 - np_1 f^{n-1}f'' + (Q_d(f))'.\end{aligned} \tag{10}$$

By eliminating $e^{\alpha_2 z}$ from the equation (5) and (10), we get

$$f^{n-2}\{\gamma f^2 - np_1\gamma_2 ff' + n(n-1)p_1\beta f'^2 + np_1\beta ff''\} = T_d(f), \tag{11}$$

where

$$\gamma_1 = (\beta' + \beta\alpha_2)(p_1' + p_1\alpha_1) - \beta({p_1}' + p_1\alpha_1)',$$
$$\gamma_2 = \beta' + \alpha_1\beta + \alpha_2\beta,$$

and

$$T_d(f) = \beta(Q_d(f))' - (\beta' + \beta\alpha_2)Q_d(f). \tag{12}$$

And we set

$$\phi = \gamma_1 f^2 - np_1\gamma_2 ff' + n(n-1)p_1\beta f'^2 + np_1\beta ff'', \tag{13}$$

which is a differential polynomial in $f(z)$. We rewrite the equation (11) as the following form

$$f^{n-2}\phi = T_d(f).$$

It follows the fact that $T_d(f)$ is a differential polynomial in $f(z)$ with degree at most $n-2$ and Lemma 2 that

$$T(r,\phi) = m(r,\phi) = S(r,f).$$

Now we claim $\phi \equiv 0$. In fact, we rewrite the equation (13) as the following form

$$\phi = f^2 A(z),$$

where

$$A(z) = \gamma_1 - np_1\gamma_2\frac{f'}{f} + n(n-1)p_1\beta(\frac{f'}{f})^2 + np_1\beta\frac{f''}{f}.$$

Then $m(r, A) = S(r, f)$. If $\phi \not\equiv 0$, then $A \not\equiv 0$. For any small $\epsilon > 0$, we have

$$\begin{aligned} 2T(r,f) &= m(r,f^2) = m(r,\frac{\phi}{A}) \\ &\leq m(r,\phi) + m(r,\frac{1}{A}) \leq S(r,f) + T(r,A) \\ &\leq S(r,f) + N(r,A) \leq S(r,f) + 2\overline{N}(r,\tfrac{1}{f}) \\ &\leq 2(1 - \Theta(0,f) + \epsilon)T(r,f). \end{aligned}$$

This is impossible for $0 < \epsilon < \Theta(0, f)$. Hence $A \equiv 0$, and $T_d(f) \equiv \phi \equiv 0$. Next, we discuss two cases.

Case 1. $Q_d(f) \not\equiv 0$. At this case, the equation (12) implies

$$Q_d(f) = c_1\beta e^{\alpha_2 z}, \tag{14}$$

where $c_1 \neq 0$. We substitute (14) into (5) and get

$$f^{n-1}\{(p_1\alpha_1 + {p_1}')f - np_1f'\} = -(1 - \frac{1}{c_1})Q_d(f). \tag{15}$$

Setting

$$\varphi = (p_1\alpha_1 + {p_1}')f - np_1f'$$

and noting that the degree of $Q_d(f)$ is at most $n-2$, we get by Lemma 2

$$m(r,\varphi) = S(r,f) \text{ and } m(r,f\varphi) = S(r,f).$$

If $\varphi \not\equiv 0$, then

$$T(r,f) = m(r,f) \leq m(r,f\varphi) + m(r,\frac{1}{\varphi}) \leq S(r,f) + T(r,\varphi) = S(r,f).$$

It is impossible. Thus $\varphi \equiv 0$. By the equation (15), we get $c_1 = 1$ and $Q_d(f) = \beta e^{\alpha_2 z}$. On the other hand, we solute the equation $\varphi \equiv 0$ and get

$$f^n(z) = c_2 p_1 e^{\alpha_1 z}. \tag{16}$$

By substituting the equation (16) into the equation (1), we get

$$(1 - \frac{1}{c_2})f^n(z) = \frac{p_2}{\beta}Q_d(f) - P_d(f),$$

where $\frac{p_2}{\beta}Q_d(f) - P_d(f)$ is a differential polynomial in f with degree at most $n-2$. By Lemma 2 again, we deduce $c_2 = 1$ and

$$f^n(z) = p_1 e^{\alpha_1 z}, \quad P_d(f) = p_2 e^{\alpha_2 z}.$$

Thus

$$f(z) = (p_1)^{\frac{1}{n}} e^{\frac{\alpha_1}{n} z},$$

and

$$P_d(f) = h(e^{\frac{\alpha_1}{n} z}),$$

where $h(e^{\frac{\alpha_1}{n} z})$ is a polynomial of $e^{\frac{\alpha_1}{n} z}$ with degree d and the small functions of $h(e^{\frac{\alpha_1}{n} z})$ as its coefficients. Thus, by Lemma 6, we have $\frac{d\alpha_1}{n} = \alpha_2$, i.e. $\frac{\alpha_1}{\alpha_2} = \frac{n}{d}$, which is a contradiction.

Case 2. $Q_d(f) \equiv 0$. Then from the equation (8), we get

$$(p_1\alpha_1 + p_1{}')P_d(f) - p_1(P_d(f))' = 0.$$

If $P_d(f) \equiv 0$, then $f^n(z) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z}$. And we rewrite this equation as the following form

$$\frac{1}{p_2}(f(z) \cdot e^{-\frac{\alpha_2}{n} z})^n + (\frac{-p_1}{p_2})(e^{\frac{(\alpha_1-\alpha_2)z}{m}})^m = 1,$$

where m is any positive integer. And Lemma 4 implies $\alpha_1 = \alpha_2$, which is a contraction. Hence, $P_d(f) \not\equiv 0$. Thus we deduce that

$$P_d(f) = c_3 p_1 e^{\alpha_1 z}, \ c_3 \neq 0. \tag{17}$$

From the equation (1), we get

$$f^n(z) + (c_3 - 1)p_1 e^{\alpha_1 z} = p_2 e^{\alpha_2 z}. \tag{18}$$

By Lemma 4 again, we get $c_3 = 1$ and $P_d(f) = p_1 e^{\alpha_1 z}$. Thus

$$f^n(z) = p_2 e^{\alpha_2 z}. \tag{19}$$

Then $f(z) = (p_2)^{\frac{1}{n}} e^{\frac{\alpha_2}{n} z}$. By the same arguments as above, we have again $\frac{d\alpha_2}{n} = \alpha_1$ and $\frac{\alpha_1}{\alpha_2} = \frac{d}{n}$, which is a contradiction again. The proof of theorem 1 is completed. ■

3.2. Proof of theorem 2

Suppose that $f(z)$ is a transcendental entire solutions of the equation (2). By differentiating the equation (2), we get

$$3f^2 f' + af'' = \lambda P_1 e^{\lambda z} - \lambda P_2 e^{-\lambda z}. \tag{20}$$

By taking both squares of (2) and (20) and eliminating $e^{\pm\lambda z}$, we deduce

$$4\lambda^2 P_1 P_2 = \lambda^2(f^3 + af')^2 - (3f^2 f' + af'')^2. \tag{21}$$

We set

$$\alpha = \lambda^2 f^2 - 9f'^2. \tag{22}$$

It is obvious that α is an entire function. We set

$$Q(f) = 4\lambda^2 P_1 P_1 - \lambda^2 a^2 f'^2 - 2a\lambda^2 f' f^3 + a^2 f''^2 + 6af' f'' f^2,$$

which is a differential polynomial in $f(z)$ with degree 4. Then we rewrite (21) as the following form

$$f^4 \alpha = Q(f). \tag{23}$$

By Lemma 2, we get $m(r, \alpha) = S(r, f)$ and $T(r, \alpha) = S(r, f)$. Thus, α is a small function of $f(z)$. We consider two cases.

Case 1. $\alpha \equiv 0$. Then the equation (22) implies that $f(z) = ce^{\pm\frac{1}{3}\lambda z}$. By substituting this into (2) and simple calculation, we get

$$(c^3 - P_1)e^{\lambda z} + \frac{1}{3}a\lambda c e^{\frac{1}{3}\lambda z} = P_2 e^{-\lambda z},$$

or

$$(c^3 - P_2)e^{-\lambda z} - \frac{1}{3}a\lambda c e^{-\frac{1}{3}\lambda z} = P_1 e^{\lambda z}.$$

By Lemma 6, we get $P_1 = 0$ or $P_2 = 0$, which is a contradiction.

Case 2. $\alpha \not\equiv 0$. Then Lemma 3 implies α is a non-zero constant. Thus

$$f'(\lambda^2 f - 9f'') = 0.$$

Since $f(z)$ is transcendental, then

$$\lambda^2 f - 9f'' = 0. \tag{24}$$

The general solutions of the equation (24) are

$$f(z) = c_1 e^{\frac{1}{3}\lambda z} + c_2 e^{-\frac{1}{3}\lambda z}, \tag{25}$$

where c_1, c_2 are constants. Since $\alpha \not\equiv 0$, we have $c_1 c_2 \neq 0$. Then by substituting (25) into (2) and simple calculation, we get

$$\begin{aligned}&({c_1}^3 - P_1)e^{\lambda z} + ({c_2}^3 - P_2)e^{-\lambda z} + (3{c_1}^2 c_2 + \frac{a\lambda c_1}{3})e^{\frac{1}{3}\lambda z}\\ &+(3c_1{c_2}^2 - \frac{a\lambda c_2}{3})e^{-\frac{1}{3}\lambda z} = 0.\end{aligned} \tag{26}$$

By Lemma 6, we deduce

$$c_1{}^3 = P_1,\ c_2{}^3 = P_2,\ 9c_1c_2 + a\lambda = 0,\ 9c_1c_2 = a\lambda.$$

Hence, $c_1c_2 = 0$. This is a contraction. The proof of theorem 2 is completed. ■

3.3. Proof of theorem 3

The proof of Theorem 3 is very similar to that of Theorem 1. We just give a main framework of the proof.

Suppose that $f(z)$ is a transcendental entire solution with finite order $\rho(f) = \rho$ of the equation (1). By using the same arguments as those in Theorem 1, we can get the corresponding equation (4)-(13) and $f^{n-2}\phi = T_d(f)$, where $\phi = \gamma_1 f^2 - np_1\gamma_2 ff' + n(n-1)p_1\beta f'^2 + np_1\beta ff''$ and $T_d(f)$ is a differential-difference polynomial in $f(z)$ with total degree at most $n-3$. By Lemma 1, we get

$$m(r,\phi) = S(r,f) + O(r^{\rho-1+\varepsilon}) \text{ and } m(r,f\phi) = S(r,f) + O(r^{\rho-1+\varepsilon}).$$

If $\phi \not\equiv 0$, then

$$\begin{aligned} T(r,f) = m(r,f) &\le m(r,\frac{1}{\phi}) + m(r,f\phi) \le T(r,\phi) + S(r,f) + O(r^{\rho-1+\varepsilon}) \\ &\le m(r,\phi) + S(r,f) + O(r^{\rho-1+\varepsilon}) = S(r,f) + O(r^{\rho-1+\varepsilon}). \end{aligned}$$

This is impossible. Hence, $\phi \equiv 0$. Similarly, we can deduce $\varphi = (p_1\alpha_1 + p_1{}')f - np_1f' \equiv 0$. By using similar arguments to the remained part of the proof of Theorem 1, we can get our conclusion easily. We omit the detail. ■

3.4. Proof of theorem 4

Suppose that $f(z)$ is a transcendental entire solution of the equation (3) with finite order $\rho(f) = \rho$. By differentiating (3), we get

$$3f^2(z)f'(z) + a(z)f'(z+1) + a'(z)f(z+1) = \lambda P_1 e^{\lambda z} - \lambda P_2 e^{-\lambda z}. \tag{27}$$

Similarly, by taking both squares of (3) and (27) and eliminating $e^{\pm\lambda z}$, we deduce

$$\begin{aligned} 4\lambda^2 P_1P_2 = \lambda^2(f^3(z) + a(z)f(z+1))^2 - (3f^2(z)f'(z) + \\ a(z)f'(z+1) + a'(z)f(z+1))^2. \end{aligned} \tag{28}$$

We set

$$\alpha(z) = \lambda^2 f^2(z) - 9f'^2(z), \tag{29}$$

which is a differential polynomial in $f(z)$. Thus $\alpha(z)$ is an entire function. And we set

$$\begin{aligned} Q(f) &= 4\lambda^2 P_1 P_1 - \lambda^2 a^2(z)(f(z+1))^2 - 2a(z)\lambda^2 f(z+1)f^3(z) \\ &\quad (a'(z))^2 f^2(z) + 6a'(z)f^2(z)f'(z+1) + 2a(z)a'(z)f(z+1)f'(z+1) \\ &\quad + a^2(z)(f'(z+1))^2 + 6a(z)f'(z)f'(z+1)f^2(z), \end{aligned}$$

which is a differential-difference polynomial in $f(z)$ with total degree 4. Then we rewrite (28) as the following form

$$f^4\alpha = Q(f). \tag{30}$$

By Lemma 1, we get

$$m(r,\alpha) = S(r,f) + O(r^{\rho-1+\varepsilon})$$

and $T(r,\alpha) = m(r,\alpha) = S(r,f) + O(r^{\rho-1+\varepsilon})$. Thus, α is a small function of $f(z)$. Next, we consider two cases.

Case 1. $\alpha \equiv 0$. Then $f(z) = ce^{\pm\frac{1}{3}\lambda z}$. By substituting this into (3) and simple calculation, we get

$$(c^3 - P_1)e^{\lambda z} + a(z)ce^{\frac{\lambda}{3}}e^{\frac{1}{3}\lambda z} = P_2 e^{-\lambda z},$$

or

$$(c^3 - P_2)e^{-\lambda z} + a(z)ce^{-\frac{\lambda}{3}}e^{-\frac{1}{3}\lambda z} = P_1 e^{\lambda z}.$$

By Lemma 6, we get $P_1 = 0$ or $P_2 = 0$, which is a contradiction.

Case 2. $\alpha \not\equiv 0$. Then Lemma 3 implies α is a non-zero constant. Thus

$$f'(\lambda^2 f - 9f'') = 0.$$

Since $f(z)$ is transcendental, then

$$\lambda^2 f - 9f'' = 0. \tag{31}$$

The general solution of the equation (31) is

$$f(z) = c_1 e^{\frac{1}{3}\lambda z} + c_2 e^{-\frac{1}{3}\lambda z}, \tag{32}$$

where c_1, c_2 are both non-zero constants. Then by substituting (32) into (3) and simple calculation, we get

$$\begin{aligned} &({c_1}^3 - P_1)e^{\lambda z} + ({c_2}^3 - P_2)e^{-\lambda z} + (3{c_1}^2 c_2 + a(z)c_1 e^{\frac{1}{3}\lambda})e^{\frac{1}{3}\lambda z} \\ &+(3c_1{c_2}^2 + a(z)c_2 e^{-\frac{1}{3}\lambda})e^{-\frac{1}{3}\lambda z} = 0. \end{aligned} \tag{33}$$

By Lemma 6, we deduce

$$c_1^{\,3} = P_1,\ c_2^{\,3} = P_2,\ 3c_1c_2 + a(z)e^{\frac{1}{3}\lambda} = 3c_1c_2 + a(z)e^{-\frac{1}{3}\lambda} = 0.$$

Therefore, if $a(z)$ is a nonconstant polynomial, then we can deduce a contraction and the equation (3) does not admit any transcendental entire solutions of finite order. And if $a(z)$ is a nonzero constant a, then

$$e^{\frac{1}{3}\lambda} = \mp 1 \text{ and } P_1P_2 = \pm(\frac{a}{3})^3.$$

Thus, c_1 can assume $\varrho_j, (j = 1, 2, 3)$, where ϱ_j satisfies $\varrho_j^{\,3} = P_1, (j = 1, 2, 3)$, and $c_2 = \pm\frac{a}{3c_1}$. Hence, $f(z)$ is of the following forms

$$f(z) = \varrho_j e^{2k\pi iz} - \frac{a}{3\varrho_j} e^{-2k\pi iz}$$

or

$$f(z) = \varrho_j e^{2k\pi iz+\pi iz} + \frac{a}{3\varrho_j} e^{-(2k\pi iz+\pi iz)}.$$

The proof of theorem 4 is completed. ■

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