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Δ-STATISTICAL BOUNDEDNESS FOR SEQUENCES OF FUZZY NUMBERS

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Abstract. In this article we introduce the notion of Δ-statistical boundedness for fuzzy real numbers and examine its some properties. We also give some relations related to this concept and construct some interesting examples.

1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [29] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [21] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. In addition, sequences of fuzzy numbers have been discussed in [2, 5, 7, 9, 11, 13, 19, 23, 24] and many others.

There are many applications of the sequences and difference sequences of numbers (real, complex and fuzzy numbers). For example sequences of numbers have unexpected and practical uses in many areas of science and engineering, including acoustics. They find application in measuring concert hall acoustics, radar echoes from planets, the travel times of deep-ocean sound waves for monitoring ocean temperature, and improving synthetic speech and the sounds associated with computer music. Furthermore, it is shown by Kawamura *et al*. [17] that the earthquake ground motions have very simple conditioned fuzzy set rules with non-fuzzy parameters of the first and second order differences ΔX_i and $\Delta^2 X_i$ defined by membership functions μ's. Therefore the difference sequences of fuzzy numbers are used, for example in the prediction of earthquake waves. For more detail see [17].

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The notion of statistical convergence was introduced by Fast [14] and Schoenberg [27], independently. Over the years and under different names statistical convergence has been discussed in the theory of fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy [15], Fridy and Orhan [16], Salát [25] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Moreover, statistical convergence is closely related to the concept of convergence in probability.

The main purpose of this paper is to study $\Delta-$statistical boundedness of fuzzy numbers so as to fill up the existing gap in the literature.

2. Definitions and Preliminaries

For easy understanding of the material incorporated in this article, we reproduce some known definitions and notions of fuzzy numbers in this section.

The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset E of $\mathbb{N}$ is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ provided the limit exists,}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

A sequence (x_k) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$. In this case we write $S - \lim x_k = L$.

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [29] a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \longrightarrow [0, 1]$. The function u itself is often used for the fuzzy set.

Let $C(\mathbb{R}^n)$ denote the family of all nonempty, compact, convex subsets of $\mathbb{R}^n$. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in C(\mathbb{R}^n)$, then

$$(A + B) = \alpha A + \alpha B, \qquad (\alpha\beta)A = \alpha(\beta A), \qquad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$. The distance between A and B is defined by the Haussdorff metric

$$\delta_\infty(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \parallel a - b \parallel, \sup_{b\in B} \inf_{a\in A} \parallel a - b \parallel\},$$

where $\| \, . \, \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete metric space.

Denote

$$L(\mathbb{R}) = \{u : \mathbb{R} \longrightarrow [0,1] \mid u \text{ satisfies } (i) - (iv) \text{ below}\},$$

where

(i) u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \le \lambda \le 1, u(\lambda x + (1-\lambda)y) \ge \min[u(x), u(y)]$;

(iii) u is upper semicontinuous;

(iv) the closure of $\{x \in \mathbb{R} : u(x) > 0\}$, denoted by $[u]^0$, is compact.

If $u \in L(\mathbb{R})$, then u is called a fuzzy number, and $L(\mathbb{R})$ is said to be a fuzzy number space.

For $0 < \alpha \le 1$, the α-level set $[u]^\alpha$ is defined by

$$[u]^\alpha = \{x \in \mathbb{R} : u(x) \ge \alpha\}.$$

Then from $(i) - (iv)$, it follows that the α-level sets $[u]^\alpha \in C(\mathbb{R})$.

Some arithmetic operations for $\alpha-$level sets are defined as follows:

Let $u, v \in L(\mathbb{R}^n)$, $k \in \mathbb{R}$ and the $\alpha-$level sets be $[u]^a = [u_1^\alpha, u_2^\alpha]$, $[v]^a = [v_1^\alpha, v_2^\alpha]$, $\alpha \in [0,1]$. Then we have

$$\begin{aligned} [u+v]^a &= [u_1^\alpha + v_1^\alpha, u_2^\alpha + v_2^\alpha] \\ [u-v]^a &= [u_1^\alpha - v_2^\alpha, u_2^\alpha - v_1^\alpha] \\ [ku]^\alpha &= \begin{cases} [ku_1^\alpha, ku_2^\alpha], & \text{if } k \ge 0 \\ [ku_2^\alpha, ku_1^\alpha], & \text{otherwise} \end{cases} . \end{aligned}$$

The set of all real numbers can be embedded in $L(\mathbb{R})$. For $a \in \mathbb{R}$, $\bar{a} \in L(\mathbb{R})$ is defined by

$$\bar{a}(x) = \begin{cases} 1, & \text{for } x = a \\ 0, & \text{for } x \ne a \end{cases} .$$

Let D denote the set of all closed and bounded intervals $u = [a_1, a_2]$ on $\mathbb{R}$. For $u, v \in D$, we define

$$d(u,v) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$$

where $u = [a_1, a_2]$ and $v = [b_1, b_2]$. Here (D, d) is a complete metric space. It follows that the $\alpha-$level sets $[u]^\alpha \in D$ for $\alpha \in [0,1]$.

Now, define a map $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$\bar{d}(u,v) = \sup_{\alpha\in[0,1]} \max\{|a_1^\alpha - b_1^\alpha|, |a_2^\alpha - b_2^\alpha|\}.$$

Then $\bar{d}$ defines a metric on $L(\mathbb{R})$.

For any $\alpha \in [0,1]$, the order relation on $L(\mathbb{R})$ is defined by

$$u \leq v \text{ if and only if } a_1^\alpha \leq b_1^\alpha \text{ and } a_2^\alpha \leq b_2^\alpha.$$

Let u and v be two fuzzy numbers. Then u and v are said to be incomparable if neither $u \leq v$ nor $v \leq u$. In this case, we will use the notation $u \not\sim v$ [12], [20].

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R})$. Thus, a sequence of fuzzy numbers X is a correspondence from the set of positive integers to a set of fuzzy numbers, i.e., to each positive integer k there corresponds a fuzzy number $X(k)$. It is more common to write X_k rather than $X(k)$ and to denote the sequence by (X_k) rather than X. The fuzzy number X_k is called the k-th term of the sequence.

Let $X = (X_k)$ be a sequence of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded. i.e. if a sequence (X_k) is bounded, then there are two fuzzy numbers u, v such that $u \leq X_k \leq v$. A sequence $X = (X_k)$ is convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $d(X_k, X_0) < \varepsilon$ for $k > k_0$. Let $\ell_\infty(F)$ and $c(F)$ denote the set of all bounded sequences and all convergent sequences of fuzzy numbers, respectively [21].

The difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, consisting of all real valued sequences $x = (x_k)$ such that $\Delta x = (x_k - x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0, were defined by Kzmaz [18].

Let $w(F)$ be the set of all sequences of fuzzy numbers. The operator $\Delta : w(F) \to w(F)$ is defined by

$$\left(\Delta^0 X\right)_k = X_k, (\Delta X)_k = \Delta X_k = X_k - X_{k+1}, \text{ for all } k \in \mathbb{N}.$$

Definition 2.1. Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ is said to be $\Delta-$bounded if the set $\{\Delta X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded, and $\Delta-$ convergent to the fuzzy number X_0 , written as $\lim_k \Delta X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $\bar{d}(\Delta X_k, X_0) < \varepsilon$ for all $k > k_0$. By $\ell_\infty(\Delta, F)$ and $c(\Delta, F)$ denote the sets of all $\Delta-$ bounded sequences and all $\Delta-$ convergent sequences of fuzzy numbers, respectively, see ([9, 10, 26]).

If $X = (X_k)$ is a sequence that satisfies a property P for all k except a set of natural density zero, then we say that X_k satisfies P for almost all k and we write by $a.a.k$.

Definition 2.2. Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ of fuzzy numbers is said to be $\Delta-$statistically convergent to fuzzy number X_0 if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \leq n : \bar{d}\,(\Delta X_k, X_0) \geq \varepsilon\right\}\right| = 0.$$

In this case we write $X_k \to X_0\,(S_F\,(\Delta))$ or $S_F\,(\Delta) - \lim X_k = X_0$. The set of all statistically convergent sequences of fuzzy numbers is denoted by $S_F\,(\Delta)$, (see [9, 10].

Theorem 2.3. [22]. *If $X = (X_k)$ is a sequence of fuzzy numbers for which there is a convergent sequence $Y = (Y_k)$ such that $\Delta X_k = Y_k$ for $a.a.k$, then X is $\Delta-$statistically convergent.*

3. Main Results

The statistical boundedness for sequences of fuzzy numbers has been defined by Aytar and Pehlivan [7]. In this study, we will define $\Delta-$statistical boundedness for sequences of fuzzy numbers and give some relations and theorems.

Definition 3.1. Let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence $X = (X_k)$ is said to be $\Delta-$statistically bounded above if there exists a fuzzy number u such that

$$\delta\left(\{k \in \mathbb{N} : \Delta X_k > u\} \cup \{k \in \mathbb{N} : \Delta X_k \not\sim u\}\right) = 0.$$

Similarly, $X = (X_k)$ is said to be $\Delta-$statistically bounded below if there exists a fuzzy number v such that

$$\delta\left(\{k \in \mathbb{N} : \Delta X_k < v\} \cup \{k \in \mathbb{N} : \Delta X_k \not\sim v\}\right) = 0.$$

If a sequence $X = (X_k)$ of fuzzy numbers is both $\Delta-$statistically bounded above and $\Delta-$statistically bounded below, then it is called $\Delta-$statistically bounded.

It is also stated this definition as follows: A sequence $X = (X_k)$ of fuzzy numbers is called $\Delta-$statistically bounded if there exists a real number T such that $\bar{d}\,(\Delta X_k, \bar{0}) < T$ for a.a.k.

Since the set $L\,(\mathbb{R})$ is partially ordered set, it must be considered the incomparable elements in $L\,(\mathbb{R})$. Therefore we have added the elements of the set $\{k \in \mathbb{N} : \Delta X_k \not\sim u\}$ to the set $\{k \in \mathbb{N} : \Delta X_k > u\}$.

We can easily see that if a sequence $X = (X_k)$ of fuzzy numbers is $\Delta-$bounded, it is also $\Delta-$statistically bounded. Generally, the converse of this claim is not true. This case can be seen in the following example.

Example 3.2. Let $X = (X_k)$ be a sequence of fuzzy numbers as follows:

$$X_k(x) = \begin{cases} \left.\begin{array}{cc} x-k, & x \in [k, k+1] \\ -x+k+2, & x \in (k+1, k+2] \\ 0, & \text{otherwise} \end{array}\right\} & \begin{array}{c}\text{if } k = 3^n \\ (n = 0, 1, 2, ...)\end{array} \\ u_1, & \text{if } k \neq 3^n \text{ and } k \text{ is odd} \\ u_2, & \text{if } k \neq 3^n \text{ and } k \text{ is even} \end{cases}$$

where

$$u_1(x) = \begin{cases} x+3, & x \in [-3, -2] \\ -x-1, & x \in (-2, -1] \\ 0, & \text{otherwise} \end{cases}$$

and

$$u_2(x) = \begin{cases} x-6, & x \in [6, 7] \\ -x+8, & x \in (7, 8] \\ 0, & \text{otherwise} \end{cases}.$$

Then, for $\alpha \in (0, 1]$, $\alpha-$level sets of X_k and ΔX_k are respectively

$$[X_k]^\alpha = \begin{cases} [k+\alpha, k+2-\alpha], & \text{if } k = 3^n \\ [-3+\alpha, -1-\alpha], & \text{if } k \neq 3^n \text{ and } k \text{ is odd} \\ [6+\alpha, 8-\alpha], & \text{if } k \neq 3^n \text{ and } k \text{ is even} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [k-8+2\alpha, k-4-2\alpha], & \text{if } k = 3^n \\ [-k+3+2\alpha, -k+7-2\alpha], & \text{if } k+1 = 3^n \\ [-11+2\alpha, -7-2\alpha], & \text{if } k \neq 3^n,\ k+1 \neq 3^n \text{ and } k \text{ is odd} \\ [7+2\alpha, 11-2\alpha], & \text{if } k \neq 3^n,\ k+1 \neq 3^n \text{ and } k \text{ is even} \end{cases}.$$

By helping the arithmetic operations, we get the sequence ΔX_k as follows:

$$\Delta X_k(x) = \begin{cases} \left.\begin{array}{cc} \frac{1}{2}(x-k+8), & x \in [k-8, k-6] \\ -\frac{1}{2}(x-k+6)+1, & x \in (k-6, k-4] \\ 0, & \text{otherwise} \end{array}\right\}, & \text{if } k = 3^n \\ \left.\begin{array}{cc} \frac{1}{2}(x+k-3), & x \in [-k+3, -k+5] \\ -\frac{1}{2}(x+k-5)+1, & x \in (-k+5, -k+7] \\ 0, & \text{otherwise} \end{array}\right\}, & \text{if } k+1 = 3^n \\ v, & \text{if } k \neq 3^n \text{ and } k \text{ is odd} \\ u, & \text{if } k \neq 3^n \text{ and } k \text{ is even} \end{cases}$$

where

$$u(x) = \begin{cases} \frac{1}{2}(x-7), & x \in [7,9] \\ -\frac{1}{2}(x-11), & x \in (9,11] \\ 0, & \text{otherwise} \end{cases}$$

and

$$v(x) = \begin{cases} \frac{1}{2}(x+11), & x \in [-11,-9] \\ -\frac{1}{2}(x+7), & x \in (-9,-7] \\ 0, & \text{otherwise} \end{cases} .$$

Hence the sequence (X_k) is $\Delta-$statistically bounded since

$$\delta(\{k \in \mathbb{N} : \Delta X_k > u\} \cup \{k \in \mathbb{N} : \Delta X_k \not\prec u\}) = \delta(\{27, 81, 243, ...\} \cup \{\emptyset\}) = 0$$

and

$$\delta(\{k \in \mathbb{N} : \Delta X_k < v\} \cup \{k \in \mathbb{N} : \Delta X_k \not\succ v\}) = \delta(\{26, 80, 242, ...\} \cup \{\emptyset\}) = 0$$

However, the sequence (X_k) is not $\Delta-$bounded. (See Fig. 1).

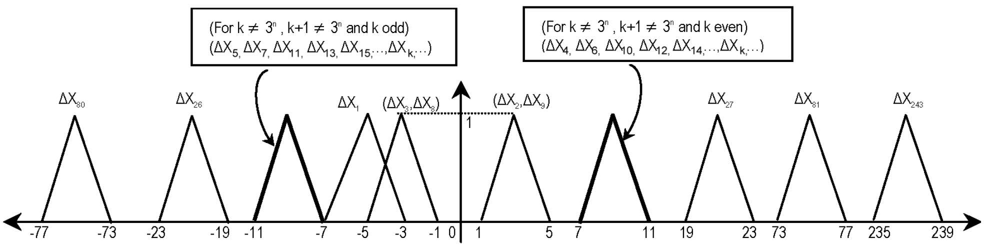

Fig. 1. A sequence (X_k) of fuzzy numbers which is $\Delta-$statistically bounded, but not $\Delta-$bounded.

Theorem 3.3. *If a sequence $X = (X_k)$ of fuzzy numbers is $\Delta-$statistically convergent, then it is $\Delta-$statistically bounded.*

Proof. Let $X = (X_k)$ be a sequence of fuzzy numbers and be $\Delta-$statistically convergent to the fuzzy number X_0, i.e. $S_F(\Delta) - \lim X_k = X_0$.

Then, we can write for every $\varepsilon > 0$, $\bar{d}(\Delta X_k, X_0) < \varepsilon$, $a.a.k$. Since X_0 is a fuzzy number, then there exists a number $T \in \mathbb{R}$ such that $\bar{d}(X_0, \bar{0}) < T$. Thus, for $a.a.k$

$$\bar{d}(\Delta X_k, \bar{0}) \leq \bar{d}(\Delta X_k, X_0) + \bar{d}(X_0, \bar{0}) < \varepsilon + T.$$

Hence, it follows that the sequence (X_k) is $\Delta-$statistically bounded.

In general, the converse of Theorem 3.3 is not true as shown in the following example.

Example 3.4. We define the sequence $X = (X_k)$ which is $\Delta-$statistically bounded as in Example 3.2. Then for every $\varepsilon > 0$ and $\alpha \in (0,1]$, we get

$$\delta\left(\left\{k \in \mathbb{N} : \bar{d}\left(\Delta X_k^\alpha, u_1^\alpha\right) \geq \varepsilon\right\}\right) = \frac{1}{2}$$
$$\delta\left(\left\{k \in \mathbb{N} : \bar{d}\left(\Delta X_k^\alpha, u_2^\alpha\right) \geq \varepsilon\right\}\right) = \frac{1}{2},$$

where $u_1^\alpha = [7+2\alpha, 11-2\alpha]$ and $u_2^\alpha = [-11+2\alpha, -7-2\alpha]$.

Thus, the sequence $X = (X_k)$ is not $\Delta-$statistically convergent.

Theorem 3.5. *If a sequence $X = (X_k)$ of fuzzy numbers is $\Delta-$statistically Cauchy, then there is a convergent sequence $Y = (Y_k)$ such that $\Delta X_k = Y_k$ for $a.a.k$.*

Proof. Assume that (X_k) is a $\Delta-$statistically Cauchy so that the closed ball $B = \bar{B}\left(\Delta X_{N(1)}, 1\right)$ contains ΔX_k for $a.a.k$ for some positive number $N(1)$. Also apply hypothesis to choose M so that $B' = \bar{B}\left(\Delta X_M, \frac{1}{2}\right)$ contains ΔX_k for a.a.k. It is clear that $B_1 = B \cap B'$ contains ΔX_k for and *a.a.k.*

Therefore B_1 is a closed set of diameter less than or equal to 1 that contains ΔX_k for *a.a.k.*

Now we proceed by choosing $N(2)$ so that $B'' = \bar{B}\left(\Delta X_{N(2)}, \frac{1}{4}\right)$ contains ΔX_k for $a.a.k$, and by the preceeding argument $B_2 = B_1 \cap B''$ contains ΔX_k for $a.a.k$ and B_2 has diameter less than or equal to $\frac{1}{2}$.

Continuing this process we construct a sequence $\{B_m\}_{m=1}^\infty$ of closed balls such that for each m, $B_m \supset B_{m+1}$, the diameter of B_m is not greater than $\frac{1}{2^{m-1}}$ and $\Delta X_k \in B_m$ for $a.a.k$.

By the nest of closed set theorem in a complete metric space we have $\cap_{m=1}^\infty B_m \neq \emptyset$ and contains exactly one element. So there is a fuzzy number L which is $L \in \cap_{m=1}^\infty B_m$. Using the fact that $\Delta X_k \in B_m$ for $a.a.k$, we choose an increasing positive integer sequence $\{H_m\}_{m=1}^\infty$ such that

$$\text{(1)} \qquad \lim_n \frac{1}{n}\left|\{k \leq n : \Delta X_k \notin B_m\}\right| < \frac{1}{m} \text{ if } n > H_m.$$

Now we construct a subsequence (Z_k) of (ΔX_k) consisting of terms of ΔX_k such that if $H_m < k \leq H_{m+1}$ and $\Delta X_k \notin B_m$ then ΔX_k is a term of Z_k.

Next define the sequence (Y_k) by

$$Y_k(x) = \begin{cases} L, & \text{if } \Delta X_k \text{ is a term of } Z_k \\ X_k, & \text{otherwise} \end{cases}.$$

Then $\lim_{k\to\infty} Y_k = L$ for if $\varepsilon > \frac{1}{m} > 0$ and $k > H_m$ then either ΔX_k is a term of (Z_k), which means $Y_k = L$ or $Y_k = \Delta X_k \in B_m$ and $\bar{d}(Y_k, L) \leq$ diameter of $B_m \leq \frac{1}{2^{m-1}}$.

We also assert that $\Delta X_k = Y_k$ for $a.a.k$. To verify this we observe that if $H_m < n \leq H_{m+1}$ then

$$\{k \leq n : Y_k \neq \Delta X_k\} \subseteq \{k \leq n : \Delta X_k \notin B_m\}$$

so by (1)

$$\begin{aligned} & \frac{1}{n} \left|\{k \leq n : Y_k \neq \Delta X_k\}\right| \\ \leq & \frac{1}{n} \left|\{k \leq n : \Delta X_k \notin B_m\}\right| < \frac{1}{m}. \end{aligned}$$

Hence, the limit as $n \to \infty$, is 0 and $\Delta X_k = Y_k$ for *a.a.k.* This completes the proof of the theorem.

Theorem 3.6. *If the fuzzy number sequence* $X = (X_k)$ *is* $\Delta-$*statistically Cauchy, then it is* $\Delta-$*statistically bounded.*

Proof. Let $X = (X_k)$ be $\Delta-$statistically Cauchy sequence. Therefore by Theorem 2.3 and Theorem 3.5, it is $\Delta-$statistically convergent. On the other hands, from Theorem 3.3 we conclude that $X = (X_k)$ is $\Delta-$statistically bounded.

Theorem 3.7. *If* $X = (X_k)$ *is a* $\Delta-$ *statistically bounded sequence, then we can write* $\Delta X = Y + Z$, *where* $Y = (Y_k)$ *is bounded and* $Z = (Z_k)$ *is a statistically null sequence.*

Proof. Assume that $X = (X_k)$ is a $\Delta-$statistically bounded sequence. For $T > 0$ sufficiently large, narural density of the set $M = \left\{k \in \mathbb{N} : \bar{d}\,(\Delta X_k, \bar{0}) \geq T\right\}$ is zero. Now, we define the sequences (X_k) and (Y_k) such that

$$Y_k = \begin{cases} \Delta X_k, & \text{for } k \in M' \\ \bar{0}, & \text{otherwise} \end{cases}$$

and

$$Z_k = \begin{cases} \Delta X_k, & \text{for } k \in M \\ \bar{0}, & \text{otherwise} \end{cases},$$

where M' is complement set of M. It is easy to see that $Y = (Y_k)$ is bounded from the definition of $\Delta-$statistically boundedness. However, Z_k is statistically null sequence. Hence for all $k \in \mathbb{N}$, it is clear that $(\Delta X_k) = (Y_k) + (Z_k)$.

We give an example to demonstrate this theorem as follows:

Example 3.8. Consider the sequence of fuzzy numbers $X = (X_k)$ as follows:

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} x-k, & x \in [k,k+1] \\ -x+k+2, & x \in (k+1,k+2] \\ 0, & \text{otherwise} \end{array}\right\} & \text{for } k = 5^n \ (n=0,1,2,...) \\ \left.\begin{array}{ll} \frac{k}{k+1}x - \frac{k-1}{k+1}, & x \in \left[\frac{k-1}{k},2\right] \\ -\frac{k}{k+1}x + \frac{3k+1}{k+1}, & x \in \left(2,\frac{3k+1}{k}\right] \\ 0, & \text{otherwise} \end{array}\right\} & \text{otherwise} \end{cases}$$

Then, for $\alpha \in (0,1]$, $\alpha-$level sets of (X_k) and (ΔX_k) are respectively

$$[X_k]^\alpha = \begin{cases} [k+\alpha, k+2-\alpha], & \text{if } k = 5^n \\ \left[\frac{k-1}{k} + \frac{\alpha(k+1)}{k}, \frac{3k+1}{k} - \frac{\alpha(k+1)}{k}\right], & \text{otherwise} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} \left[\frac{k^2-2k+\alpha(2k+3)-4}{k+1}, \frac{k^2+2k-\alpha(2k+3)+2}{k+1}\right], & \text{if } k = 5^n \\ \left[\frac{-k^2-2k+\alpha(2k+1)-1}{k}, \frac{-k^2+2k-\alpha(2k+1)+1}{k}\right], & \text{if } k+1 = 5^n \\ \left[\frac{-2k^2-4k+\alpha\left(2k^2+4k+1\right)-1}{k(k+1)}, \frac{2k^2+4k-\alpha\left(2k^2+4k+1\right)+1}{k(k+1)}\right], & \text{otherwise} \end{cases}.$$

It is seen that the sequence (ΔX_k) is statistically convergent to fuzzy number L, where $[L]^\alpha = [-2+2\alpha, 2-2\alpha]$. On the other hands, it is written that the set $[\Delta X_k]^\alpha$ is summation of $[Y_k]^\alpha$ and $[Z_k]^\alpha$, where

$$[Y_k]^\alpha = \begin{cases} \bar{0}, & \text{if } k = 5^n \\ \bar{0}, & \text{if } k+1 = 5^n \\ \left[\frac{-2k^2-4k+\alpha\left(2k^2+4k+1\right)-1}{k(k+1)}, \frac{2k^2+4k-\alpha\left(2k^2+4k+1\right)+1}{k(k+1)}\right], & \text{otherwise} \end{cases}$$

and

$$[Z_k]^\alpha = \begin{cases} \left[\frac{k^2-2k+\alpha(2k+3)-4}{k+1}, \frac{k^2+2k-\alpha(2k+3)+2}{k+1}\right], & \text{if } k = 5^n \\ \left[\frac{-k^2-2k+\alpha(2k+1)-1}{k}, \frac{-k^2+2k-\alpha(2k+1)+1}{k}\right], & \text{if } k+1 = 5^n \\ \bar{0}, & \text{otherwise} \end{cases}.$$

Here, we get the sequence (Y_k) is bounded and the sequence (Z_k) is statistically convergent to zero.

Lemma 3.9. *A sequence $X = (X_k)$ of fuzzy numbers is $\Delta-$statistically bounded iff there exists a subset $K = (k_1 < k_2 < k_3 < ...) \subset \mathbb{N}$ such that $\delta(K) = 1$ and (ΔX_{k_n}) is a bounded sequence.*

Proof. It is clear from definition of $\Delta-$statistically bounded and from Theorem 3.7.

The following is an inclusion relation between statistically bounded and $\Delta-$statistically bounded sequences .

Theorem 3.10. *If a sequence $X = (X_k)$ of fuzzy numbers is statistically bounded, then it is $\Delta-$statistically bounded.*

Proof. It is easy to see that if a sequence of fuzzy numbers is statistically bounded, it is also $\Delta-$statistically bounded. But, the converse of this claim does not hold in general as in following example.

Example 3.11. Define the sequence $X = (X_k)$ as follows:

$$X_k(x) = \begin{cases} \left.\begin{array}{cc} x+3, & x \in [-3,-2] \\ -x-1, & x \in (-2,-1] \\ 0, & \text{otherwise} \end{array}\right\} & \begin{array}{c} \text{for } k = 3^n \\ (n = 0,1,2,...) \end{array} \\ \left.\begin{array}{cc} x-2k+1,, & x \in [2k-1,2k] \\ -x+2k+1,, & x \in (2k,2k+1] \\ 0, & \text{otherwise} \end{array}\right\} & \text{otherwise} \end{cases}$$

For $\alpha \in (0,1]$, we get

$$[X_k]^\alpha = \begin{cases} [-3+\alpha, -1-\alpha], & \text{if } k = 3^n \\ [2k-1+\alpha, 2k+1-\alpha], & \text{otherwise} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [-2k+2\alpha-6, -2k-2\alpha-2], & \text{if } k = 3^n \\ [2k+2\alpha, 2k-2\alpha+4], & \text{if } k+1 = 3^n \\ [2\alpha-4, -2\alpha], & \text{otherwise} \end{cases}.$$

Then, it follows that (ΔX_k) is a statistically bounded sequence. On the other hands, (X_k) is not statistically bounded.

4. Conclusion

The notion of statistical boundedness in sequence of fuzzy numbers was introduced and studied by Aytar and Pehlivan [7]. Now, in this paper we study $\Delta-$statistical boundedness of sequence of fuzzy numbers using a difference operator and examine relations between $\Delta-$statistical boundedness, $\Delta-$statistical convergence and $\Delta-$statistical cauchy convergence by helping some interesting examples.

Here, we would like to specify that, for $m \in \mathbb{N}$, Δ^m-statistical boundedness for sequence of fuzzy numbers can be studied by researchers, which is an open problem.

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