POSITIVE SOLUTIONS FOR A PREDATOR-PREY INTERACTION MODEL WITH HOLLING-TYPE FUNCTIONAL RESPONSE AND DIFFUSION

Yunfeng Jia, Jianhua Wu and Hong-Kun Xu*

Abstract. We deal with a predator-prey interaction model with Holling-type monotonic functional response and diffusion and which is endowed with a second homogeneous boundary condition. Via spectrum analysis and bifurcation theory, we investigate the local and global bifurcation solutions of the model which emanate from a positive constant solution by taking the growth rate as a bifurcation parameter. Basing on the fixed point index theory, we prove the existence of positive steady-state solutions of the model.

1. INTRODUCTION

Both mathematicians and ecologists are concerned with the dynamics of biological populations via the partial differential equation method, in particular, of the reaction-diffusion systems derived from the interactions among several species which have extensively been investigated recently. Predator-prey models are however an important branch of the reaction-diffusion systems (see, for example, [3, 4, 5, 11, 12, 14, 17, 20, 24]). In predator-prey models, functional responses of the predator to the prey density play a critical role, which refers to the change in the density of prey attached per time unit and predator unit as the prey density changes. In general, the functional response, denoted by $p(u)$, is monotone (see [4, 6, 7, 9, 10]) and continuously differentiable on $[0, \infty)$. Examples of functional response functions are

$$p(u) = \frac{au}{b}, \quad p(u) = \frac{au}{b+u}, \quad p(u) = \frac{au^2}{b+u^2}, \quad p(u) = \frac{au^2}{b+ru+u^2},$$

where a, b, r are positive constants, with a denoting the maximal growth rate of species, and b the half-saturation constant. These functional responses are usually

said to be of Holling-type I, II, III and IV, respectively. The dynamics of Holling-type predator-prey models are interesting and have therefore been extensively studied. For more details of response functions, the reader can consult the references [2, 5, 8, 10, 20].

In [1, 5], the authors used the Holling-type II response $p(u) = \frac{au}{b+u}$ for studying the dynamic behavior of the system of the ordinary differential equations (ODEs)

$$(1.1) \quad \begin{cases} u_t = r_1 u(1 - \dfrac{u}{k}) - \dfrac{auv}{b+u}, & t > 0, \\ v_t = r_2 v(1 - \dfrac{v}{cu}), & t > 0, \end{cases}$$

where the variables u and v represent the concentrations or densities of the two species (and so are usually assumed to be non-negative), the constants r_1 and r_2 are the birth rates of u and v respectively, k is the prey environmental carrying capacity, c is a measure of the food quality of prey for conversion into predator births, a is the maximum number of prey that can be eaten by a predator per time unit, and b is the saturation value that corresponds to the number of prey necessary to achieve one half of the maximum rate a. All these parameters r_1, r_2, c, k, a, b are assumed to be positive.

Mathematical properties and ecological meaning of the ODE model (1.1) have been investigated qualitatively and numerically in order to explain mutual interactions between populations such as mite and spider mite, lynx and hare, sparrow and sparrow hawk, and so on. A lot of interesting phenomena, such as stable limit cycles, semi-stable limit cycles, bifurcations, global stability of constant positive solutions and of periodic solutions have also been studied (see [1, 4, 5, 13, 23]).

Note that the densities of prey and predator are spatially inhomogeneous in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. Thus, instead of (1.1), we are led to the following reaction-diffusion system

$$(1.2) \quad \begin{cases} u_t - d_1 \Delta u = r_1 u(1 - \dfrac{u}{k}) - \dfrac{auv}{b+u}, & x \in \Omega, & t > 0, \\ v_t - d_2 \Delta v = r_2 v(1 - \dfrac{v}{cu}), & x \in \Omega, & t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega, & t > 0, \\ u = u_0 \geq 0, \not\equiv 0, v = v_0 \geq 0, \not\equiv 0, & x \in \Omega, & t = 0, \end{cases}$$

where d_1, d_2 denote the diffusion rates of u, v, respectively, and ∂_n is the derivative in the direction of outer normal to $\partial\Omega$.

The main purpose of this paper is to study the positive steady-state solutions of (1.2), that is, the existence of nonconstant positive classical solutions of the following elliptic system

$$(1.3) \quad \begin{cases} -d_1\Delta u = r_1 u(1 - \dfrac{u}{k}) - \dfrac{auv}{b+u}, & x \in \Omega, \\[2mm] -d_2\Delta v = r_2 v(1 - \dfrac{v}{cu}), & x \in \Omega, \\[2mm] \partial_n u = \partial_n v = 0, & x \in \partial\Omega. \end{cases}$$

In [21], the authors investigated system (1.3) for a simplified version. They gave a priori estimates of the upper and lower bounds, the non-existence and existence of positive non-constant solutions of the model. However the existence conditions for positive solutions in the literature seem complicated. In the present paper, we shall focus our attention on the local and global bifurcation solutions which emanate from a positive constant solution by taking the growth rate as a bifurcation parameter. Furthermore, we also discuss the stability of the constant positive solution. The existence result of positive steady-state solutions of the model is proved via the fixed point index theory.

The organization of this paper is as follows. In Section 2, we analyze the long time behavior of system (1.2). Section 3 is devoted to the stability of the unique constant positive solution. The existence and the structure of bifurcation solutions emanating from the constant positive solution are investigated in Section 4. Finally, in Section 5, by using the fixed point index theory, we prove the existence of positive steady-state solutions of the model.

2. Long Time Behavior of Solutions of System (1.2)

For system (1.2), we analyze the long time behavior of its solutions.

Theorem 2.1. *Suppose that (u, v) satisfies (1.2). Then, for any $\varepsilon > 0$, the rectangle $[0, k + \varepsilon) \times [0, ck + \varepsilon)$ is a global attractor for all solutions of (1.2) in $\mathbb{R}^2$.*

Proof. Since $(u, v) = (u(x, t), v(x, t))$ is a solution of (1.2), u satisfies

$$\begin{cases} u_t - d_1\Delta u \le r_1 u(1 - \dfrac{u}{k}), & x \in \Omega, & t > 0, \\[2mm] \partial_n u = 0, & x \in \partial\Omega, & t > 0, \\[2mm] u = u_0 \ge 0, \not\equiv 0, & x \in \Omega, & t = 0. \end{cases}$$

Let $z_1(t)$ be a solution of the ODE system

$$\begin{cases} z_1'(t) = r_1 z_1(1 - \dfrac{z_1}{k}), & t > 0, \\[2mm] z_1(t) = \max_{x \in \overline{\Omega}} u_0(x, t), & t = 0. \end{cases}$$

Then $\lim_{t \to \infty} z_1(t) = k$. From the comparison principle of parabolic equations, it follows that $u(x, t) \le z_1(t)$; hence

$$\limsup_{t \to +\infty} \max_{x \in \overline{\Omega}} u(x, t) \le k.$$

As a result, for any small enough $\varepsilon > 0$, there exists $T > 0$, such that $u(x, t) \leq k + \varepsilon$ for all $x \in \overline{\Omega}$ and $t \geq T$. It then follows that v satisfies

$$
\begin{cases}
v_t - d_2 \Delta v \leq r_2 v \left(1 - \dfrac{v}{c(k + \varepsilon)} \right), & x \in \Omega, & t \geq T, \\
\partial_n v = 0, & x \in \partial\Omega, & t \geq T, \\
v(x, t) > 0, & x \in \Omega, & t = T.
\end{cases}
$$

Let $w_1(t)$ be a solution of the ODE system

$$
\begin{cases}
w_1'(t) = r_2 w_1 (1 - \dfrac{w_1}{c(k + \varepsilon)}), & t > T, \\
w_1(t) = \max_{x \in \overline{\Omega}} v(x, t), & t = T.
\end{cases}
$$

Then $\lim_{t \to \infty} w_1(t) = c(k + \varepsilon)$. By the comparison principle we know that $v(x, t) \leq w_1(t)$, which leads to

$$
\limsup_{t \to +\infty} \max_{x \in \overline{\Omega}} v(x, t) \leq c(k + \varepsilon).
$$

Since $\varepsilon > 0$ is arbitrary, we get that $\limsup_{t \to +\infty} \max_{x \in \overline{\Omega}} v(x, t) \leq ck$ and the result of the theorem then follows. ∎

On the other hand, we also have the following persistent property.

Theorem 2.2. *Suppose that $\frac{ack}{br_1} < 1$. If (u, v) satisfies (1.2), then*

$$
\liminf_{t \to +\infty} \min_{x \in \overline{\Omega}} u(x, t) \geq k \left(1 - \frac{ack}{br_1} \right), \quad \liminf_{t \to +\infty} \min_{x \in \overline{\Omega}} v(x, t) \geq ck \left(1 - \frac{ack}{br_1} \right).
$$

Proof. Obviously, we see that u satisfies

$$
\begin{cases}
u_t - d_1 \Delta u \geq r_1 u \left(1 - \dfrac{ack}{br_1} - \dfrac{u}{k} \right), & x \in \Omega, & t > 0, \\
\partial_n u = 0, & x \in \partial\Omega, & t > 0, \\
u = u_0 \geq 0, \not\equiv 0, & x \in \Omega, & t = 0.
\end{cases}
$$

Assume that $z_2(t)$ is a solution of the ODE system

$$
\begin{cases}
z_2'(t) = r_1 z_2 \left(1 - \dfrac{ack}{br_1} - \dfrac{z_2}{k} \right), & t > 0, \\
z_2(t) = \min_{x \in \overline{\Omega}} u_0(x, t), & t = 0.
\end{cases}
$$

Then $\lim_{t \to \infty} z_2(t) = k \left(1 - \frac{ack}{br_1} \right)$ in view of $\frac{ack}{br_1} < 1$. It follows from the comparison principle that $u(x, t) \geq z_2(t)$, and then

$$
\liminf_{t \to +\infty} \min_{x \in \overline{\Omega}} u(x, t) \geq k \left(1 - \frac{ack}{br_1} \right).
$$

As a result, for any $0 < \varepsilon < k\left(1 - \frac{ack}{br_1}\right)$, there exists $T > 0$, such that

$$u(x,t) \geq k\left(1 - \frac{ack}{br_1}\right) - \varepsilon$$

for all $x \in \overline{\Omega}$ and $t \geq T$. Thus, v satisfies

$$\begin{cases} v_t - d_2\Delta v \geq r_2 v\left(1 - \dfrac{v}{c\left(k\left(1 - \frac{ack}{br_1}\right) - \varepsilon\right)}\right), & x \in \Omega, \quad t \geq T, \\ \partial_n v = 0, & x \in \partial\Omega, \quad t \geq T, \\ v(x,t) > 0, & x \in \Omega, \quad t = T. \end{cases}$$

Assume that $w_2(t)$ is a solution of the ODE system

$$\begin{cases} w_2'(t) = r_2 w_2\left(1 - \dfrac{w_2}{c\left(k\left(1 - \frac{ack}{br_1}\right) - \varepsilon\right)}\right), & t \geq T, \\ w_2(t) = \min_{x \in \overline{\Omega}} v(x,t), & t = T. \end{cases}$$

Then $\lim_{t\to\infty} w_2(t) = c\left(k\left(1 - \frac{ack}{br_1}\right) - \varepsilon\right)$. By the comparison principle we get $v(x,t) \geq w_2(t)$, and then

$$\liminf_{t\to+\infty} \min_{x\in\overline{\Omega}} v(x,t) \geq c\left(k\left(1 - \frac{ack}{br_1}\right) - \varepsilon\right).$$

Setting $\varepsilon \to 0$ yields

$$\liminf_{t\to+\infty} \min_{x\in\overline{\Omega}} v(x,t) \geq ck\left(1 - \frac{ack}{br_1}\right),$$

and the proof is then completed.   ∎

3. STABILITY OF CONSTANT POSITIVE SOLUTION

In this section, we establish the stability of the constant positive solution of the system (1.3). Note that the system (1.3) has a unique constant positive solution (u_1, v_1) given by

$$(3.1) \quad u_1 = \frac{(r_1 - ack - br_1) + \sqrt{(r_1 - ack - br_1)^2 + 4bkr_1^2}}{2r_1}, \qquad v_1 = cu_1.$$

Suppose $\{\mu_i : i = 0, 1, 2, \cdots\}$ is the set of eigenvalues of $-\Delta$ on Ω with homogenous Neumann boundary condition. Let X_i be the eigenspace corresponding to the eigenvalue μ_i. We now show that the constant positive solution (u_1, v_1) is asymptotically stable under a mild condition (see condition (3.2) below).

Theorem 3.1. *Let $\alpha = u_1 \left(\frac{r_1}{k} - \frac{acu_1}{(b+u_1)^2} \right)$ and $\beta = \frac{au_1}{b+u_1}$. If*

$$(3.2) \qquad \max \left\{ -r_2, d_2^{-1} \left(d_1 r_2 - 2\sqrt{d_1 d_2 r_2 c \beta} \right) \right\} < \alpha,$$

then the constant positive solution (u_1, v_1) is asymptotically stable.

Proof. The linearized operator of system (1.3) at (u_1, v_1) is

$$(3.3) \qquad \begin{pmatrix} -d_1 \Delta - r_1 + \dfrac{2r_1}{k} u_1 + \dfrac{abv_1}{(b+u_1)^2} & \dfrac{au_1}{b+u_1} \\ -\dfrac{r_2 v_1^2}{cu_1^2} & -d_2 \Delta - r_2 + \dfrac{2r_2 v_1}{cu_1} \end{pmatrix}.$$

Since $r_1(1 - \frac{u_1}{k}) - \frac{av_1}{b+u_1} = 0$ and $1 - \frac{v_1}{cu_1} = 0$, the matrix (3.2) can equivalently be rewritten as

$$(3.4) \quad \begin{pmatrix} -d_1 \Delta + u_1 \left(\dfrac{r_1}{k} - \dfrac{acu_1}{(b+u_1)^2} \right) & \dfrac{au_1}{b+u_1} \\ -cr_2 & -d_2 \Delta + r_2 \end{pmatrix} = \begin{pmatrix} -d_1 \Delta + \alpha & \beta \\ -cr_2 & -d_2 \Delta + r_2 \end{pmatrix}.$$

Thus, for each $i \in \{0, 1, 2, \cdots\}$, the operator (3.4) is invariant on X_i, and $\mu \in \mathbb{R}$ is an eigenvalue of (3.4) on X_i for some i if and only if μ is the eigenvalue of the matrix

$$A_i := \begin{pmatrix} d_1 \mu_i + \alpha & \beta \\ -cr_2 & d_2 \mu_i + r_2 \end{pmatrix}.$$

The determinant and trace of A_i are

$$\det A_i = d_1 d_2 \mu_i^2 + (d_2 \alpha + d_1 r_2)\mu_i + r_2(\alpha + c\beta)$$

and, respectively,

$$\operatorname{tr} A_i = (d_1 + d_2)\mu_i + r_2 + \alpha.$$

So, we find that $\det A_i > 0$ and $\operatorname{tr} A_i > 0$ provided there hold the following two inequalities:

(i) $(d_2 \alpha + d_1 r_2)^2 - 4d_1 d_2 r_2(\alpha + c\beta) < 0$;

(ii) $\alpha > -r_2$.

The inequality (i) holds if and only if there hold

$$d_2^{-1}(d_1 r_2 - 2\sqrt{d_1 d_2 r_2 c \beta}) < \alpha < d_2^{-1}(d_1 r_2 + 2\sqrt{d_1 d_2 r_2 c \beta}).$$

Meanwhile, it is obvious that $\det A_i > 0$ for $\alpha > 0$. Therefore, if condition (3.2) holds, then we must have $\det A_i > 0$ and $\operatorname{tr} A_i > 0$. In this case, the eigenvalues of A_i have positive real parts, and therefore (u_1, v_1) is asymptotically stable. ∎

Remark 3.2. Suppose that condition (3.2) holds. Then

(i) For $i = 0$, $\det A_0 = r_2(\alpha + c\beta)$ and $\operatorname{tr} A_0 = \alpha + r_2$. If

$$(\alpha + r_2)^2 < 4r_2(\alpha + c\beta),$$

then the real parts of eigenvalues of A_0 are $\operatorname{Re} \mu = \frac{1}{2}(\alpha + r_2) > 0$. While if

$$(\alpha + r_2)^2 \geq 4r_2(\alpha + c\beta),$$

then

$$\operatorname{Re} \mu_+ = \frac{1}{2}\left((\alpha + r_2) + \sqrt{(\alpha + r_2)^2 - 4r_2(\alpha + c\beta)}\right) > 0,$$
$$\operatorname{Re} \mu_- = \frac{1}{2}\left((\alpha + r_2) - \sqrt{(\alpha + r_2)^2 - 4r_2(\alpha + c\beta)}\right) > 0.$$

(ii) For $i \geq 1$, if $(\operatorname{tr} A_i)^2 < 4 \det A_i$, then

$$\operatorname{Re}\mu_\pm = \frac{1}{2}\operatorname{tr} A_i = \frac{1}{2}((d_1 + d_2)\mu_i + r_2 + \alpha) > 0.$$

If $(\operatorname{tr} A_i)^2 \geq 4 \det A_i$, then

$$\operatorname{Re} \mu_+ = \frac{1}{2}\left(\operatorname{tr} A_i + \sqrt{(\operatorname{tr} A_i)^2 - 4 \det A_i}\right) > 0,$$
$$\operatorname{Re} \mu_- = \frac{1}{2}\left(\operatorname{tr} A_i - \sqrt{(\operatorname{tr} A_i)^2 - 4 \det A_i}\right) > 0.$$

The above facts show that if condition (3.2) holds, there must exist a constant $\delta > 0$ independent of i and such that $\operatorname{Re} \mu \geq \delta$ for all $i = 0, 1, 2, \cdots$. Therefore, all eigenvalues of (3.4) are in the half plane $\{\mu : \operatorname{Re} \mu \geq \delta\}$.

4. Existence of Bifurcation Solution Emanating From (u_1, v_1)

In order to apply the bifurcation theory to study the existence of positive solutions, we take r_1 as a parameter and discuss the local bifurcation solutions of (1.3) which bifurcate from (u_1, v_1). The local bifurcation theory will be used to give a precise description for the structure of a positive solution near the bifurcation point, and the global bifurcation analysis describes the curve trend as the bifurcation parameter varies. Theorem 3.1 shows that (u_1, v_1) is asymptotically stable when condition (3.1) holds. So, in this case, there exist no bifurcation solutions emanating from (u_1, v_1). In view of this reason, in order to discuss bifurcation solutions of (1.3), we assume that α satisfies the condition

$$(4.1) \qquad \alpha \leq \max\left\{-r_2, d_2^{-1}\left(d_1 r_2 - 2\sqrt{d_1 d_2 r_2 c\beta}\right)\right\}.$$

Theorem 4.1. *Set*

(4.2)
$$\alpha_i = -\frac{d_1 d_2 \mu_i^2 + d_1 r_2 \mu_i + r_2 c\beta}{r_2 + d_2 \mu_i}, \quad i \geq 0.$$

Assume that

(i) $\alpha = \alpha_{i_0}$ *for some i_0, where α is given as in Theorem 3.1.*

(ii) $\alpha_i \neq \alpha_j$ *for all $i \neq j$.*

Then $(U_1; \alpha)$ is a bifurcation point of the system (1.3) as long as μ_{i_0} is a simple eigenvalue of $-\Delta$ on Ω with the homogenous Neumann boundary condition. [Here $U_1 = (u_1, v_1)^T$ is the constant positive solution of (1.3) with u_1 and v_1 being given in (3.1)].

Proof. For fixed r_2, k, a, b, c, we define a nonlinear operator $F : X \times \mathbb{R} \longrightarrow Y$ by

$$F(U; r_1) = \begin{pmatrix} -d_1 \Delta u - r_1 u + \dfrac{r_1}{k} u^2 + \dfrac{auv}{b+u} \\ -d_2 \Delta v - r_2 v + \dfrac{r_2 v^2}{cu} \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix},$$

where $X = C_0^{2,\alpha}(\overline{\Omega}) \times C_0^{2,\alpha}(\overline{\Omega}), Y = C^\alpha(\overline{\Omega}), C_0^{2,\alpha}(\overline{\Omega}) = \{u \in C^{2,\alpha}(\overline{\Omega}) : \partial_n u|_{\partial\Omega} = 0\}, \alpha \in (0,1), U \in X$. Thus, we see that U is the solution of the boundary value problem

$$F(U; r_1) = 0, \quad x \in \Omega; \quad \partial_n U = 0, \quad x \in \partial\Omega$$

if and only if U is a solution of (1.3). Therefore $F(U_1; r_1) = 0$, and the Fréchet derivative F_U of F at $(U_1; r_1)$ is

$$F_U(U_1; r_1) = \begin{pmatrix} -d_1\Delta - r_1 + \dfrac{2r_1}{k}u_1 + \dfrac{abv_1}{(b+u_1)^2} & \dfrac{au_1}{b+u_1} \\ -\dfrac{r_2 v_1^2}{cu_1^2} & -d_2\Delta - r_2 + \dfrac{2r_2 v_1}{cu_1} \end{pmatrix}$$

$$= \begin{pmatrix} -d_1\Delta + \alpha & \beta \\ -cr_2 & -d_2\Delta + r_2 \end{pmatrix}$$

$$=: F_U(U_1; \alpha),$$

where α and β are given as in Theorem 3.1.

We next show that $\lambda = 0$ is an eigenvalue of $F_U(U_1; \alpha)$. Indeed, assume that λ is an eigenvalue of $F_U(U_1; \alpha)$ and $(\varphi, \psi)^T$ is a corresponding eigenfunction with expansions

$$\varphi = \sum_{i=0}^{\infty} \sum_{j=1}^{\tau_i} a_{ij}\varphi_{ij}, \quad \psi = \sum_{i=0}^{\infty} \sum_{j=1}^{\tau_i} b_{ij}\varphi_{ij},$$

where $\tau_i \geq 1$ is the multiplicity of the eigenvalue μ_i of $-\Delta$, and φ_{ij} are the normalized eigenfunctions corresponding to μ_i. The sequence $\{\varphi_{ij}\}$, $i \geq 0$, $1 \leq j \leq \tau_i$ constitutes a complete orthonormal basis for $L^2(\Omega)$. Noticing $-\Delta\varphi_{ij} = \mu_i\varphi_{ij}$, we have

$$\sum_{i=0}^{\infty}\sum_{j=1}^{\tau_i} \left(\begin{array}{cc} \lambda - d_1\mu_i - \alpha & -\beta \\ cr_2 & \lambda - d_2\mu_i - r_2 \end{array} \right) \left(\begin{array}{c} a_{ij} \\ b_{ij} \end{array} \right) \varphi_{ij} = 0.$$

Since λ is an eigenvalue of $F_U(U_1; \alpha)$, λ solves the characteristic equation

$$(4.3) \quad \lambda^2 - ((d_1+d_2)\mu_{i_0}+\alpha_{i_0}+r_2)\lambda + d_1d_2\mu_{i_0}^2 + (d_2\alpha_{i_0}+d_1r_2)\mu_{i_0}+r_2(\alpha_{i_0}+c\beta) = 0.$$

By assumption (i), we find that the constant term of (4.3) equals 0. Hence 0 is an eigenvalue of $F_U(U_1; \alpha)$. This shows that $F_U(U_1; \alpha)$ is degenerate and $(\varphi, \psi)^T \in N(F_U(U_1; \alpha))$, where $N(F_U(U_1; \alpha))$ is the kernel space of $F_U(U_1; \alpha)$.

Since μ_{i_0} is simple, $\tau_{i_0} = 1$. In this case, $\varphi_{i_0 j} = \varphi_{i_0 1} =: \varphi_{i_0}$. Simple calculations give that

$$N(F_U(U_1; \alpha)) = \text{span}\left\{ \left(\begin{array}{c} \varphi \\ \psi \end{array} \right) \right\} = \text{span}\left\{ \left(\begin{array}{c} -\beta \\ d_1\mu_{i_0} + \alpha_{i_0} \end{array} \right) \varphi_{i_0} \right\},$$

and $\dim N(F_U(U_1; \alpha)) = 1$.

Now we consider $\text{codim}\, R(F_U(U_1; \alpha))$, where $R(F_U(U_1; \alpha))$ is the range space of $F_U(U_1; \alpha)$.

Since the conjugate operator $F_U^*(U_1; \alpha)$ of $F_U(U_1; \alpha)$ is

$$F_U^*(U_1; \alpha) = \left(\begin{array}{cc} -d_1\Delta + \alpha & -cr_2 \\ \beta & -d_2\Delta + r_2 \end{array} \right),$$

the matrices $F_U^*(U_1; \alpha)$ and $F_U(U_1; \alpha)$ have the same characteristic polynomial. Thus, $F_U^*(U_1; \alpha)$ is also degenerate. By direct calculations, we get

$$N(F_U^*(U_1; \alpha)) = \text{span}\left\{ \left(\begin{array}{c} \varphi^* \\ \psi^* \end{array} \right) \right\} = \text{span}\left\{ \left(\begin{array}{c} cr_2 \\ d_1\mu_{i_0} + \alpha_{i_0} \end{array} \right) \varphi_{i_0} \right\},$$

and $\dim N(F_U^*(U_1; \alpha)) = 1$. Therefore, noticing

$$R(F_U(U_1; \alpha)) = (N(F_U^*(U_1; \alpha)))^{\perp},$$

we must have $\text{codim}\, R(F_U(U_1; \alpha)) = 1$.

On the other hand, since the Fréchet derivative $F_{Ur_1}(U_1; \alpha)$ is $\left(\begin{array}{cc} \frac{2u_1}{k} - 1 & 0 \\ 0 & 0 \end{array} \right)$,

$$F_{Ur_1}(U_1; \alpha) \left(\begin{array}{c} \varphi \\ \psi \end{array} \right) = \left(\begin{array}{cc} \frac{2u_1}{k} - 1 & 0 \\ 0 & 0 \end{array} \right) \left(\begin{array}{c} \varphi \\ \psi \end{array} \right) = \left(\frac{2u_1}{k} - 1 \right) \left(\begin{array}{c} \varphi \\ 0 \end{array} \right).$$

We claim that $\left(\frac{2u_1}{k} - 1\right) \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \notin R(F_U(U_1; \alpha))$. As a matter of fact, suppose on the contrary that $\left(\frac{2u_1}{k} - 1\right) \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \in R(F_U(U_1; \alpha))$. Then the system

$$\begin{cases} -d_1 \Delta u + \alpha u + \beta v = \left(\frac{2u_1}{k} - 1\right)\varphi, \\ -d_2 \Delta v + r_2 v - cr_2 u = 0 \end{cases}$$

is solvable. Multiplying the first and second equations by φ and ψ, respectively, and integrating over Ω, we get

$$(4.4) \qquad \begin{aligned} \left(\frac{2u_1}{k} - 1\right)\int_\Omega \varphi^2 dx &= \int_\Omega (-d_1\Delta u + \alpha u + \beta v)\varphi dx \\ &= \int_\Omega (-d_1\Delta\varphi + \alpha\varphi)u dx + \beta\int_\Omega v\varphi dx \\ &= \beta\int_\Omega (v\varphi - u\psi)dx, \end{aligned}$$

and

$$(4.5) \qquad \begin{aligned} 0 &= \int_\Omega (-d_2\Delta v + r_2 v - cr_2 u)\psi dx \\ &= \int_\Omega (-d_2\Delta\psi + r_2\psi)v dx - cr_2\int_\Omega u\psi dx \\ &= cr_2\int_\Omega (v\varphi - u\psi)dx. \end{aligned}$$

By (4.4) and (4.5) we know that $\int_\Omega \varphi^2 dx = 0$, and so $\varphi \equiv 0$. This is a contradiction since φ is an eigenfunction. We have therefore proved that $\left(\frac{2u_1}{k} - 1\right)\begin{pmatrix} \varphi \\ 0 \end{pmatrix} \notin R(F_U(U_1; \alpha))$. Consequently, we can apply the Crandall-Rabinowitz's bifurcation theorem [16] to get our assertion that $(U_1; \alpha)$ is a bifurcation point of (1.3). ∎

The Crandall-Rabinowitz's bifurcation theorem implies that there exist $s_0 > 0$, $\gamma : (-s_0, s_0) \to \mathbb{R}$, $(\omega_1, \omega_2)^T : (-s_0, s_0) \to X$, $\gamma, \omega_1, \omega_2 \in C^1(-s_0, s_0)$ satisfying

$$\gamma(0) = 0, \quad (\omega_1, \omega_2)^T \in R(F_U(U_1; \alpha)), \quad \omega_1(0) = \omega_2(0) = 0.$$

Now let $\underline{u}(s) = u_1 + s(\varphi + \omega_1(s)) = u_1 - s\beta\varphi_{i_0} + o(s)$, $\underline{v}(s) = v_1 + s(\psi + \omega_2(s)) = v_1 + s(d_1\mu_{i_0} + \alpha_{i_0})\varphi_{i_0} + o(s)$, $r_1(s) = r_1 + \gamma(s)$, $\underline{U}(s) = (\underline{u}, \underline{v})^T$. Then $(\underline{U}(s); r_1(s))$ is the unique positive solution of the equation

$$F(U; r_1) = 0, \quad x \in \Omega; \quad \partial_n U = 0, \quad x \in \partial\Omega$$

near $(U_1; \alpha)$ and $\underline{U}(s)$ is the solution of (1.3). That is, the zero set of F only consists of two curves $(U_1; \alpha)$ and $(\underline{U}(s); r_1(s))$ in a neighborhood of the bifurcation point $(U_1; \alpha)$.

Remark 4.2. In Theorem 4.1, the condition $\alpha_i \neq \alpha_j$ for any integer $i \neq j$ is essential. In fact, if $\alpha_i = \alpha_j$ for some $i \neq j$, then it can't ensure that $\dim N(F_U(U_1; \alpha)) = 1$. This can be easily seen from the proof above. By (4.2) we can find that $\alpha_i = \alpha_j$ for $i \neq j$ if and only if

$$d_1 d_2 r_2 (\mu_i + \mu_j) + d_1 d_2^2 \mu_i \mu_j + d_1 r_2^2 = c\beta d_2 r_2.$$

Remark 4.3. If Ω is one dimensional, then all μ_i are simple and $(U_1; \alpha)$ is always a bifurcation point of (1.3) when $\alpha = \alpha_{i_0}$ for some i_0 and $\alpha_i \neq \alpha_j$ for any $i \neq j$.

Theorem 4.1 gives a precise description for the structure of positive solutions near the bifurcation point. But it provides no information on the bifurcating curve far from the equilibrium. In the following, we investigate the positive solutions of (1.3) by considering global bifurcation. The global bifurcation result shows that the bifurcation curve reaches to infinity. For simplicity, we suppose that Ω is one dimensional, say $\Omega = (0, 1)$.

Theorem 4.4. *Suppose that the assumptions of Theorem 4.1 hold. Then the bifurcation curve Γ_{i_0} of the positive bifurcation solution $(\underline{u}, \underline{v})$ of (1.3) which occurs at $(U_1; \alpha)$ tends to infinity.*

Proof. For $\Omega = (0, 1)$, system (1.3) becomes the following ordinary differential equations system

$$(4.6) \quad \begin{cases} -d_1 u'' = r_1 u(1 - \frac{u}{k}) - \dfrac{auv}{b + u}, & x \in (0, 1), \\ -d_2 v'' = r_2 v(1 - \dfrac{v}{cu}), & x \in (0, 1), \\ u' = v' = 0, & x = 0, 1. \end{cases}$$

Obviously, system (4.6) also has the unique positive constant solution (u_1, v_1).

Consider the eigenvalue problem

$$(4.7) \quad \begin{cases} -\varphi'' = \mu\varphi, & x \in (0, 1), \\ \varphi' = 0, & x = 0, 1. \end{cases}$$

It is well know that all eigenvalues $\mu_i, i = 0, 1, 2, \cdots$, of (4.7) are simple and all eigenfunctions $\{\varphi_i\}_{i=0}^{\infty}$ of (4.7) constitute an orthonormal basis in $L^2((0, 1))$.

Let $\tilde{u} = u - u_1, \tilde{v} = v - v_1$. Then system (4.6) can be written as

$$(4.8) \quad \begin{cases} -d_1 \tilde{u}'' + \alpha \tilde{u} + \beta \tilde{v} + \tilde{f}(\tilde{u}, \tilde{v}) &= 0, & x \in (0, 1), \\ -d_2 \tilde{v}'' - cr_2 \tilde{u} + r_2 \tilde{v} + \tilde{g}(\tilde{u}, \tilde{v}) &= 0, & x \in (0, 1), \\ \tilde{u}' = \tilde{v}' &= 0, & x = 0, 1, \end{cases}$$

where $\tilde{f}(\tilde{u}, \tilde{v})$ and $\tilde{g}(\tilde{u}, \tilde{v})$ are higher order terms of $\tilde{u}$ and $\tilde{v}$. Thus, the unique constant solution (u_1, v_1) of (4.6) shifts to the unique constant solution $(0, 0)$ of (4.8).

For $w(x) \in C([0, 1])$, assume that $u = G_1(w)$ and $v = G_2(w)$ are solutions of the problems

$$d_1 u'' + \alpha u = w, \quad x \in (0, 1), \quad u' = 0, \quad x = 0, 1,$$

and

$$d_2 v'' - r_2 v = w, \quad x \in (0, 1), \quad v' = 0, \quad x = 0, 1,$$

respectively, where $\alpha < 0$ in view of Theorem 4.1. Then $(u, v) \in C^2([0, 1]) \times C^2([0, 1])$ is unique, and the operators $G_1 = \left(d_1 \frac{d^2}{dx^2} + \alpha \right)^{-1}, G_2 = \left(d_2 \frac{d^2}{dx^2} - r_2 \right)^{-1}$ are compact.

Set

$$\widetilde{U} = (\tilde{u}, \tilde{v}), \quad E = \left\{ (u, v) | u, v \in C^2([0, 1]), \quad u' = v' = 0, \quad x = 0, 1 \right\}.$$

Then (4.8) can be interpreted as the equation

$$(4.9) \qquad \widetilde{U} = G(\alpha)(\widetilde{U}) + H(\widetilde{U}),$$

where

$$G(\alpha) = \begin{pmatrix} 2\alpha G_1 & \beta G_1 \\ -cr_2 G_2 & 0 \end{pmatrix}$$

and H are compact on E, $G(\alpha)(\widetilde{U}) = (2\alpha G_1(\tilde{u}) + \beta G_1(\tilde{v}), -cr_2 G_2(\tilde{u}))$, $H(\widetilde{U}) = (G_1(\tilde{f}), G_2(\tilde{g})) = o(|\widetilde{U}|)$.

In order to apply Rabinowitz's global bifurcation theorem [15], we should verify following facts:

  (i) 1 is an eigenvalue of $G(\alpha)$ with algebraic multiplicity one;

  (ii) for any sufficiently small $\varepsilon > 0$, the fixed point index $\text{index}(I - G(\alpha) - H, (0, 0))$ satisfies

$$(4.10) \quad \text{index}(I - G(\alpha - \varepsilon) - H, (0, 0)) \neq \text{index}(I - G(\alpha + \varepsilon) - H, (0, 0)).$$

For any $\varphi, \psi \in C^2([0,1]) \times C^2([0,1])$, it is easy to verify that $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in N(G(\alpha) - I)$ if and only if $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in N(F_U(U_1; \alpha))$. Thus,

$$N(G(\alpha) - I) = N(F_U(U_1; \alpha)) = \text{span}\left\{ \begin{pmatrix} -\beta \\ d_1\mu_{i_0} + \alpha_{i_0} \end{pmatrix} \varphi_{i_0} \right\}.$$

This shows that 1 is an eigenvalue of $G(\alpha)$ indeed and $\dim N(G(\alpha) - I) = 1$.

Now, we consider $N(G^*(\alpha) - I)$, where $G^*(\alpha)$ is the conjugate operator of $G(\alpha)$.

Let $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in N(G^*(\alpha) - I)$. Then we have

$$2\alpha G_1\varphi - cr_2 G_2\psi = \varphi, \quad \beta G_1\varphi = \psi.$$

Using the definitions of G_1, G_2 we get

$$d_2\varphi'' = (2\alpha d_1^{-1}d_2 + r_2)\varphi + (2\alpha^2\beta^{-1}d_1^{-1}d_2 - 2\alpha\beta^{-1}r_2 - cr_2)\psi, \quad d_1\psi'' = \beta\varphi - \alpha\psi.$$

Write $\varphi = \sum_{i=0}^{\infty} a_i\varphi_i, \psi = \sum_{i=0}^{\infty} b_i\varphi_i$, where $\{\varphi_i\}_{i=0}^{\infty}$ is the set of eigenfunctions of (4.7) in $L^2((0,1))$. Then $\sum_{i=0}^{\infty} C_i \begin{pmatrix} a_i \\ b_i \end{pmatrix} \varphi_i = 0$, where

$$C_i = \begin{pmatrix} 2\alpha d_1^{-1}d_2 + r_2 + d_2\mu_i & 2\alpha^2\beta^{-1}d_1^{-1}d_2 - 2\alpha\beta^{-1}r_2 - cr_2 \\ \beta & -\alpha + d_1\mu_i \end{pmatrix}.$$

A direct calculation yields

$$\det C_i = d_1 d_2 \mu_i^2 + (d_1 r_2 + d_2\alpha)\mu_i + r_2(c\beta + \alpha).$$

Take $i = i_0$ in $\det C_i$. Then it is exactly the case that $\lambda = 0$ in (4.3). Therefore C_i is degenerate for $i = i_0$. By a simple calculation we get

$$N(G^*(\alpha) - I) = \text{span}\left\{ \begin{pmatrix} -\alpha_{i_0} + d_1\mu_{i_0} \\ -\beta \end{pmatrix} \varphi_{i_0} \right\}.$$

Clearly, $\begin{pmatrix} -\alpha_{i_0} + d_1\mu_{i_0} \\ -\beta \end{pmatrix} \varphi_{i_0}$ and $\begin{pmatrix} -\beta \\ d_1\mu_{i_0} + \alpha_{i_0} \end{pmatrix} \varphi_{i_0}$ are not orthogonal to each other, so $\begin{pmatrix} -\beta \\ d_1\mu_{i_0} + \alpha_{i_0} \end{pmatrix} \varphi_{i_0} \notin (N(G^*(\alpha) - I))^{\perp} = R(G(\alpha) - I)$, and therefore,

$$N(G(\alpha) - I) \cap R(G(\alpha) - I) = \{0\}.$$

Further, $N((G(\alpha) - I)^2) = N(G(\alpha) - I)$, and for any positive integer n, we have

$$N((G(\alpha) - I)^n) = N(G(\alpha) - I).$$

By the definition of algebraic multiplicity of eigenvalues, we know that the algebraic multiplicity of eigenvalue 1 is just $\dim \cup_{n=1}^{\infty} N(G(\alpha) - I)^n$. This proves that 1 is an eigenvalue of $G(\alpha)$ with algebraic multiplicity one.

If $\alpha \neq \alpha_i$ is in a neighborhood of α_i, then the operator $I - G(\alpha) : E \to E$ is non-degenerate and $(0, 0)$ is an isolate fixed point of (4.9). Thus, by the definition of the fixed point index in [22], the index of $I - G(\alpha) - H$ at $(0, 0)$ is given by

$$\text{index}(I - G(\alpha) - H, (0, 0)) = \deg(I - G(\alpha), B, (0, 0)) = (-1)^{\nu},$$

where B is a sufficiently small ball centering at $(0, 0)$, and ν is the sum of the algebraic multiplicities of all positive eigenvalues of $G(\alpha) - I$.

In the following, we claim that (4.10) holds.

Let μ be an eigenvalue of $G(\alpha)$ with eigenfunction $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$, then we have

$$\mu d_1 \varphi'' = (2 - \mu)\alpha\varphi + \beta\psi, \quad \mu d_2 \psi'' = -cr_2\varphi + \mu r_2 \psi.$$

Write $\varphi = \sum_{i=0}^{\infty} a_i \varphi_i, \psi = \sum_{i=0}^{\infty} b_i \varphi_i$. Then

$$\sum_{i=0}^{\infty} \begin{pmatrix} (2 - \mu)\alpha + \mu d_1 \mu_i & \beta \\ -cr_2 & \mu r_2 + \mu d_2 \mu_i \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix} \varphi_i = 0.$$

Thus, all the eigenvalues of $G(\alpha)$ consist of the roots of the characteristic equation

$$(4.11) \qquad (d_1 \mu_i - \alpha)\mu^2 + 2\alpha\mu + \frac{c\beta r_2}{d_2 \mu_i + r_2} = 0, \quad i \geq 0.$$

If 1 is a root of (4.11), taking $\mu = \mu_j$ in (4.11), then we get $\alpha_i = \alpha_j$, and so $i = j$ by our assumption. Therefore, without counting the eigenvalues corresponding to $i = j$ in (4.11), $G(\alpha) - I$ has the same number of positive eigenvalues for all $\alpha \to \alpha_i$, and they also have the same multiplicities. For the case $i = j$, the two roots of (4.11) are

$$\mu(\alpha) = 1, \qquad \tilde{\mu}(\alpha) = \frac{-d_1 \mu_i - \alpha}{d_1 \mu_i - \alpha} < 1.$$

(Note that $\alpha < 0$). So $\tilde{\mu}(\alpha) < 1$ as $\alpha \to \alpha_i$. Since the root of (4.11) is increasing in α, we must have

$$\mu(\alpha - \varepsilon) < 1, \quad \mu(\alpha + \varepsilon) > 1.$$

for small ε. Consequently, $G(\alpha + \varepsilon) - I$ has exactly one more positive eigenvalue than $G(\alpha - \varepsilon) - I$ does. By a similar argument above we can show that this eigenvalue has algebraic multiplicity one. This proves (4.10).

Using the bifurcation theorem in [15], we know that the bifurcation curve Γ_{i_0} which occurs at $(U_1; \alpha)$ either tends to infinity or meets some other bifurcation point. In the following, we claim that the first alternative must occur. For convenience of use later, we denote such bifurcation point by $(U_1; \alpha_{i_0})$.

Assume that Γ_{i_0} dose not reach to infinity. Then Γ_{i_0} must meet another bifurcation point, say $(U_1; \alpha_{i_1})$, and can not meet other bifurcation point $(U_1; \alpha_{i_2})$ for $i_2 > i_1$.

Consider the problem

$$
(4.12) \qquad
\begin{cases}
d_1 u'' + r_1 u(1 - \dfrac{u}{k}) - \dfrac{auv}{b + u} = 0, & x \in (0, \dfrac{1}{k}), \\
d_2 v'' + r_2 v(1 - \dfrac{v}{cu}) = 0, & x \in (0, \dfrac{1}{k}), \\
u' = v' = 0, & x = 0, \dfrac{1}{k}.
\end{cases}
$$

If $\overline{U}$ is the solution of (4.12), then using $\overline{U}$, we may construct a solution of (4.6) by a reflective and periodic extension. For example, let $x_n = \frac{n}{k}, n = 0, 1, 2, \cdots, k$ and

$$
U(x) =
\begin{cases}
\overline{U}(x - x_{2n}), & x_{2n} \le x \le x_{2n+1}, \\
\overline{U}(x_{2n+2} - x), & x_{2n+1} \le x \le x_{2n+2}.
\end{cases}
$$

Then $x \in [0, 1]$ and $U(x)$ is a solution of (4.6). Clearly, $(U_1; \alpha_{i_1})$ is also a bifurcation point of (4.12). Denote by Γ_{i_1} the bifurcation curve which emanates from $(U_1; \alpha_{i_1})$. Then by the same argument above it is easy to show that Γ_{i_1} either reaches to infinity or meets some other bifurcation point $(U_1; a_{i_3}), i_3 > i_1$. If the latter alternative occurs, then it shows that Γ_i meets $(U_1; a_{i_3})$ too. It is an obvious contradiction. Therefore, Γ_{i_1} reaches to infinity, and then by the extension again we know that Γ_{i_0} reaches to infinity too. The proof is accomplished. ∎

5. Nonexistence and Existence of Nonconstant Positive Solutions

This section is devoted to the study of nonexistence and existence of nonconstant positive solutions of system (1.3). We shall prove that system (1.3) has no nonconstant positive solutions if the effective diffusion rates are suitably large.

For any $\varphi \in L^1(\Omega)$, we denote by $\overline{\varphi}$ the averaged value of φ over Ω, that is,

$$
\overline{\varphi} = \frac{1}{|\Omega|} \int_{\Omega} \varphi \, dx.
$$

Theorem 5.1. *Assume the following conditions*

(5.1)
$$d_1 > \frac{r_1}{\mu_1} + \left(\frac{ak}{2b}\right)^2 + \left(\frac{cr_2 b^2 r_1^2}{2(br_1 - ack)^2}\right)^2$$

and

(5.2)
$$d_2 > \frac{1}{\mu_1}\left(r_2 + \frac{2}{\mu_1}\right).$$

Then system (1.3) *does not admit any nonconstant positive solutions.*

Proof. For any $y, z \in \mathbb{R}$ and $s > 0$, using the Cauchy inequality [18] we get

(5.3)
$$yz \leq \frac{1}{4s}y^2 + sz^2.$$

In particular, taking $s = \frac{1}{\mu_1}$, we get

$$yz \leq \frac{\mu_1}{4}y^2 + \frac{1}{\mu_1}z^2.$$

Let (u, v) be a nonconstant positive solution of (1.3). Then we have $u \leq k$ and $v \leq ck$. Multiplying the two equations of (1.3) by $u - \overline{u} =: \xi$ and $v - \overline{v} =: \eta$, respectively, and then integrating over Ω and using the inequality (5.3), we obtain

$$
\begin{aligned}
d_1 \int_\Omega |\nabla \xi|^2 dx &= \int_\Omega \left(r_1 u\left(1 - \frac{u}{k}\right) - \frac{auv}{b+u}\right)\xi dx \\
&= \int_\Omega \left(\left(r_1 u\left(1 - \frac{u}{k}\right) - \frac{auv}{b+u}\right) - \left(r_1\overline{u}\left(1 - \frac{\overline{u}}{k}\right) - \frac{a\overline{u}\,\overline{v}}{b+\overline{u}}\right)\right)\xi dx \\
&= \int_\Omega \left(r_1\xi - \frac{r_1}{k}(u+\overline{u})\xi - \frac{ab\overline{v}\xi}{(b+\overline{u})(b+u)} - \frac{au\eta}{b+u}\right)\xi dx \\
&= \int_\Omega \left(r_1 - \frac{r_1}{k}(u+\overline{u}) - \frac{ab\overline{v}}{(b+\overline{u})(b+u)}\right)\xi^2 dx - \int_\Omega \frac{au}{b+u}\xi\eta dx \\
&\leq \int_\Omega \left(r_1 - \frac{r_1}{k}(u+\overline{u}) - \frac{ab\overline{v}}{(b+\overline{u})(b+u)}\right)\xi^2 dx + \int_\Omega \left|\frac{ak}{b}\xi\eta\right| dx \\
&\leq r_1 \int_\Omega \xi^2 dx + \frac{a^2 k^2 \mu_1}{4b^2}\int_\Omega \xi^2 dx + \frac{1}{\mu_1}\int_\Omega \eta^2 dx
\end{aligned}
$$

and

$$
\begin{aligned}
d_2 \int_\Omega |\nabla \eta|^2 dx &= \int_\Omega \left(r_2 v - \frac{r_2}{c}\frac{v^2}{u}\right)\eta dx \\
&= \int_\Omega \left(\left(r_2 v - \frac{r_2}{c}\frac{v^2}{u}\right) - \left(r_2\overline{v} - \frac{r_2}{c}\frac{\overline{v}^2}{\overline{u}}\right)\right)\eta dx \\
&= \int_\Omega \left(r_2\eta - \frac{r_2}{c}\left(\frac{v+\overline{v}}{\overline{u}}\eta - \frac{v^2}{\overline{u}u}\xi\right)\right)\eta dx \\
&= \int_\Omega \left(r_2 - \frac{r_2}{c}\frac{v+\overline{v}}{\overline{u}}\right)\eta^2 dx + \frac{r_2}{c\overline{u}}\int_\Omega \frac{v^2}{u}\xi\eta dx
\end{aligned}
$$

$$\leq r_2 \int_\Omega \eta^2 dx + \int_\Omega \frac{cr_2 b^2 r_1^2}{(br_1 - ack)^2} \xi\eta dx$$

$$\leq r_2 \int_\Omega \eta^2 dx + \left(\frac{cr_2 b^2 r_1^2}{2(br_1 - ack)^2} \right)^2 \int_\Omega \xi^2 dx + \frac{1}{\mu_1} \int_\Omega \eta^2 dx.$$

Therefore, by the Poincaré inequality [19], we have

$$d_1\mu_1 \int_\Omega \xi^2 dx + d_2\mu_1 \int_\Omega \eta^2 dx \leq d_1 \int_\Omega |\nabla\xi|^2 dx + d_2 \int_\Omega |\nabla\eta|^2 dx$$

$$\leq \left(r_1 + \frac{a^2 k^2 \mu_1}{4b^2} + \mu_1 \left(\frac{cr_2 b^2 r_1^2}{2(br_1 - ack)^2} \right)^2 \right) \int_\Omega \xi^2 dx + \left(r_2 + \frac{2}{\mu_1} \right) \int_\Omega \eta^2 dx,$$

which contradicts our assumptions. Therefore, (1.3) does not admit any nonconstant positive solutions under the assumptions (5.1) and (5.2). ∎

We next turn to investigate the existence of positive solutions via the fixed point index theory. Put $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and $U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$. Let

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad F(U) = \begin{pmatrix} r_1 u(1 - \frac{u}{k}) - \frac{auv}{b+u} \\ r_2 v(1 - \frac{v}{cu}) \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta \\ -cr_2 & r_2 \end{pmatrix}.$$

Then the Fréchet derivative of F at U_1 is $F_U(U_1) = B$. Hence (1.3) can be rewritten as

(5.4) $$-\Delta U = D^{-1}F(U), \quad x \in \Omega; \qquad \partial_n U = 0, \quad x \in \partial\Omega.$$

Thus, U is a solution of (5.4) if and only if U satisfies the equation

$$U - (I - \Delta)^{-1}(D^{-1}F(U) + U) = 0.$$

Let $H((d_1, d_2); U) = U - (I - \Delta)^{-1}(D^{-1}F(U) + U)$. Then the Fréchet derivative of H is

$$H_U((d_1, d_2); U_1) = I - (I - \Delta)^{-1}(D^{-1}B + I).$$

As the same as that of in Section 3, we see that, for each $i \in \{0, 1, 2, \cdots\}$, $H_U((d_1, d_2); U_1)$ is invariant on X_i; moreover, ξ is the eigenvalue of $H_U((d_1, d_2); U_1)$ on X_i if and only if $\xi(1 + \mu_i)$ is the eigenvalue of the matrix

$$\mu_i I - D^{-1}B = \begin{pmatrix} \mu_i - \alpha d_1^{-1} & -\beta d_1^{-1} \\ cr_2 d_2^{-1} & \mu_i - r_2 d_2^{-1} \end{pmatrix} =: M_i.$$

The determinant and the trace of M_i are given by

$$\det M_i = \frac{1}{d_1 d_2}(d_1 d_2 \mu_i^2 - (d_1 r_2 + d_2\alpha)\mu_i + r_2(\alpha + c\beta))$$

and respectively,

$$\operatorname{tr} M_i = 2\mu_i - \alpha d_1^{-1} - r_2 d_2^{-1}.$$

Now let

$$h((d_1, d_2); \mu) = \frac{1}{d_1 d_2}(d_1 d_2 \mu^2 - (d_1 r_2 + d_2 \alpha)\mu + r_2(\alpha + c\beta)).$$

Suppose that

(5.5)
$$(d_1 r_2 + d_2 \alpha)^2 > 4d_1 d_2 r_2(\alpha + c\beta).$$

Then the equation $h((d_1, d_2); \mu) = 0$ has two different real roots denoted by $\mu_+(d_1, d_2)$, $\mu_-(d_1, d_2)$, where

$$\mu_+(d_1, d_2) = \frac{(d_1 r_2 + d_2 \alpha) + \sqrt{(d_1 r_2 + d_2 \alpha)^2 - 4d_1 d_2 r_2(\alpha + c\beta)}}{2d_1 d_2},$$

$$\mu_-(d_1, d_2) = \frac{(d_1 r_2 + d_2 \alpha) - \sqrt{(d_1 r_2 + d_2 \alpha)^2 - 4d_1 d_2 r_2(\alpha + c\beta)}}{2d_1 d_2}.$$

Set $\Lambda = \{\mu_0, \mu_1, \mu_2, \cdots\}$ and

$$\mathcal{R} = \mathcal{R}(d_1, d_2) = \{\mu : \mu > 0, \mu_-(d_1, d_2) < \mu < \mu_+(d_1, d_2)\}.$$

In order to calculate index$(H((d_1, d_2), \cdot); U_1)$, we first need a lemma. Recall that $\tau_i \geq 1$ is the multiplicity of each $\mu_i, i = 0, 1, 2, \cdots$.

Lemma 5.2. [20]. *Suppose that $h((d_1, d_2); \mu_i) \neq 0$ for any $\mu_i \in \Lambda$. Then*

$$\operatorname{index}(H((d_1, d_2), \cdot); U_1) = (-1)^\sigma,$$

where

$$\sigma = \begin{cases} \sum_{\mu_i \in \mathcal{R} \cap \Lambda} \tau_i, & \mathcal{R} \cap \Lambda \neq \emptyset, \\ 0, & \mathcal{R} \cap \Lambda = \emptyset. \end{cases}$$

Particularly, if $h((d_1, d_2); \mu) > 0$ for any $\mu > 0$, then $\sigma = 0$.

We are now in a position to state and prove the main result of this section.

Theorem 5.3. *Suppose that $\sum_{\mu_i \in \mathcal{R} \cap \Lambda} \tau_i$ is odd and $\frac{r_2}{d_2} \in (\mu_p, \mu_{p+1})$ for some positive integer p. Then there exists $d_0^* > 0$ such that (1.3) has nonconstant positive solutions for $d_1 > d_0^*$.*

Proof. We need to evaluate index$(H((d_1, d_2), \cdot); U_1)$. By Lemma 5.2, we know that the key ingredient in the calculation of this index is to seek the range of μ when $h((d_1, d_2); \mu) < 0$.

It is easy to see that (5.5) holds and $\mu_+(d_1, d_2) > \mu_-(d_1, d_2) > 0$ when d_1 is sufficiently large. Moreover

$$\lim_{d_1 \to +\infty} \mu_+(d_1, d_2) = \frac{r_2}{d_2}, \quad \lim_{d_1 \to +\infty} \mu_-(d_1, d_2) = 0.$$

Hence, if $\frac{r_2}{d_2} \in (\mu_p, \mu_{p+1})$ for some positive integer p, then there must exist suitably large d_0 such that

(5.6) $$\mu_p < \mu_+(d_1, d_2) < \mu_{p+1}, \quad 0 < \mu_-(d_1, d_2) < \mu_1$$

for $d_1 \geq d_0$.

By the nonexistence result above, we know that there is $d > d_0$ such that (1.3) has no nonconstant positive solutions when $d_1 \geq d, d_2 = d$. Meanwhile, we can choose d large enough so that $\frac{r_2}{d_2} < \mu_1$. Thus, there must exist some d_0^* such that

(5.7) $$0 < \mu_-(d_1, d) < \mu_+(d_1, d) < \mu_1$$

provided that $d_1 \geq d_0^*$ (In fact, d_0^* only needs to satisfy $d_0^* \geq \max\{d_0, d\}$) . We now prove that (1.3) has nonconstant positive solutions for all $d_1 \geq d_0^*$. We do this by contradiction.

Suppose on the contrary that there exists d_1^* such that (1.3) has no nonconstant positive solutions for $d_1^* \geq d_0^*$.

Take $d_1 = d_1^*$. For $t \in [0, 1]$, we construct a homotopy operator by

$$D(t) = \begin{pmatrix} td_1 + (1-t)d_0^* & 0 \\ 0 & td_2 + (1-t)d \end{pmatrix}$$

and consider the problem

(5.8) $$-\Delta U = D^{-1}(t)F(U), \quad x \in \Omega; \quad \partial_n U = 0, \quad x \in \partial\Omega.$$

Obviously, U is a solution of (1.3) if and only if U is a solution of (5.8) (in this case $t = 1$). Therefore, as the unique constant positive solution of (1.3), U_1 is also the unique constant positive solution of (5.8).

It is clear that for each $t \in [0, 1]$, U is a nonconstant positive solution of (5.8) if and only if U is a positive solution of the equation

(5.9) $$U - (I - \Delta)^{-1}(D^{-1}(t)F(U) + U) = 0.$$

In view of the previous discussions, it is easily known that the equation (5.9) has no nonconstant positive solutions for $t = 0$. Meanwhile, by our assumptions we also know that (5.9) has no nonconstant positive solutions for $t = 1, d_1 = d_1^*$.

Now we set $f(U, t) = U - (I - \Delta)^{-1}(D^{-1}(t)F(U) + U)$. Then

$$f(U, 1) = H((d_1, d_2); U), \quad f(U, 0) = H((d_0^*, d); U),$$

and we have the Fréchet derivatives

$$H_U((d_1, d_2); U_1) = I - (I - \Delta)^{-1}(D^{-1}B + I)$$

and

$$H_U((d_0^*, d); U_1) = I - (I - \Delta)^{-1}(D_1^{-1}B + I),$$

where $D_1 = \begin{pmatrix} d_0^* & 0 \\ 0 & d \end{pmatrix}$. Using (5.6) and (5.7) we have

$$\mathcal{R}(d_1, d_2) \cap \Lambda = \{\mu_1, \mu_2, \cdots \mu_p\}, \quad \mathcal{R}(d_0^*, d) \cap \Lambda = \emptyset.$$

If $\sum_{\mu_i \in \mathcal{R} \cap \Lambda} \tau_i = \sum_{i=0}^{p} \tau_i$ is odd, then by Lemma 5.2 we have

$$(5.10) \qquad \text{index}(f(\cdot, 1); U_1) = \text{index}(H((d_1, d_2), \cdot); U_1) = -1,$$

$$(5.11) \qquad \text{index}(f(\cdot, 0); U_1) = \text{index}(H((d, d_0^*), \cdot); U_1) = 1.$$

If (u, v) is a positive solution of (1.3), then there must exist some $\varepsilon > 0$ such that $u, v > \varepsilon$ for $x \in \overline{\Omega}$. Let

$$\mathcal{S} = \{(u, v) \in X : \varepsilon < u, v < \max\{k, ck\} + \varepsilon\}.$$

Then for $t \in [0, 1]$ and $(u, v) \in \partial \mathcal{S}$, where $\partial \mathcal{S}$ is the boundary of $\mathcal{S}$, we have $f(U, t) \neq 0$. Thus, the Leray-Schauder degree $\deg(f(\cdot, t), \mathcal{S}, 0)$ is well defined, and by the homotopy invariance on degree [22], we know that $\deg(f(\cdot, t), \mathcal{S}, 0)$ is a constant. Therefore we have

$$(5.12) \qquad \deg(f(\cdot, 0), \mathcal{S}, 0) = \deg(f(\cdot, 1), \mathcal{S}, 0).$$

However, the equations $f(U, 0) = 0$ and $f(U, 1) = 0$ both have equal unique positive solution U_1, by (5.10) and (5.11) we have

$$\deg(f(\cdot, 1), \mathcal{S}, 0) = \text{index}(f(\cdot, 1); U_1) = -1,$$

$$\deg(f(\cdot, 0), \mathcal{S}, 0) = \text{index}(f(\cdot, 0); U_1) = 1.$$

This obviously contradicts (5.12), and the proof is therefore complete. ∎

Remark 5.4. Theorem 5.1 shows that when $d_2 > \frac{1}{\mu_1}\left(r_2 + \frac{2}{\mu_1}\right) > \frac{r_2}{\mu_1}$ and d_1 is large, then (1.3) has no nonconstant positive solution. On the other hand, for the existence result, we also see that the condition $\frac{r_2}{d_2} \in (\mu_p, \mu_{p+1})$ in Theorem 5.3 for some $p \geq 1$ implies that $d_2 < \frac{r_2}{\mu_1}$.

Using the same techniques as above, we can similarly show that following result holds.

Theorem 5.5. *Suppose that* $\mathcal{R}(d_1, d_2) \cap \Lambda = \{\mu_j, \mu_{j+1}, \cdots \mu_{j+p-1}\}$ *for some positive integers* $j, p \geq 1$. *If* $\sum_{\mu_i \in \mathcal{R} \cap \Lambda} \tau_i$ *is odd, then* (1.3) *has nonconstant positive solutions.*

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Yunfeng Jia and Jianhua Wu
College of Mathematics and Information Science
Shaanxi Normal University
Xi'an, Shaanxi 710062
P. R. China
E-mail: jiayf@snnu.edu.cn
        jianhuaw@snnu.edu.cn

Hong-Kun Xu
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung 80424, Taiwan
and
Department of Mathematics
College of Science
King Saud University
P. O. Box 2455
Riyadh 11451, Saudi Arabia
E-mail: xuhk@math.nsysu.edu.tw