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ALGORITHMS CONSTRUCTION FOR NONEXPANSIVE MAPPINGS AND INVERSE-STRONGLY MONOTONE MAPPINGS

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Abstract. In this paper, we construct two algorithms for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an α-inverse-strongly monotone mapping in a Hilbert space. We show that the sequence converges strongly to a common element of two sets under the some mild conditions on parameters. As special cases of the above two algorithms, we obtain two schemes which both converge strongly to the minimum norm element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an α-inverse-strongly monotone mapping.

1. INTRODUCTION

Let C be a closed convex subset of a real Hilbert space H. Recall that a mapping S of C into itself is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C.$$

We denote by $F(S)$ the set of fixed points of S. Algorithms for nonexpansive mappings have been studied in the literature, See, for instance [1-16]. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0, \forall u, v \in C.$$

A is called *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \forall u, v \in C.$$

It is well known that the variational inequality problem VI(C, A) is to find $x^* \in C$ such that

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$$\langle Ax^*, v - x^* \rangle \geq 0, \forall v \in C. \tag{1.1}$$

The variational inequality has been extensively studied in the literature. See, e.g., [17-21, 26-33] and the references therein.

For finding an element of $F(S) \cap VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse-strongly monotone, Takahashi and Toyoda [22] introduced the following iterative scheme:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) SP_C(x_n - \lambda_n Ax_n), n \geq 0, \tag{1.2}$$

where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to some $z \in F(S) \cap VI(C, A)$. We note that the scheme (1.2) has only weak convergence. An interesting problem is:

Question 1.1. Could we construct an algorithm based on (1.2) such that the constructed algorithm has strong convergence?

It is the first purpose in this paper that we will study the following algorithm

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) SP_C[(1 - \alpha_n)(x_n - \lambda_n Ax_n)], n \geq 0. \tag{1.3}$$

Remark 1.2. We should point out that the scheme (1.3) is similar to the scheme (1.2). As far as we know, this appears to be the first time in the literature that the scheme (1.3) is proposed. At the same time, we can show that the scheme (1.3) has strong convergence. As a matter of fact, we will propose a general algorithm which includes the algorithm (1.3) as a special case.

In 2005, Iiduka and Takahashi [23] further considered an iterative scheme for nonexpansive mapping and α-inverse-strongly monotone mapping:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \ \ and \ \ \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ defined by (1.4) converges strongly to $P_{F(S)\cap VI(C,A)}(x)$.

It is the second purpose in this paper that we will introduce a unified algorithm which includes (1.4) as a special case. Furthermore, we prove the strong convergence of the proposed algorithm under some more weaker assumptions on algorithm parameters.

On the other hand, we also notice that it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. For instance, given a closed convex subset C of a Hilbert space H_1 and a bounded linear operator $R : H_1 \to H_2$, where H_2 is another Hilbert space. The C-constrained pseudoinverse of R, $R_C^\dagger$, is then defined as the minimum-norm solution of the constrained minimization problem

$$R_C^\dagger(b) := \arg\min_{x \in C} \|Rx - b\|$$

which is equivalent to the fixed point problem

$$x = P_C(x - \lambda R^*(Rx - b))$$

where P_C is the metric projection from H_1 onto C, R^* is the adjoint of R, $\lambda > 0$ is a constant, and $b \in H_2$ is such that $P_{\overline{R(C)}}(b) \in R(C)$.

It is therefore another interesting problem to invent some algorithms that can generate schemes which converge strongly to the minimum-norm solution of a given problem.

It is the third purpose in this paper that we want to construct some algorithms for finding the minimum norm element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an α-inverse-strongly monotone mapping.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a closed convex subset of H. It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\|u - u_0\| = \inf\{\|u - x\| : x \in C\}.$$

We denote u_0 by $P_C u$, where P_C is called the *metric projection* of H onto C. The metric projection P_C of H onto C has the following basic properties:

(i) $\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in H$;

(ii) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$ for every $x, y \in H$;

(iii) $\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H$, $y \in C$;

(iv) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$ for all $x \in H$, $y \in C$.

Such properties of P_C will be crucial in the proof of our main results. Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see from property (iv) that

$$x^* \in VI(C, A) \Leftrightarrow x^* = P_C(x^* - \lambda A x^*), \forall \lambda > 0.$$

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [24]-[25]).

We need the following lemmas for proving our main results.

Lemma 2.1. ([9]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n+\beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.2. ([5]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section we will state and prove our main results. Throughout, we assume that:

(a) H is a real Hilbert space and C is a closed convex subset of H;

(b) $A : C \to H$ is an α-inverse-strongly monotone mapping and $S : C \to C$ is a nonexpansive mapping with $F(S) \cap VI(C, A) \neq \emptyset$;

(c) $\{\alpha_n\}$ is a real number sequence in $(0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$; $\{\beta_n\}$ is a real number sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ satisfying $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$. We assume the one of the following conditions is satisfied:

(C1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ and $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$;

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\limsup_{n\to\infty} \beta_n < 1$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(C3) $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$, $\limsup_{n\to\infty} \beta_n < 1$, $\lim_{n\to\infty} \frac{\beta_{n+1}-\beta_n}{\alpha_n} = 0$ and $\lim_{n\to\infty} \frac{\lambda_{n+1}-\lambda_n}{\alpha_n} = 0$.

At the same time, we also need the following facts:

(1) If $x^* \in F(S) \cap VI(C, A)$, then $x^* = P_C(x^* - \lambda_n Ax^*) = Sx^*$ for all $n \ge 0$;

(2) $I - \lambda_n A$ is nonexpansive and for all $x, y \in C$

$$\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \le \|x - y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2. \tag{3.1}$$

The aim of this section is to introduce some iterative methods for solving our three purposes in the first section. For these purpose, we first introduce the following iterative method which is based on Iiduka and Takahashi's algorithm (1.4).

Algorithm 3.1. For fixed $u \in H$ and given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)P_C[\alpha_n u + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n)], \forall n \ge 0. \tag{3.2}$$

In particular, if we take $u = 0$, then (3.2) reduces to

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)P_C[(1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n)], \forall n \ge 0. \tag{3.3}$$

We are now in a position to prove the strong convergence of the algorithm (3.2). As a special case, we obtain the strong convergence of the algorithm (3.3) to the minimum-norm element in $F(S) \cap VI(C, A)$.

Theorem 3.2. *The sequence $\{x_n\}$ defined by (3.2) converges strongly to $P_{F(S)\cap VI(C,A)}(u)$. If $u = 0$, then sequence $\{x_n\}$ defined by (3.3) converges strongly to $P_{F(S)\cap VI(C,A)}(0)$ which is the minimum norm element in $F(S) \cap VI(C, A)$.*

Proof. Set $y_n = P_C[\alpha_n u + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n)]$ for all $n \ge 0$. Pick $x^* \in F(S) \cap VI(C, A)$ to obtain,

$$\begin{aligned}
\|y_n - x^*\| &\le \|\alpha_n u + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) - x^*\| \\
&\le \alpha_n\|u - x^*\| + (1 - \alpha_n)\|SP_C(x_n - \lambda_n Ax_n) - SP_C(x^* - \lambda_n Ax^*)\| \\
&\le \alpha_n\|u - x^*\| + (1 - \alpha_n)\|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\| \\
&\le \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\|.
\end{aligned}$$

From (3.2), we get

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \beta_n\|x_n-x^*\|+(1-\beta_n)\|y_n-x^*\|\\ &\le \beta_n\|x_n-x^*\|+(1-\beta_n)\alpha_n\|u-x^*\|+(1-\beta_n)(1-\alpha_n)\|x_n-x^*\|\\ &= \alpha_n(1-\beta_n)\|u-x^*\|+[1-\alpha_n(1-\beta_n)]\|x_n-x^*\|\\ &\le \max\{\|u-x^*\|,\|x_n-x^*\|\}.\end{aligned}$$

By induction, we can deduce that the sequence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{Ax_n\}$.

Next we will show $\|x_{n+1}-x_n\|\to 0$. We will divide into two cases to prove this fact.

Case 1. Assume the conditions (C1) is satisfied.

By the definition of y_n, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|P_C[\alpha_{n+1}u+(1-\alpha_{n+1})SP_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})]\\ &\quad -P_C[\alpha_n u+(1-\alpha_n)SP_C(x_n-\lambda_n Ax_n)]\|\\ &\le \|(\alpha_{n+1}-\alpha_n)u+(1-\alpha_{n+1})SP_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})\\ &\quad -(1-\alpha_n)SP_C(x_n-\lambda_n Ax_n)\|\\ &\le \alpha_n(\|u\|+\|SP_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})\|)\\ &\quad +\alpha_n(\|u\|+\|SP_C(x_n-\lambda_n Ax_n)\|)+\|SP_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})\\ &\quad -SP_C(x_n-\lambda_n Ax_n)\|\\ &\le \alpha_n(\|u\|+\|SP_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})\|)+\alpha_n(\|u\|\\ &\quad +\|SP_C(x_n-\lambda_n Ax_n)\|)+\|(I-\lambda_{n+1}A)x_{n+1}\\ &\quad -(I-\lambda_{n+1}A)x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|\\ &\le \alpha_n(\|u\|+\|SP_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})\|)+\alpha_n(\|u\|\\ &\quad +\|SP_C(x_n-\lambda_n Ax_n)\|)+\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|.\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Therefore by Lemma 2.1, we obtain $\lim_{n\to\infty}\|y_n-x_n\|=0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|y_n-x_n\|=0.$$

Case 2. Assume (C2) or (C3) is satisfied.

From (3.2), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq \beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|y_{n-1}\|)\\ &\quad +(1-\beta_n)\|y_n-y_{n-1}\|\end{aligned}$$

and

$$\begin{aligned}\|y_n-y_{n-1}\| &\leq |\alpha_n-\alpha_{n-1}|\|u\|+(1-\alpha_n)\|SP_C(x_n-\lambda_nAx_n)\\ &\quad -SP_C(x_{n-1}-\lambda_{n-1}Ax_{n-1})\|\\ &\quad +|\alpha_n-\alpha_{n-1}|\|SP_C(x_{n-1}-\lambda_{n-1}Ax_{n-1})\|\\ &\leq |\alpha_n-\alpha_{n-1}|(\|u\|+\|SP_C(x_{n-1}-\lambda_{n-1}Ax_{n-1})\|)\\ &\quad +(1-\alpha_n)\|(x_n-\lambda_nAx_n)-(x_{n-1}-\lambda_nAx_{n-1})\|\\ &\quad +|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|\\ &\leq |\alpha_n-\alpha_{n-1}|(\|u\|+\|SP_C(x_{n-1}-\lambda_{n-1}Ax_{n-1})\|)\\ &\quad +(1-\alpha_n)\|x_n-x_{n-1}\|\\ &\quad +|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq [1-(1-\beta_n)\alpha_n]\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\|u\|\\ &\quad +\|SP_C(x_{n-1}-\lambda_{n-1}Ax_{n-1})\|)\\ &\quad +|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|y_{n-1}\|)+|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|.\end{aligned}$$

This together with Lemma 2.2 imply that$\|x_{n+1}-x_n\|\to 0$.

By the convexity of the norm and (3.1), we have

$$\begin{aligned}\|y_n-x^*\|^2 &= \|P_C[\alpha_nu+(1-\alpha_n)SP_C(x_n-\lambda_nAx_n)]-x^*\|^2\\ &\leq \|\alpha_n(u-x^*)+(1-\alpha_n)(SP_C(x_n-\lambda_nAx_n)-x^*)\|^2\\ &\leq \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|SP_C(x_n-\lambda_nAx_n)-Sx^*\|^2\\ &\leq \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|P_C(x_n-\lambda_nAx_n)-x^*\|^2\\ &= \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|P_C(x_n-\lambda_nAx_n)-P_C(x^*-\lambda_nAx^*)\|^2\\ &\leq \alpha_n\|u-x^*\|^2+\|(x_n-\lambda_nAx_n)-(x^*-\lambda_nAx^*)\|^2\\ &\leq \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Ax^*\|^2.\end{aligned}$$

Hence, we obtain

$$\|x_{n+1}-x^*\|^2 \leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2$$

$$\begin{aligned}&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\ &\quad +(1-\beta_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ax^*\|^2 \\ &= \|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 \\ &\quad +(1-\beta_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ax^*\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&-(1-\beta_n)a(b-2\alpha)\|Ax_n - Ax^*\|^2 \\ \leq\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 \\ \leq\ & (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \times (\|x_n - x_{n+1}\|) + (1-\beta_n)\alpha_n\|u - x^*\|^2.\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain $\|Ax_n - Ax^*\| \to 0$ as $n \to \infty$.

Set $z_n = P_C(x_n - \lambda_n Ax_n)$ for all $n \geq 0$. From property (ii) of the metric projection, we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(x^* - \lambda_n Ax^*)\|^2 \\ &\leq \langle (x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*), z_n - x^*\rangle \\ &= \frac{1}{2}\Big\{\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\|^2 + \|z_n - x^*\|^2 \\ &\quad -\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*) - (z_n - x^*)\|^2\Big\} \\ &\leq \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(x_n - z_n) - \lambda_n(Ax_n - Ax^*)\|^2\Big\} \\ &\leq \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 \\ &\quad +2\lambda_n\langle x_n - z_n, Ax_n - Ax^*\rangle - \|\lambda_n(Ax_n - Ax^*)\|^2\Big\} \\ &\leq \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 \\ &\quad +2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|\Big\}.\end{aligned}$$

So, we obtain

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|$$

and hence

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2]\end{aligned}$$

$$\begin{aligned}&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\alpha_n\|u - x^*\|^2 + (1-\alpha_n)(\|x_n - x^*\|^2\\&\quad -\|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|)]\\&\le \|x_n - x^*\|^2 + \alpha_n\|u - x^*\|^2 - (1-\alpha_n)(1-\beta_n)\|x_n - z_n\|^2\\&\quad +2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|,\end{aligned}$$

which implies that

$$\begin{aligned}&(1-\alpha_n)(1-\beta_n)\|x_n - z_n\|^2\\&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|u - x^*\|^2\\&\quad +2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|\\&\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| + \alpha_n\|u - x^*\|^2\\&\quad +2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|.\end{aligned}$$

Since $\alpha_n \to 0, \|x_n - x_{n+1}\| \to 0$ and $\|Ax_n - Ax^*\| \to 0$ as $n \to \infty$, we have $\|x_n - z_n\| \to 0$ as $n \to \infty$. At the same time, we note that

$$\|y_n - Sz_n\| = \|P_C[\alpha_n u + (1-\alpha_n)Sz_n] - P_C[Sz_n]\| \le \alpha_n\|u - Sz_n\| \to 0.$$

Then we have

$$\|Sz_n - z_n\| \le \|Sz_n - y_n\| + \|y_n - x_n\| + \|x_n - z_n\| \to 0.$$

Next we show that

$$\limsup_{n\to\infty}\langle u - z_0, z_n - z_0\rangle \le 0, \tag{3.4}$$

where $z_0 = P_{F(S)\cap VI(C,A)}(u)$.

To show it, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, Sz_n - z_0\rangle = \lim_{i\to\infty}\langle u - z_0, Sz_{n_i} - z_0\rangle.$$

As $\{z_{n_i}\}$ is bounded, we have that a subsequence $\{z_{n_{ij}}\}$ of $\{z_{n_i}\}$ converges weakly to z. We may assume without loss of generality that $z_{n_i} \rightharpoonup z$. Since $\|Sz_n - z_n\| \to 0$, we obtain $Sz_{n_i} \rightharpoonup z$ as $i \to \infty$. By the similar argument as that of [23], we can deduce $z \in F(S) \cap VI(C, A)$. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z_0, z_n - z_0\rangle &= \limsup_{n\to\infty}\langle u - z_0, Sz_n - z_0\rangle\\&= \lim_{i\to\infty}\langle u - z_0, Sz_{n_i} - z_0\rangle\\&= \langle u - z_0, z - z_0\rangle\\&\le 0.\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle \le 0.$$

From property (iii) of the metric projection, we have

$$\begin{aligned}\|y_n - z_0\|^2 &= \langle y_n - [\alpha_n u + (1-\alpha_n)Sz_n], y_n - z_0\rangle \\ &\quad + \langle \alpha_n(u - z_0) + (1-\alpha_n)(Sz_n - z_0), y_n - z_0\rangle \\ &\le \langle \alpha_n(u - z_0) + (1-\alpha_n)(Sz_n - z_0), y_n - z_0\rangle \\ &\le (1-\alpha_n)\|Sz_n - z_0\|\|y_n - z_0\| + \alpha_n\langle u - z_0, y_n - z_0\rangle \\ &\le \frac{1-\alpha_n}{2}(\|z_n - z_0\|^2 + \|y_n - z_0\|^2) + \alpha_n\langle u - z_0, y_n - z_0\rangle \\ &\le \frac{1-\alpha_n}{2}(\|x_n - z_0\|^2 + \|y_n - z_0\|^2) + \alpha_n\langle u - z_0, y_n - z_0\rangle.\end{aligned}$$

Hence,

$$\|y_n - z_0\|^2 \le (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, y_n - z_0\rangle.$$

Therefore,

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &\le \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|y_n - z_0\|^2 \\ &\le \beta_n\|x_n - z_0\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - z_0\|^2 \\ &\quad + 2\alpha_n(1-\beta_n)\langle u - z_0, y_n - z_0\rangle \\ &= [1 - \alpha_n(1-\beta_n)]\|x_n - z_0\|^2 + 2\alpha_n(1-\beta_n)\langle u - z_0, y_n - z_0\rangle \\ &= (1-\gamma_n)\|x_n - z_0\|^2 + \delta_n,\end{aligned}$$

where $\gamma_n = (1-\beta_n)\alpha_n$ and $\delta_n = (1-\beta_n)\alpha_n\{2\langle u - z_0, y_n - z_0\rangle\}$. It is easily seen that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and

$$\limsup_{n\to\infty}\delta_n/\gamma_n = \limsup_{n\to\infty}\{2\langle u - z_0, y_n - z_0\rangle\} \le 0.$$

Hence, all conditions of Lemma 2.2 are satisfied. Therefore, we immediately deduce that $x_n \to z_0$, $y_n \to z_0$ and $z_n \to z_0$. Finally, if we take $u = 0$, then $z_0 = P_{F(S)\cap VI(C,A)}(0)$. This clearly implies that z_0 is a minimum-norm element in $F(S)\cap VI(C,A)$. This completes the proof. ∎

Remark 3.3. If we take $u \in C$ and $\beta_n = 0$ for all $n \ge 0$, then (3.2) reduces to

$$x_{n+1} = \alpha_n u + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n), \forall n \ge 0$$

which is exactly (1.4) studied by Iiduka and Takahashi[23]. Therefore, Theorem 3.2 includes the main result in Iiduka and Takahashi [23] as a special case.

Next, we introduce another interesting algorithm which is different from those in the literature.

Algorithm 3.4. For fixed $u \in H$ and given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

(3.5) $$x_{n+1} = \beta_n x_n + (1-\beta_n)SP_C[\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)], \forall n \geq 0.$$

In particular, if we take $u = 0$, then (3.5) reduces to

(3.6) $$x_{n+1} = \beta_n x_n + (1-\beta_n)SP_C[(1-\alpha_n)(x_n - \lambda_n Ax_n)], \forall n \geq 0.$$

Theorem 3.5. *The sequence $\{x_n\}$ defined by (3.5) converges strongly to $P_{F(S)\cap VI(C,A)}(u)$. If $u = 0$, then sequence $\{x_n\}$ defined by (3.6) converges strongly to $P_{F(S)\cap VI(C,A)}(0)$ which is the minimum norm element in $F(S)\cap VI(C,A)$.*

Proof. Set $y_n = P_C[\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)]$ for all $n \geq 0$. Take $x^* \in F(S)\cap VI(C,A)$. From (3.5), we have

$$\begin{aligned}
\|y_n - x^*\| &= \|P_C[\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)] - P_C[x^* - \lambda_n Ax^*]\| \\
&\leq \|\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\| \\
&\leq \alpha_n\|u - x^* + \lambda_n Ax^*\| + (1-\alpha_n)\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\| \\
&\leq \alpha_n(\|u - x^*\| + b\|Ax^*\|) + (1-\alpha_n)\|x_n - x^*\|.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\|Sy_n - x^*\| \\
&\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\alpha_n(\|u - x^*\| + b\|Ax^*\|) \\
&\quad +(1-\beta_n)(1-\alpha_n)\|x_n - x^*\| \\
&= \alpha_n(1-\beta_n)(\|u - x^*\| + b\|Ax^*\|) + [1 - \alpha_n(1-\beta_n)]\|x_n - x^*\| \\
&\leq \max\{\|u - x^*\| + b\|Ax^*\|, \|x_n - x^*\|\}.
\end{aligned}$$

By induction, we can deduce that the sequence $\{x_n\}$ is bounded.

Next we will show $\|x_{n+1} - x_n\| \to 0$. We will divide into two cases to prove this fact.

Case 1. Assume (C1) holds. By (3.5), we have

$$\begin{aligned}
\|Sy_{n+1} - Sy_n\| &= \|P_C[\alpha_{n+1}u + (1-\alpha_{n+1})(x_{n+1} - \lambda_{n+1}Ax_{n+1})] \\
&\quad -P_C[\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)]\| \\
&\leq \|(1-\alpha_{n+1})(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (1-\alpha_n)(x_n - \lambda_n Ax_n)\| \\
&\quad +(\alpha_{n+1} + \alpha_n)\|u\| \\
&\leq \|x_{n+1} - \lambda_{n+1}Ax_{n+1} - (x_n - \lambda_{n+1}Ax_n)\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\
&\quad +\alpha_{n+1}(\|u\| + \|x_{n+1} - \lambda_{n+1}Ax_{n+1}\|) + \alpha_n(\|u\| + \|x_n - \lambda_n Ax_n\|)
\end{aligned}$$

$$\begin{aligned}&\leq \|x_{n+1}-x_n\|+\alpha_{n+1}(\|u\|+\|x_{n+1}-\lambda_{n+1}Ax_{n+1}\|)\\&\quad+\alpha_n(\|u\|+\|x_n-\lambda_nAx_n\|)+|\lambda_{n+1}-\lambda_n|\|Ax_n\|.\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}(\|Sy_{n+1}-Sy_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Therefore by Lemma 2.1, we obtain $\lim_{n\to\infty}\|Sy_n-x_n\|=0.$ Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|Sy_n-x_n\|=0.$$

Case 2. Assume (C2) or (C3) holds. From (3.5), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq \beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|Sy_{n-1}\|)\\&\quad+(1-\beta_n)\|Sy_n-Sy_{n-1}\|\\&\leq \beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|Sy_{n-1}\|)\\&\quad+(1-\beta_n)\|y_n-y_{n-1}\|\end{aligned}$$

and

$$\begin{aligned}\|y_n-y_{n-1}\| &\leq |\alpha_n-\alpha_{n-1}|\|u\|+(1-\alpha_n)\|(x_n-\lambda_nAx_n)-(x_{n-1}\\&\quad-\lambda_{n-1}Ax_{n-1})\|\\&\quad+|\alpha_n-\alpha_{n-1}|\|x_{n-1}-\lambda_{n-1}Ax_{n-1}\|\\&\leq |\alpha_n-\alpha_{n-1}|(\|u\|+\|x_{n-1}-\lambda_{n-1}Ax_{n-1}\|)\\&\quad+(1-\alpha_n)\|(x_n-\lambda_nAx_n)-(x_{n-1}-\lambda_nAx_{n-1})\|\\&\quad+|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|\\&\leq |\alpha_n-\alpha_{n-1}|(\|u\|+\|x_{n-1}-\lambda_{n-1}Ax_{n-1}\|)\\&\quad+(1-\alpha_n)\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq [1-(1-\beta_n)\alpha_n]\|x_n-x_{n-1}\|\\&\quad+|\alpha_n-\alpha_{n-1}|(\|u\|+\|x_{n-1}-\lambda_{n-1}Ax_{n-1}\|)\\&\quad+|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|Sy_{n-1}\|)+|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|.\end{aligned}$$

This together with Lemma 2.2 imply that$\|x_{n+1}-x_n\|\to 0.$ Next we will use $M>0$ to denote some possible constants appearing in the following.

For $x^*\in F(S)\cap VI(C,A)$, we obtain

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\le\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|Sy_n - x^*\|^2 \\
\le\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\
=\ & (1-\beta_n)\|P_C[\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)] - P_C[x^* - \lambda_n Ax^*]\|^2 + \beta_n\|x_n - x^*\|^2 \\
\le\ & (1-\beta_n)\|\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\|^2 + \beta_n\|x_n - x^*\|^2 \\
\le\ & (1-\beta_n)[\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\| \\
& +\alpha_n\|u - x_n + \lambda_n Ax_n\|]^2 + \beta_n\|x_n - x^*\|^2 \\
\le\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\|^2 + \alpha_n M] \\
\le\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ax^*\|^2 + \alpha_n M] \\
=\ & \|x_n - x^*\|^2 + \lambda_n(1-\beta_n)(\lambda_n - 2\alpha)\|Ax_n - Ax^*\|^2 + \alpha_n M.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& -(1-\beta_n)a(b-2\alpha)\|Ax_n - Ax^*\|^2 \\
\le\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M \\
\le\ & (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \times (\|x_n - x_{n+1}\|) + \alpha_n M.
\end{aligned}
$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain $\|Ax_n - Ax^*\| \to 0$ as $n \to \infty$.

From property (ii) of the metric projection P_C, we have

$$
\begin{aligned}
& \|y_n - x^*\|^2 \\
=\ & \|P_C[\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n) - P_C[x^* - \lambda_n Ax^*]\|^2 \\
\le\ & \langle \alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*), y_n - x^* \rangle \\
=\ & \frac{1}{2}\Big\{\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*) - \alpha_n[u - (I - \lambda_n A)x_n]\|^2 + \|y_n - x^*\|^2 \\
& -\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*) - (y_n - x^*) - \alpha_n[u - (I - \lambda_n A)x_n]\|^2\Big\} \\
\le\ & \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|y_n - x^*\|^2 + \alpha_n M \\
& -\|(x_n - y_n) - \lambda_n(Ax_n - Ax^*) - \alpha_n[u - (I - \lambda_n A)x_n]\|^2\Big\} \\
\le\ & \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|y_n - x^*\|^2 + \alpha_n M - \|x_n - y_n\|^2 \\
& +2\lambda_n\langle x_n - y_n, Ax_n - Ax^* \rangle + 2\alpha_n\langle u - x_n + \lambda_n Ax_n, x_n - y_n \rangle \\
& -\|\lambda_n(Ax_n - Ax^*) + \alpha_n[u - (I - \lambda_n A)x_n]\|^2\Big\} \\
\le\ & \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|y_n - x^*\|^2 - \|x_n - y_n\|^2 + (\alpha_n + \|Ax_n - Ax^*\|)M\Big\}.
\end{aligned}
$$

So, we obtain

$$\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + (\alpha_n + \|Ax_n - Ax^*\|)M$$

and hence

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2 - (1-\beta_n)\|x_n - y_n\|^2 + (\alpha_n + \|Ax_n - Ax^*\|)M,\end{aligned}$$

which implies that

$$\begin{aligned}&(1-\beta_n)\|x_n - y_n\|^2 \\ \le\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (\alpha_n + \|Ax_n - Ax^*\|)M \\ \le\ & \|x_n - x_{n+1}\| \times (\|x_n - x^*\| + \|x_{n+1} - x^*\|) + (\alpha_n + \|Ax_n - Ax^*\|)M.\end{aligned}$$

Since $\alpha_n \to 0, \|x_n - x_{n+1}\| \to 0$ and $\|Ax_n - Ax^*\| \to 0$ as $n \to \infty$, we have $\|x_n - y_n\| \to 0$ as $n \to \infty$. Then we have

$$\lim_{n\to\infty} \|Sy_n - y_n\| = 0.$$

Next we show that

$$\limsup_{n\to\infty} \langle u - z_0, y_n - z_0\rangle \le 0,$$

where $z_0 = P_{F(S)\cap VI(C,A)}u$.

To show it, we choose a subsequence $\{y_{n_i}\}$ of $\{zy_n\}$ such that

$$\limsup_{n\to\infty} \langle u - z_0, Sy_n - z_0\rangle = \lim_{i\to\infty} \langle u - z_0, Sy_{n_i} - z_0\rangle.$$

As $\{y_{n_i}\}$ is bounded, we have that a subsequence $\{y_{n_{ij}}\}$ of $\{y_{n_i}\}$ converges weakly to z. We may assume without loss of generality that $y_{n_i} \rightharpoonup z$. Since $\|Sy_n - y_n\| \to 0$, we obtain $Sy_{n_i} \rightharpoonup z$ as $i \to \infty$. Then we can obtain $z \in F(S) \cap VI(C,A)$. In fact, let us first show that $z \in VI(C,A)$.

Let

$$Tv = \begin{cases} Av + N_C v, v \in C, \\ \qquad \emptyset, v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v,w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, w - Av\rangle \ge 0$. On the other hand, from $y_n = P_C[\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)]$, we have $\langle v - y_n, y_n - \alpha_n u - (1-\alpha_n)(x_n - \lambda_n Ax_n)\rangle \ge 0$, that is,

$$\langle v - y_n, \frac{y_n - x_n}{\lambda_n} + Ax_n + \frac{\alpha_n}{\lambda_n}(x_n - \lambda_n Ax_n - u)\rangle \ge 0.$$

Therefore, we have

$$
\begin{aligned}
&\langle v - y_{n_i}, w\rangle \\
&\geq \langle v - y_{n_i}, Av\rangle \\
&\geq \langle v - y_{n_i}, Av\rangle - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i} + \frac{\alpha_{n_i}}{\lambda_{n_i}}(x_{n_i} - \lambda_{n_i}Ax_{n_i} - u)\rangle \\
&= \langle v - y_{n_i}, Av - Ax_{n_i} - \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} - \frac{\alpha_{n_i}}{\lambda_{n_i}}(x_{n_i} - \lambda_{n_i}Ax_{n_i} - u)\rangle \\
&= \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle \\
&\quad -\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle - \langle v - y_{n_i}, \frac{\alpha_{n_i}}{\lambda_{n_i}}(x_{n_i} - \lambda_{n_i}Ax_{n_i} - u)\rangle \\
&\geq \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle \\
&\quad -\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle - \frac{\alpha_{n_i}}{\lambda_{n_i}}\langle v - y_{n_i}, (x_{n_i} - \lambda_{n_i}Ax_{n_i} - u)\rangle.
\end{aligned}
$$

Note that $\|y_{n_i} - x_{n_i}\| \to 0, \|Ay_{n_i} - Ax_{n_i}\| \to 0$ and $\alpha_{n_i} \to 0$. Hence we obtain $\langle v - z, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Let us show that $z \in F(S)$. Assume $z \notin F(S)$. From Opial's condition, we have

$$
\begin{aligned}
\liminf_{i\to\infty} \|y_{n_i} - z\| &< \liminf_{i\to\infty} \|y_{n_i} - Sz\| \\
&= \liminf_{i\to\infty} \|y_{n_i} - Sy_{n_i} + Sy_{n_i} - Sz\| \\
&\leq \liminf_{i\to\infty} \|Sy_{n_i} - Sz\| \\
&\leq \liminf_{i\to\infty} \|y_{n_i} - z\|.
\end{aligned}
$$

This is a contradiction. Thus, we obtain $z \in F(S)$. Then we have

$$
\begin{aligned}
&\limsup_{n\to\infty} \langle u - z_0, y_n - z_0\rangle \\
&\qquad = \limsup_{n\to\infty} \langle u - z_0, Sy_n - z_0\rangle \\
&\qquad = \lim_{i\to\infty} \langle u - z_0, Sy_{n_i} - z_0\rangle \\
&\qquad = \langle u - z_0, z - z_0\rangle \\
&\qquad \leq 0.
\end{aligned}
$$

Note the fact $z_0 \in F(S) \cap VI(C, A)$ and $(1 - \alpha_n)\lambda_n > 0$. Then, we have

$$
z_0 = P_C[z_0 - (1 - \alpha_n)\lambda_n Az_0] = P_C[\alpha_n z_0 + (1 - \alpha_n)(z_0 - \lambda_n Az_0)].
$$

By the property (ii) of the metric projection, we have

$$\begin{aligned}
&\|y_n - z_0\|^2 \\
=\ & \|P_C[\alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n)] - P_C[\alpha_n z_0 + (1-\alpha_n)(z_0 - \lambda_n A z_0)]\|^2 \\
\le\ & \langle \alpha_n(u - z_0) + (1-\alpha_n)((x_n - \lambda_n A x_n) - (z_0 - \lambda_n A z_0)), y_n - z_0\rangle \\
\le\ & (1-\alpha_n)\|(x_n - \lambda_n A x_n) - (z_0 - \lambda_n A z_0)\|\|y_n - z_0\| + \alpha_n\langle u - z_0, y_n - z_0\rangle \\
\le\ & (1-\alpha_n)\|x_n - z_0\|\|y_n - z_0\| + \alpha_n\langle u - z_0, y_n - z_0\rangle \\
\le\ & \frac{1-\alpha_n}{2}(\|x_n - z_0\|^2 + \|y_n - z_0\|^2) + \alpha_n\langle u - z_0, y_n - z_0\rangle.
\end{aligned}$$

Hence,

$$\|y_n - z_0\|^2 \le (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, y_n - z_0\rangle.$$

Therefore,

$$\begin{aligned}
&\|x_{n+1} - z_0\|^2 \\
\le\ & \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|y_n - z_0\|^2 \\
\le\ & \beta_n\|x_n - z_0\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n(1-\beta_n)\langle u - z_0, y_n - z_0\rangle \\
=\ & [1 - \alpha_n(1-\beta_n)]\|x_n - z_0\|^2 + 2\alpha_n(1-\beta_n)\langle u - z_0, y_n - z_0\rangle \\
=\ & (1-\gamma_n)\|x_n - z_0\|^2 + \delta_n,
\end{aligned}$$

where $\gamma_n = (1-\beta)\alpha_n$ and $\delta_n = (1-\beta)\alpha_n\{2\langle u - z_0, y_n - z_0\rangle\}$. It is easily seen that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and

$$\limsup_{n\to\infty}\delta_n/\gamma_n = \limsup_{n\to\infty}\{2\langle u - z_0, y_n - z_0\rangle\} \le 0.$$

Hence, all conditions of Lemma 2.2 are satisfied. Therefore, we immediately deduce that $x_n \to z_0$ and $y_n \to z_0$. Finally, if we take $u = 0$, then $z_0 = P_{F(S)\cap VI(C,A)}(0)$. This clearly implies that z_0 is a minimum-norm element in $F(S) \cap VI(C,A)$. This completes the proof. ■

Remark 3.6. We note that the algorithm (3.2) is a natural extension of the algorithm (1.4). We observe that the algorithm (3.2) contains two projection operators P_C. However, projection operator P_C is only used once in algorithm (3.5). As far as we know, this appears to be the first time in the literature that the scheme (3.5) is studied.

As direct consequence of Theorem 3.2 and Theorem 3.5, we obtain the following corollaries.

Corollary 3.7. *The sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_C[\alpha_n u + (1-\alpha_n)P_C(x_n - \lambda_n A x_n)], \forall n \ge 0,$$

converges strongly to $P_{VI(C,A)}(u)$.

In particular, if we take $u = 0$*, then the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_C[(1-\alpha_n)P_C(x_n - \lambda_n A x_n)], \forall n \geq 0,$$

converges strongly to $P_{VI(C,A)}(0)$ *which is the minimum norm element in* $VI(C,A)$.

Corollary 3.8. *The sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_C[\alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n)], \forall n \geq 0,$$

converges strongly to $P_{VI(C,A)}(u)$.

In particular, if we take $u = 0$*, then the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_C[(1-\alpha_n)(x_n - \lambda_n A x_n)], \forall n \geq 0, \tag{3.7}$$

converges strongly to $P_{VI(C,A)}(0)$ *which is the minimum norm element in* $VI(C,A)$.

Remark 3.9. We note in the literature, there exists a classical algorithm for solving variational inequality (1.1):

$$x_{n+1} = P_C(x_n - \lambda A x_n), n \geq 0.$$

However the operator A must be strongly monotone and Lipschitz continuous. It is still an open problem: whether or not the strongly monotonicity or Lipschitz continuity of the operator A can be dropped? In this article, we propose a general algorithm (3.7) for solving variational inequality (1.1). We prove the strong convergence of the algorithm (3.7) without the continuity assumption on the operator A.

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