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THE SOLUTION OF 3D-PHOTON TRANSPORT PROBLEM IN INTERSTELLAR CLOUD

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Abstract. In this paper we study a problem for photon transport in a host medium (e.g. an interstellar cloud where a localized source is present), that occupies a compact convex region V in R^3. We find the generalized solution of the photon transport problem by means of the theory of equicontinuous semigroup of bounded linear operators on a sequentially complete locally convex topological vector space.

1. INTRODUCTION

The interstellar medium is mainly composed by molecular gases (mainly hydrogen), by more complex molecules and by grains of ”dust” of silicon and carbonate. It is concentrated in big clouds, whose dimensions are of the order of ten light years. We note that the diameter of the solar system is of the order of parsec.

A simple mathematical model of photon transport in a cloud which assume that the photon transport phenomenon is one-dimensional has been well considered in [5, 6, 11]. In this paper we consider a more complex model that the photon transport problem in a three-dimensional space. Let an interstellar cloud occupy the closed and convex region $V \subset R^3$, bounded by the closed regular surface Σ, and let V_i be the interior of V so that $V = V_i \cup \Sigma$ and V is a compact set. We assume the scattering cross-section $\sigma_s(x)$, the total cross-section $\sigma(x)$, and the photon source $q(x)$ satisfying

$$\begin{aligned} &0 < \sigma_s \equiv \sigma_s(x) < \sigma(x) \equiv \sigma \forall x \in V, \sigma_s(x) = \sigma(x) = 0 \quad \forall x \notin V, \\ &q(x) = q_0 \delta(x - x_0) \quad \forall x \in R^3, \end{aligned} \tag{1.1}$$

where σ_s and σ are constants, δ is the Dirac delta functional, and $x_0 \in R^3$. We denote $N(x, u, t)$ be the photon number density, so that $N(x, u, t_0)\, dxdu$ (t_0 is a

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fixed number) is the expected number of photons in the volume element dx centered at x and having velocity within the solid angle du around the unit vector $u \in S$, where S is the surface of the unit sphere.

Morante, Lamb and Mcbride [1] described the photon transport equation in interstellar space as follows:

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}N(x,u,t) &= -u\cdot\bigtriangledown_x N(x,u,t) - \sigma N(x,u,t)\\ &\quad +\frac{\sigma_s}{4\pi}\int_S N(x,u',t)\,du'\\ &\quad +q(x,t) \quad \forall x\in V\end{aligned} \tag{1.2}$$

where $q(x,t) = q_0\delta(x-x_0)$ and x_0 is a fixed element in V.

This equation accompany with non re-entry boundary condition

$$N(y,u,\cdot) = 0 \text{ if } \ y\in\Sigma \text{ and } u\cdot n(y) < 0, \tag{1.3}$$

where $n(y)$ is the outward normal at $y \in \Sigma$. We denote Σ_1 be the portion of Σ which satisfies (1.3), and denote $\Sigma_2 = \Sigma - \Sigma_1$. Note that the non re-entry boundary condition (1.3) implies that there are no photon sources outside V, and hence $\sigma_s(x) = 0$ if x is at outside of V.

Throughout this paper, we denote $X = L^1(V\times S)$ be the Banach space endow with the norm

$$\|\Psi\|_1 = \int_V\int_S |\Psi(x,u)|\,dudx \text{ for each } \Psi\in X.$$

We define the "free-streaming" operators $A : D(A) \to X$ and $K : X \to X$ by

$$A\Psi(x,u) = -cu\cdot\nabla_x\Psi(x,u),$$

with $D(A) = \{\Psi\in X : -cu\cdot\nabla_x\Psi\in X \text{ and } \Psi(y,u) = 0 \text{ if } \ y\in\Sigma \text{ and } u\cdot n(y) < 0\}$

$$K\Psi(x) = \tfrac{\sigma_s}{4\pi}\int_S\Psi(x,w)\,dw.$$

We also let B and T be the linear operators on X which are defined by

$$B\Psi(x,u) = -c\sigma\Psi(x,u),\, T\Psi = (A+B+K)\Psi \ \text{ for every } \Psi\in X.$$

Under these notations, system (1.2)-(1.3) can be transformed as:

$$\begin{cases}\dfrac{d}{dt}N(t) = T(N(t)) + Q(t), & \forall\, t>0;\\ N(0) = N_0;\end{cases} \tag{1.4}$$

where $N(t) = N(\cdot,\cdot,t)$ is a function from $[0,\infty)$ into X and $Q(t) = q(x,t) = q_0\delta(x - x_0)$. Since the particle density is low, it is reasonable to assume that the number of photons inside the cloud changes slowly in time, i.e.,$\frac{d}{dt}N(t)$ is small. Some people solved equation (1.4) for the special case (see e.g. [1, 2, 11]). This new system

$$\begin{cases} T(N(t)) + Q(t) = 0, \quad \forall\, t > 0; \\ N(0) = N_0; \end{cases} \tag{1.5}$$

is so-call quasi-static equation (moreover, for physical reasons we can assume $Q(t) = Q$ be a constant) and it is a good approximation to equation of system (1.4). However, neither of the initial value problem (1.4) nor (1.5) has a solution in the Banach space $X = L^1(V \times S)$ since δ does not belong to X. For solving (1.4), we have to consider more general space. Let $D(V \times S)$ is the vector space consisted of all continuous functions defined on some open set containing $V \times S$. We also assume that every element in $D(V \times S)$ is vanish outside $V \times S$, and it has continuous partial derivatives of all orders. We will consider the perturbed Cauchy problem in a sequentially complete locally convex space $\widetilde{X} = D'(V \times S)$ (the dual space of $D(V \times S)$) rather than in the Banach space X. We will give further descriptions about these spaces in section 3.

Although system (1.4) is stemming from the photon transport problem, but it is not only limited in this problem. The basis of radionuclide imaging technique in cardiology is the notion of a tracer in the form of a radiopharmaceutical which emits gamma rays and which can be administered to patients usually by intravenous injection. The distribution of the radiopharmaceutical depends on the physical processes within the body, which differ in health and disease. The problem (1.4) also can be considered as a mathematical model for gamma ray in a medium composed of three time-dependent regions (see [3, 4]).

2. Preliminaries

Through out this paper we will use following notations. We always denote Z be a sequentially complete locally convex space (hereafter, we denote it as $sclcs$) under a suitable family of seminorms Γ, and we denote Z' to be the dual space of Z. We also denote $\mathcal{L}(Z)$ be the space of all continuous linear operators on Z. For each $L \in \mathcal{L}(Z)$ and $v \in Z'$, we use (Lx, v) to represent the scalar value $v(Lx)$. We suppose there is a collection of bounded subsets $\digamma$ of Z such that $(\cup_{M\in\digamma} M) = Z$. Under these notations, for each $M \in \digamma$ and N be any equicontinuous subset of Z' there is a seminorm $p^*_{M,N}$ on $\mathcal{L}(Z)$, which is given by

$$p^*_{M,N}(L) = \sup_{x\in M, v\in N} |(Lx, v)|, \quad \text{for every } L \in \mathcal{L}(Z). \tag{2.1}$$

It is well known that the family $\Gamma^* = \left\{ p^*_{M,N} : M \in \digamma,\ N \text{ is equicontinuous in } Z' \right\}$ induces a locally convex topology for $\mathcal{L}(Z)$(see e.g. [10], p. 131). If $\digamma$ is a collection of finite subsets of X, then the topology induced by Γ^* is called the simple topology. We denote the space $\mathcal{L}(Z)$ with this topology by $\mathcal{L}_s(Z)$. If $\digamma$ is a collection of all bounded subsets of X, then the topology induced by Γ^* is called the bounded convergence topology. We denote the space $\mathcal{L}(Z)$ with this topology by $\mathcal{L}_b(Z)$.

We say a family $\Im$ of linear operators on X is equicontinuous if for each $p \in \Gamma$, there is a continuous seminorm $q = q(p) \in \Gamma$ such that $p\,(Lx) \le q\,(x)$, for all $L \in \Im$ and $x \in X$. For each $p \in \Gamma$ and L be a linear operator on Z, we define a corresponding seminorm for the linear operator L by

$$\widehat{p}\,(L) = \sup\left\{p\,(Lx) : x \in X \text{ with } p\,(x) \le 1\right\} \tag{2.2}$$

(2.2) is identical with (2.1), if we take the seminorm $p(x) = \sup_{x \in M, v \in N} |(x, v)|$ in (2.2).

A linear operator L on Z is said to be p-continuous if

$$\widehat{p}\,(L) = \sup\left\{p\,(Lx) :\ x \in X \text{ with } p\,(x) \le 1\right\} < \infty.$$

The family seminorm $\widehat{\Gamma} = \{\widehat{p} : \widehat{p} < \infty, p \in \Gamma\}$ on $\mathcal{L}(Z)$ is actually a locally m-convex algebra. A linear operator $L \in \mathcal{L}(Z)$ is said to be Γ-continuous if it is p-continuous for every $p \in \Gamma$. Let $\mathcal{L}_\Gamma(Z)$ denote the space of all Γ-continuous linear operators on Z and denote $\mathcal{B}_\Gamma(X)$ the subspace of $\mathcal{L}_\Gamma(Z)$ whose elements L satisfies

$$\|L\|_{\widehat{\Gamma}} = \sup\left\{\widehat{p}\,(L) : p \in \Gamma\right\} < \infty. \tag{2.3}$$

$\mathcal{B}_\Gamma(X)$ with norm $\|.\|_{\widehat{\Gamma}}$ is a Banach algebra. Under these notations, one may have the relation

$$\mathcal{B}_\Gamma(X) \subset \mathcal{L}_\Gamma(X) \subset \mathcal{L}(X).$$

As long as $K \in \mathcal{B}_\Gamma(X)$, we can define the operator e^{tK} by

$$e^{tK} = \sum_{i=0}^{\infty} \frac{t^i}{i!} K^i \text{ for each } t > 0 \text{ and } e^{0K} = I \text{ for } t = 0.$$

By similar way, we can define a topology on Z'. The simple topology ($\sigma\,(Z', Z)$-topology) of Z' (denote by Z'_s) is defined by the family of seminorms of the form

$$p\left(x'\right) = p\left(x'; x_1, x_{2,\ldots,}x_n\right) = \sup_{1 \le j \le n} \left|\left(x_j, x'\right)\right|,$$

where $x_1, x_2, \ldots, x_n$ are an arbitrary finite system of elements of Z.

The bounded convergence topology ($\beta(Z', Z)$- topology) of Z' (denote by Z'_b) is defined by the family of seminorms of the form

$$p(x') = p(x'; F) = \sup_{x \in F} \left|(x, x')\right|,$$

where F is an arbitrary bounded subset of Z. (2.2) also can define the seminorms on $\mathcal{L}(Z'_s)$ or $\mathcal{L}(Z'_b)$ if Z is seen as the subspace of Z''. In general, $\mathcal{L}(Z'_s) \subset \mathcal{L}(Z'_b)$.

Let A' be the adjoint operator of $A \in \mathcal{L}(Z_s)$. The seminorm $\widetilde{p}_{N,M}$ on $\mathcal{L}(Z')$ is defined by

$$\widetilde{p}_{N,M}(A') = \sup_{v \in N, x \in M} \left|(x, A'v)\right|, \text{ for every } A' \in \mathcal{L}(Z').$$

Then the family $\widetilde{\Gamma} = \{\widetilde{p}_{N,M}; M \in F,\ N$ is equicontinuous in $Z'\}$ induces a locally convex topology for $\mathcal{L}(Z')$. Moreover, we have

$$\widetilde{p}_{N,M}(A') = \widehat{p}_{M,N}(A).$$

This show that if $A \in \mathcal{L}_{\widehat{\Gamma}}(z)$ implies $A' \in \mathcal{L}_{\widetilde{\Gamma}}(Z')$. Furthermore, if $A \in \mathcal{B}_{\widehat{\Gamma}}(Z)$, then $A' \in \mathcal{B}_{\widetilde{\Gamma}}(Z')$ and $\|A\|_{\widehat{\Gamma}} = \|A'\|_{\widetilde{\Gamma}}$ (please see [9, p135]).

Definition 1. Let Z be a $sclcs$. The family of continuous linear operators $\{T(t)\}_{t \geq 0}$ on Z is called a strongly continuous C_0-semigroup (abbreviated as C_0-semigroup) if following three conditions hold:

(1) $T(0) = I$,

(2) $T(t)T(s) = T(t+s)$ for all $s, t \geq 0$, and

(3) $T(t)x \longrightarrow x$ as $t \downarrow 0$, for every $x \in X$.

The C_0-semigroup $\{T(t)\}_{t \geq 0}$ is said to be equicontinuous if for each continuous seminorm p on X, there exists a continuous seminorm q on Z such that $p(T(t)x) \leq q(x)$ for all $t \geq 0$ and all $x \in Z$. Moreover, if there exists a number $\beta \geq 0$ such that $\left\{e^{-\beta t}T(t)\right\}_{t \geq 0}$ is an equicontinuous C_0-semigroup, then it is called a quasi-equicontinuous C_0-semigroup.

Definition 2. Let $\beta \geq 0$, $M \geq 1$ be given and let Γ be a calibration (seminorm family) for Z. We will denote the set of all densely defined linear operators A satisfying the condition

$$\left\|(\lambda - A)^{-k}\right\|_{\Gamma} \leq M(\lambda - \beta)^{-k}, \lambda > \beta, k = 0, 1, 2, ... \tag{2.4}$$

by $G(X, \Gamma, M, \beta)$. We also write

$$G(X, \Gamma, M) = \cup_{\beta \geq 0} G(X, \Gamma, M, \beta) \text{ and } G(X, \Gamma) = \cup_{M \geq 1} G(X, \Gamma, M).$$

Lemma 1. ([7, p. 307, Corollary 4.4]). *Let Z be a sclcs space. A linear operator A in Z is the generator of a quasi-equicontinuous C_0-semigroup $\{Q(t)\}_{t\geq 0}$, that is, is an equicontinuous for some $\beta \geq 0$, if and only if*

- (a) *the domain $D(A)$ of A is dense in Z, and*
- (b) *the resolvent $(\lambda I - A)^{-1}$ exists for all $\lambda \geq \beta$ and there is a calibration Γ such that*

$$\left\| (\lambda - A)^{-1} \right\|_{\Gamma} \leq (\lambda - \beta)^{-1}. \tag{2.5}$$

Follows from Lemma 1 if $\{Q(t)\}_{t\geq 0}$ is a quasi-equicontinuous C_0-semigroup with the generator A, then there is a calibration such that A satisfies (2.5).

Main Theorem 1. Let $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(Z)$ be a quasi-equicontinuous C_0-semigroup with the generator A, and let Z^+ denote the closure of the domain $D(A')$ in the topology induced by $\widetilde{\Gamma}$, where A' is the adjoint operator of A. If $T^+(t)$ be the restriction of $T(t)$ to Z^+, then $\{T^+(t)\}_{t\geq 0} \subset \mathcal{L}(Z^+)$ and $\{T^+(t)\}_{t\geq 0}$is a C_0-semigroup with the generator A^+ which is the largest restriction of A' with domain and range in Z^+.

Proof. From Lemma 1, there is a calibration Γ such that A satisfies (2.5). If the resolvent $(\lambda I - A)^{-1}$ exist for some λ, then the operator $(\lambda I' - A')^{-1}$ exists and

$$\left(\lambda' I' - A'\right)^{-1} = \left((\lambda I - A)^{-1}\right)' \text{ (pleasesee[12], p.273.proposition2).}$$

This implies that

$$\left\| \left(\lambda' I' - A'\right)^{-1} \right\|_{\widetilde{\Gamma}} = \left\| \left((\lambda I - A)^{-1}\right)' \right\|_{\widetilde{\Gamma}} = \left\| \left((\lambda I - A)^{-1}\right) \right\|_{\widehat{\Gamma}} \leq (\lambda - \beta)^{-1} \text{ in } Z^+.$$

Follows from Lemma 1, there exists a quasi-equicontinuous C_0-semigroup $\{T^+(t)\}_{t\geq 0}$ generated by A^+ in Z^+.

For any positive integer m, $x \in Z$ and $y' \in Z^+$, we have

$$\left(\left(I - m^{-1}tA\right)^{-m} x, y'\right) = \left(x, \left(I - m^{-1}tA^+\right)^{-m} y'\right).$$

And so we obtain, by letting $m \to \infty$, the equality $(T(t)x, y') = (x, T^+(t)y')$. Hence $T'(t)y' = T^+(t)y'$ that is, $T^+(t)$ is the restriction to Z^+ of $T'(t)$.

Suppose $x' \in D(A')$ and $x' \in Z^+$, $A'x' \in Z^+$, then $(\lambda I' - A')x' \in Z^+$ and hence

$$\left(\lambda I' - A^+\right)^{-1} \left(\lambda I' - A'\right) x' = x'.$$

Applying $(\lambda I - A^{+})$ from the left on both side, we obtain $A'x' = A^{+}x'$. This implies that A^{+} is the largest restriction of A' with domain as well as range in Z^{+}.

For solving (1.6) and (1.7) we need one more Lemma, which was proved in [9].

Lemma 2. ([9] Corollary p. 212). *Let Z be a barreled locally convex space. Then the following collections of subsets of its dual $Z\,'$ are identical:*

(a) *the equicontinuous sets,*

(b) *the $\beta\,(Z', Z)$-bounded sets.*

(c) *the $\sigma\,(Z', Z)$-bounded sets.*

3. Generalized Solution of Photon Transport Problem

Now we are able to consider the photon transport problem (1.4). We will show that this problem has a unique generalized (or weak) solution U' in the space $\widetilde{X}\,(= D'\,(V \times S))$. For describing the space $\widetilde{X}$, we will use following notations. Let the Schwartz space $D\,(V \times S)$ consisting of those C^{∞} functions which, together with all their derivatives, vanish outside of $V \times S$. $D\,(V \times S)$ is a Frechet space [9, Example 10 ,p. 90] with the calibration of seminorms $\Gamma = \{q_{\alpha}\}$ for any multi-index such that

$$q_{\alpha}\,(\psi) = \sup_{(x,u)\in V\times S} \left|(\partial^{\alpha}\psi)\,(x,u)\right| \ , \ \psi \in \ D\,(V \times S) \tag{3.1}$$

where $x \in V$, $u \in S$, $|(x,u)| = \left(|x|^2 + |u|^2\right)^{\frac{1}{2}}$ and

$$\begin{aligned} \partial^{\alpha} &= \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_6^{\alpha_6}\left(\partial_i = \frac{\partial}{\partial x_i}\right) \\ &= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_6^{\alpha_6}}\left(|\alpha| = \sum_{i=1}^{6}\alpha_i\right) \end{aligned}$$

Since every Frechet space is a barrel space, this implies $D\,(V \times S)$ is a barrel space. Let $\widetilde{X}\,(= D'\,(V \times S))$ be the dual space of $D\,(V \times S)$.

It can be shown that (see e.g. [8]) $X = L^1\,(V \times S)$ embed in $\widetilde{X}$ in the following sense.

We say that $f \in D'\,(V \times S)$ can be identified with $f \in L^1\,(V \times S)$ if,

$$(\phi, f) = \int_V \int_S f\,(x,u)\,\phi\,(x,u)\,dudx \ \forall\phi \in D\,(V \times S)\,.$$

and denote $\widetilde{f}$ be the element embed with f in $\widetilde{X}$.

We also can appropriately extend the operator

$$T\Psi(x,u) = -c\sigma\Psi(x,u) - cu\cdot\nabla_x\Psi(x,u) + \frac{\sigma_s}{4\pi}\int_S \Psi(x,w)\,dw,$$

with domain

$$\begin{aligned} D(T) = \{&\Psi\in X : -cu\cdot\nabla_x\Psi\in X \\ &\text{and it satisfies the non-entry boundary condition}\}\end{aligned}$$

to the operator $\widetilde{T}:\widetilde{X}\to\widetilde{X}$. In an analogous way, give $Tf\in X$, it is not hard to see

$$\left(\phi,\widetilde{T}\,\widetilde{f}\right) = \left(\phi,\widetilde{Tf}\right) = \textstyle\int_V\int_S (Tf)(x,u)\,\phi(x,u)\,dudx \quad \forall\ \phi\in D(V\times S).$$

That is $\widetilde{T}$ agrees with T on X and $\widetilde{T}\,\widetilde{f}\in D'(V\times S)$ coincides with $\widetilde{Tf}$ for all $f\in X$.

By change the order of integration and apply Green's theorem then we have the following relation

$$\begin{aligned}
\left(\phi,\widetilde{T}\,\widetilde{f}\right) &= \int_V\int_S (Tf)(x,u)\,\phi(x,u)\,dudx \\
&= \int_V\int_S -cu\cdot\nabla_x f(x,u)\,\phi(x,u)\,dudx \\
&\quad + \int_V\int_S -c\sigma f(x,u)\,\phi(x,u)\,dudx \\
&\quad + \frac{\sigma_s}{4\pi}\int_V\int_S\left(\int_S f(x,w)\,dw\right)\phi(x,u)\,dudx \\
&= \int_S \left(-c\left(f\phi\mid_{\Sigma}\right)\right)du + \int_V\int_S cu\cdot(\nabla_x\phi)\,f dudx \\
&\quad + \int_V\int_S -c\sigma f(x,u)\,\phi(x,u)\,dudx \\
&\quad + \frac{\sigma_s}{4\pi}\int_V\int_S\left(\int_S f(x,w)\,dw\right)\phi(x,u)\,dudx \\
&= \int_S \left(-c\left(f\phi\mid_{\Sigma_2} + f\phi\mid_{\Sigma_1}\right)\right)du \\
&\quad + \int_V\int_S cu\cdot(\nabla_x\phi)\,f dudx \\
&\quad + \int_V\int_S -c\sigma f(x,u)\,\phi(x,u)\,dudx \\
&\quad + \frac{\sigma_s}{4\pi}\int_V\int_S\left(\int_S f(x,w)\,dw\right)\phi(x,u)\,dudx
\end{aligned} \tag{3.2}$$

It is obvious that $f \in D(A)$ implies $f\phi\mid_{\Sigma_1} = 0$. If $\phi(y,u) = 0$ for $y \in \Sigma_2$, then $\int_S (-c(f\phi\mid_{\Sigma_2} - f\phi\mid_{\Sigma_1}))\,du = 0$, and hence (3.2) becomes

$$
\begin{aligned}
\left(\phi, \widetilde{T}\,\widetilde{f}\right) &= \int_V \int_S cu \cdot (\nabla_x \phi)\, f du dx + \int_V \int_S -c\sigma f(x,u)\,\phi(x,u)\,du dx \\
&\quad + \frac{\sigma_s}{4\pi} \int_V \int_S \left(\int_S f(x,w)\,dw \right) \phi(x,u)\,du dx.
\end{aligned}
\tag{3.3}
$$

Now we can define the operator $\widehat{T}$ on $D(V \times S)$ by

$$
\widehat{T}\phi(x,u) = -c\sigma\phi(x,u) + cu \cdot \nabla_x \phi(x,u) + \tfrac{\sigma_s}{4\pi} \int_S \phi(x,w)\,dw
$$

which with domain

$$
D\left(\widehat{T}\right) = \{\phi \in D : cu \cdot \nabla_x \phi \in D(V \times S) \text{ and } \phi(y,u) = 0 \text{ for } y \in \Sigma_2\}.
$$

Under this definition, it is easy to see that $\widehat{T}$ is a linear operator from $D(V \times S)$ into itself, and we can get the equation (3.3) from (3.2). We also have the relation

$$
\left(\phi, \widetilde{T}\,\widetilde{f}\right) = \left(\widehat{T}\phi, \widetilde{f}\,\right) \quad \text{for all } \widetilde{f} \in D(\widetilde{T}),\ \phi \in D\left(\widehat{T}\right).
$$

In other word, $\widetilde{T}$ is the formal adjoint of $\widehat{T}$.

Meri Lisi and Silvia Totaro [11] showed that there exists a Γ-contraction C_0-semigroup $\{W(t)\}_{t \geq 0}$ generated by $cu\frac{\partial}{\partial x}$. With a similar method used in [7, example 4.2] and following calculation, we get the conclusion that there exists a Γ-contraction C_0-semigroup on $D(V \times S)$ generated by $\widehat{A} = cu \cdot \nabla_x$.

Let $\{W(t)\}_{t \geq 0}$ be the semigroup on $D(V \times S)$ defined by

$$
W(t)\,\Psi(x,u) = \Psi(x + cut, u)\,\chi_V(x + cut),\ t \geq 0,\ \Psi \in D(V \times S),\ u \in S.
$$

The topology of X is also induced by seminoms $\Xi = \{q_\alpha\}$ for any multi-index α, which is defined by (3.1).

Since

$$
\begin{aligned}
q_\alpha(W(t)\,\psi) &= \sup_{(x,u) \in V \times S} |(\partial^\alpha W(t)\,\psi)(x,u)| \\
&= \sup_{(x,u) \in V \times S} |\partial^\alpha \psi(x + cut, u)\,\chi_V(x + cut)| \\
&= \sup_{(x,u) \in V \times S} |\partial^\alpha \psi(x,u)\,\chi_V(x + cut)| = q_\alpha(\psi) \ \ forall \alpha.
\end{aligned}
$$

Follows from (2.2) we have

$$
\|W(t)\|_{\widehat{\Xi}} \leq 1\ .
$$

This implies that $\{W(t)\}_{t\geq 0}$ is a Ξ-contraction C_0-semigroup. The generator of $\{W(t)\}_{t\geq 0}$ is the operator $cu \cdot \nabla_x$.

Let the operators $\widehat{B}$ and $\widehat{K}$ on the space $D(V \times S)$ be defined by

$$\widehat{B}\phi(x,u) = -c\sigma\phi(x,u)$$

and

$$\widehat{K}\phi(x) = \tfrac{\sigma_s}{4\pi}\int_S \phi(x,w)\,dw \text{ for every } \phi \in D.$$

Since

$$\sup\left\{q_\alpha\left(\widehat{K}\phi\right) : \phi \in D,\ q_\alpha(\phi) \leq 1\right\} \leq c\sigma_s\ ,$$

we have $\left\|\widehat{K}\right\|_{\widehat{\Xi}} \leq c\sigma_s$, and $\left\|\widehat{B}\right\|_{\widehat{\Xi}} \leq c\sigma$. Follows from general perturbation theorem (see e.g. [7]), there exists a locally equicontinuous C_0-semigroup $\{V(t)\}_{t\geq 0}$ on $D(V \times S)$ generated by $\widehat{T}$. The Theorem implies that there exist a quasi-equicontinuous C_0-semigroup $\{V'(t)\}_{t\geq 0}$ on $D'(V \times S)$ generated by $\widetilde{T}$.

Remark. The topology on $D'(V \times S)$ determined by $D'_s(V \times S)$ is the same as the topology determined by $D'_b(V \times S)$. Since $D(V \times S)$ is a Montal space [9, Example 4, p.240], $\widetilde{X} = D'(V \times S)$, the dual space of $D(V \times S)$, is also a Montal space [9, proposition 9, p.236]. Since every Montal space is a barrel space and it is reflexive [9, p.231], this implies $D(V \times S)$ is a barrel space and $D''(V \times S) = D(V \times S)$. The seminorms $\widetilde{\Gamma} = \{\widetilde{p}_{N,M} : M \in F,\ N \text{ is equicontinuous in } X'\}$ on $D'_s(V \times S)$ is the same as on $D'_b(V \times S)$, since according Lemma 2, N is a equicontinuous set in $D'(V \times S)$ which is identical with the $\beta(D', D)$-bounded set, and hence identical with the $\sigma(D', D)$-bounded set. This implies $A' \in L_{\widetilde{\Gamma}}(D'_b(V \times S))$ if $A' \in L_{\widetilde{\Gamma}}(D'_s(V \times S))$.

Now we can consider the photon transport problem (1.4) on $D'(V \times S)$. Since $Q(t) \equiv q_0\delta(x - x_0) \in D'(V \times S)$ is integrable for $t \in [0, l]$, where l is a finite number, then (1.4) has a unique generalized solution U' which is given by

$$U'(t) = V'(t)\,N_0 + \int_0^t V'(t-s)\,Q(s)\,ds.$$

We also can consider the quasi-static equation (1.5) on $D'(V \times S)$. Since $\|W(t)\|_{\widehat{\Xi}} \leq 1$, where $\{W(t)\}_{t\geq 0}$ is the Ξ-contraction C_0-semigroup generated by $\widehat{A} = cu\cdot\nabla_x$ on $D(V \times S)$ and $\left\|\widehat{K}\right\|_{\widehat{\Xi}} \leq c\sigma_s$ this implies $\widehat{A}+\widehat{K} \in G(D(V \times S), \Gamma, 1, c\sigma_s)$ and $\left[\lambda I - \left(\widehat{A} + \widehat{K}\right)\right]^{-1}$ exist on $D(V \times S)$ for any $\lambda > c\sigma_s$. In particular, if we choose $\lambda = c\sigma$, then from (1.1), we have the existence of $\left[c\sigma I - \left(\widehat{A}+\widehat{K}\right)\right]^{-1} =$

$\left(-\widehat{T}\right)^{-1}$ and $\left(\widetilde{T}\right)^{-1} = \left(\left(\widehat{T}\right)^{-1}\right)'$. This shows that (1.5) has a quasi-static solution, which is given by

$$U' = \left(\widetilde{T}\right)^{-1}(-Q) \text{ on } D'(V \times S).$$

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