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LINEAR WEINGARTEN SURFACES FOLIATED BY CIRCLES IN MINKOWSKI SPACE

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Abstract. In this work, we study spacelike surfaces in Minkowski space $\mathbf{E}_1^3$ foliated by pieces of circles that satisfy a linear Weingarten condition of type $aH + bK = c$, where a, b and c are constants and H and K denote the mean curvature and the Gauss curvature respectively. We show that such surfaces must be surfaces of revolution or surfaces with constant mean curvature $H = 0$ or surfaces with constant Gauss curvature $K = 0$.

1. Introduction and Results

Let $\mathbf{E}_1^3$ be the Minkowski three-dimensional space, that is, the real vector space $\mathbb{R}^3$ endowed with the scalar product $\langle , \rangle = (dx_1)^2 + (dx_2)^2 - (dx_3)^2$, where (x_1, x_2, x_3) denote the usual coordinates in $\mathbb{R}^3$. An immersion $x : M \to \mathbf{E}_1^3$ of a surface M is called *spacelike* if the induced metric $x^*\langle , \rangle$ on M is a Riemannian metric. In this paper, we study spacelike surfaces that satisfy a relation of type

$$aH + bK = c, \qquad (1)$$

where H and K are the mean curvature and the Gauss curvature of M respectively, and a, b and c are constants with $a^2 + b^2 \neq 0$. In such case we say that M is a *linear Weingarten surface*. This class of surfaces includes the surfaces with constant mean curvature ($b = 0$ in (1)) and the surfaces with constant Gauss curvature ($a = 0$ in (1)). In Euclidean space, there is a great amount of literature on Weingarten surfaces, beginning with works of Chern, Hartman, Winter and Hopf in the fifties. More recently and focusing in Lorenzian spaces, we refer [1, 2, 3, 5, 8, 16], without being a complete bibliography.

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In order to look for examples of linear Weingarten spacelike surfaces in $\mathbf{E}_1^3$, it is natural to assume some hypothesis about the geometry of the surface. A simple condition is that the surface is rotational. In such case, Equation (1) is an ODE of second order given in terms of the generating curve of M. In a more general scene, we consider surfaces constructed by a foliation of circles.

Definition 1.1. A cyclic surface in Minkowski space $\mathbf{E}_1^3$ is a surface determined by a smooth uniparametric family of circles.

We also say that the surface is foliated by circles. As in Euclidean space, by a circle in $\mathbf{E}_1^3$ we mean a planar curve with constant curvature. Since each circle is included in a plane, given a cyclic surface, there exists a uniparametric family of planes of $\mathbf{E}_1^3$ whose intersection with M is the set of circles that defines the surface. Because the circles are contained in a spacelike surface, each circle of the foliation must be a spacelike curve. However, the planes containing the circles can be of any causal type.

Our work is motivated by the following fact. In Minkowski space $\mathbf{E}_1^3$ there are cyclic spacelike surfaces with $H=0$ (or $K=0$) that are not rotational surfaces. For the maximal case ($H=0$) these surfaces are foliated by circles in parallel planes and they represent in Minkowski ambient the same role as the classical Riemann examples of minimal surfaces in Euclidean space. These surfaces appeared for the first time in the literature in [12] and they have been origin of an extensive study in recent years: see for example [4, 6, 7, 9, 11]. In the same sense, non-rotational cyclic surfaces with constant Gauss curvature $K=0$ appeared in [14]. Figure 1 exhibits both types of surfaces in the case that the planes of the foliation are spacelike planes. See also Remark 3.1. Besides these examples, it is natural to ask if there exist other cyclic surfaces in the family of linear Weingarten surfaces of $\mathbf{E}_1^3$. If we compare with what happens in Euclidean space, the difficulty in $\mathbf{E}_1^3$ is the

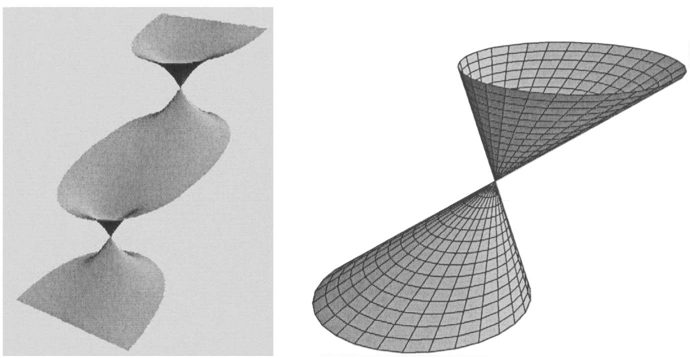

Fig. 1. Examples of non-rotational spacelike surfaces in $\mathbf{E}_1^3$ foliated by circles in parallel spacelike planes: $H=0$ (left) and $K=0$ (right).

variety of possible cases that can appear since the plane containing the circle can be of spacelike, timelike or lightlike type.

In the case that the planes of the foliation are parallel, we prove:

Theorem 1.1. *Let M be a spacelike cyclic surface in $\mathbf{E}_1^3$ and we assume that the circles of the foliation lie in parallel planes. If M is a linear Weingarten surface, then M is a surface of revolution, or $H = 0$ or $K = 0$.*

In Minkowski space $\mathbf{E}_1^3$ there are spacelike surfaces that play the same role as spheres in Euclidean space. These surfaces are the pseudohyperbolic surfaces. After an isometry of $\mathbf{E}_1^3$, a pseudohyperbolic surface of radius $r > 0$ and centered at $p \in \mathbf{E}_1^3$ is given by

$$\mathbf{H}^{2,1}(r,p) = \{x \in \mathbf{E}_1^3; \langle x-p, x-p\rangle = -r^2\}.$$

From the Euclidean viewpoint, and if p is the origin of coordinates, $\mathbf{H}^{2,1}(r,p)$ is the hyperboloid of two sheets $x_1^2 + x_2^2 - x_3^2 = -r^2$ which is obtained by rotating the hyperbola $\{x_1^2 - x_3^2 = r^2, x_2 = 0\}$ with respect to the x_3-axis. This surface is spacelike with constant mean curvature $H = 1/r$ and with constant Gauss curvature $K = 1/r^2$. In particular, $\mathbf{H}^{2,1}(r,p)$ is a linear Weingarten surface: exactly, there are many choices of constants a, b and c that satisfy (1). Although this surface is rotational, any uniparametric family of (non-parallel) planes intersects $\mathbf{H}^{2,1}(r,p)$ in circles. Taking account this fact about the pseudohyperbolic surfaces, our next result establishes:

Theorem 1.2. *Let M be a spacelike cyclic surface in $\mathbf{E}_1^3$. If M is a linear Weingarten surface, then M is a pseudohyperbolic surface or the planes of the foliation are parallel.*

As consequence of the above two theorems, we conclude

Corollary 1. *The only non-rotational spacelike cyclic surfaces that are linear Weingarten surfaces are the Riemann examples of maximal surfaces [12] and a family of surfaces with $K = 0$ described in [14].*

The proofs of Theorems 1.1 and 1.2 involve long algebraic computations that have been possible check by using a symbolic program such as Mathematica.

Finally, we point out that Theorems 1.1 and 1.2 hold for linear Weingarten cyclic *timelike* surfaces of $\mathbf{E}_1^3$. The proofs are similar and we do not included them in the present paper, although they can easily carried. In fact, a key in the proofs is that the induced metric on M is non-degenerate and so, it could be Riemannian (M is spacelike) of Lorentzian (M is timelike).

2. Preliminaries

A vector $v \in \mathbf{E}^3_1$ is said spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$. A plane $P \subset \mathbf{E}^3_1$ is said spacelike, timelike or lightlike if the induced metric on P is a Riemannian metric (positive definite), a Lorentzian metric (a metric of index 1) or a degenerated metric, respectively. This is equivalent that any orthogonal vector to P is timelike, spacelike or lightlike respectively.

Consider $\alpha : I \subset \mathbb{R} \to \mathbf{E}^3_1$ a parametrized regular curve in $\mathbf{E}^3_1$. We say that α is spacelike if $\alpha'(t)$ is a spacelike vector for all $t \in I$. It is possible to reparametrize α by the arc-length, that is by a parameter s such that $\langle \alpha'(s), \alpha'(s)\rangle = 1$ for any $s \in I$. Then one defines a Frenet trihedron at each point and whose differentiation allows to define the curvature κ and the torsion τ of α. See [10, 15]. Motivated by what happens in Euclidean ambient, we give the following definition:

Definition 2.2. A spacelike circle in Minkowski space $\mathbf{E}^3_1$ is a planar spacelike curve with constant curvature.

We describe the spacelike circles in $\mathbf{E}^3_1$. The classification depends on the causal character of the plane P containing the circle. After an isometry of the ambient space $\mathbf{E}^3_1$, a circle parametrizes as follows:

1. If P is the horizonal plane $x_3 = 0$, the circle is given by
$$\alpha(s) = r\Big(\cos(\frac{s}{r}), \sin(\frac{s}{r}), 0\Big), \quad r > 0.$$
In this case, the curve is a Euclidean horizontal circle of radius r.
2. If P is the vertical plane $x_1 = 0$, then
$$\alpha(s) = r\Big(0, \sinh(\frac{s}{r}), \cosh(\frac{s}{r})\Big), \quad r > 0.$$
The curve describes the hyperbola $x_2^2 - x_2^2 = r^2$ in a vertical plane.
3. If P is the plane $x_2 - x_3 = 0$, then
$$\alpha(s) = \Big(s, r\frac{s^2}{2}, r\frac{s^2}{2}\Big), \quad r > 0.$$
The curve is a parabola in P.

A surface M in $\mathbf{E}^3_1$ is a surface of revolution (or rotational surface) if there exists a straight line l such that M is invariant by the rotations that leave l pointwise fixed. In particular, a rotational surface in $\mathbf{E}^3_1$ is formed by a uniparametric family of circles of $\mathbf{E}^3_1$ in parallel planes.

We end this section with local formula for the mean curvature and the Gauss curvature of a spacelike surface. Given a spacelike surface M in $\mathbf{E}^3_1$, the spacelike condition is equivalent that any unit normal vector $\mathbf{G}$ to M is timelike. Since any two timelike vectors in $\mathbf{E}_1^3$ can not be orthogonal, we have $\langle \mathbf{G}, (0,0,1)\rangle \neq 0$ on the whole of M. This shows that M is an orientable surface. As in Euclidean space, one define the mean curvature H and the Gauss curvature of K of M as:

$$H = \frac{1}{2}\,\text{trace}(d\mathbf{G}), \qquad K = \det\,(-d\mathbf{G}).$$

Let $\mathbf{X} = \mathbf{X}(u,v)$ be a parametrization of M. Then the following formulae are well-known [17]:

$$H = \frac{1}{2}\,\frac{eG - 2fF + gE}{EG - F^2}, \qquad K = \frac{e\,g - f^2}{EG - F^2},$$

where $E = \langle \mathbf{X}_u, \mathbf{X}_u\rangle$, $F = \langle \mathbf{X}_u, \mathbf{X}_v\rangle$ and $G = \langle \mathbf{X}_v, \mathbf{X}_v\rangle$ are the coefficients of the first and $e = \langle \mathbf{G}, \mathbf{X}_{uu}\rangle$, $f = \langle \mathbf{G}, \mathbf{X}_{uv}\rangle$ and $g = \langle \mathbf{G}, \mathbf{X}_{vv}\rangle$ the coefficients of the second fundamental form. Moreover, $W =: EG - F^2$ is a positive function because M is spacelike. From the expressions of H and K, we have

$$G[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}] - 2F[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}] + E[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}] = 2HW^{3/2}$$

$$[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}][\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}] - [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}]^2 = KW^2$$

where $[,,]$ denotes the determinant of three vectors: $[v_1, v_2, v_3] = \det(v_1, v_2, v_3)$.

3. Proof of Theorem 1.1

We consider a spacelike surface $M \subset \mathbf{E}_1^3$ parametrized by circles in parallel planes. We discard the trivial cases $a = 0$ or $b = 0$ in (1): in such case, K or H is a constant function and we know that Theorem 1.1 is true [13, 14]. We distinguish three cases according to the causal character of the planes of the foliation.

3.1. The planes are spacelike

After a rigid motion in $\mathbf{E}_1^3$, we assume the planes are parallel to the plane $x_3 = 0$. Then the circles are horizontal Euclidean circles and M parametrizes as

$$\mathbf{X}(u,v) = (f(u), g(u), u) + r(u)(\cos v, \sin v, 0), \tag{2}$$

where $f, g, r > 0$ are smooth functions in some u-interval I. With this parametrization, M is a surface of revolution if and only if f and g are constant functions.

The Weingarten relation $aH + bK = c$ writes as:

$$a\,\frac{G[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}] - 2F[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}] + E[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}]}{2W^{3/2}} \tag{3}$$

$$+b\,\frac{[\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}][\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}] - [\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}]^2}{W^2} = c. \tag{4}$$

3.1.1. Case $c = 0$

Equation (3) writes as

$$a^2\left(G[\mathbf{X}_u,\mathbf{X}_v,\mathbf{X}_{uu}]-2F[\mathbf{X}_u,\mathbf{X}_v,\mathbf{X}_{uv}]+E[\mathbf{X}_u,\mathbf{X}_v,\mathbf{X}_{vv}]\right)^2W^2$$
$$-4b^2\left([\mathbf{X}_u,\mathbf{X}_v,\mathbf{X}_{uu}][\mathbf{X}_u,\mathbf{X}_v,\mathbf{X}_{vv}]-[\mathbf{X}_u,\mathbf{X}_v,\mathbf{X}_{uv}]^2\right)^2=0.$$

Without loss of generality, we assume $4b^2 = 1$. If we compute the above equation the parametrization given in (2), we obtain an expression

$$\sum_{j=0}^{4} A_j(u)\cos(jv)+B_j(u)\sin(jv)=0. \tag{5}$$

Then the functions A_j and B_j must vanish on I. By contradiction, we assume that M is not rotational. Then f' or g' does not vanish in some interval.

1. We consider the cases that one of the functions f or g is constant. For simplicity we consider $f' = 0$ in some interval. Then $g' \neq 0$. The coefficient A_4 writes as

$$A_4=\frac{1}{8}a^2r^6g'^2(rg''-2r'g')^2.$$

As $g' \neq 0$, we have that $rg'' - 2r'g' = 0$. Then $g' = \lambda r^2$ for some positive constant $\lambda \neq 0$. Now

$$A_2=\frac{1}{2}\lambda^2r^8(4r'^2-a^2r^2A^2),\quad B_1=2\lambda r^7r'(a^2rA^2-2r'')$$

where $A = -1 + \lambda^2r^4 + r'^2 - rr''$. From Equation $B_1 = 0$, either $r' = 0$ or $a^2rA^2 - 2r'' = 0$. If $r' = 0$ in some interval, r is constant and a computation of the coefficient E of the first fundamental form gives $E = 0$. Since this is not possible, $r' \neq 0$. A combination of $A_2 = 0$ and $B_1 = 0$ leads to that function r satisfies $2r'^2 - rr'' = 0$. The solution is

$$r(u)=\frac{c_2}{u+c_1},\quad c_1,c_2\in\mathbb{R}.$$

Now $A_2 = 0$ gives a polynomial equation on u given by

$$-4(u+c_1)^6+a^2\Big((u+c_1)^4+c_2^2-\lambda^2c_2^4\Big)^2=0.$$

In particular, the leading coefficient a^2 must vanish: contradiction. This means that the assumption that f is constant is impossible.

2. We assume that both f and g are not constant functions. Then $f', g' \neq 0$. The coefficient B_4 yields:

$$(-4f'g'r'+rf'f''+rf'g'')(-2f'^2r'+2g'^2r'+rf'f''-rf'g'')=0.$$

We distinguish two cases:

1. Assume $-4f'g'r' + rg'f'' + rf'g'' = 0$. Then

$$f'' = \frac{4f'g'r' - rf'g''}{rg'}.$$

Now $A_4 = 0$ gives $a^2r^6(f'^2 + g'^2)^2(-2g'r' + rg'')^2 = 0$, that is,

$$g'' = \frac{2g'r'}{r}. \tag{6}$$

This implies $g' = \lambda r^2$ with $\lambda > 0$. Analogously, $f' = \mu r^2$, $\mu > 0$. The computation of B_2 and B_1 leads to

$$B_2 = \lambda\mu r^8(a^2r^2A^2 - 4r'^2), \quad B_1 = 2\lambda r^7 r'(a^2rA^2 - 2r''),$$

where $A = -1 + (\lambda^2 + \mu^2)r^4 + r'^2 - r''$. Equation $B_1 = 0$ gives the possibility $r' = 0$, that is, r is a constant function. In such case, $B_2 = \lambda\mu a^2(-1 + (\lambda^2 + \mu^2)r^4)^2$. The computation of the coefficient E of the first fundamental form gives $E = 0$: contradiction. Thus, we can assume that $r' \neq 0$. By combining $B_2 = B_1 = 0$, we obtain $rr'' = 2r'^2$. Solving this equation, we have

$$r(u) = \frac{c_2}{u + c_1}, \quad c_1, c_2 \in \mathbb{R}.$$

The coefficient B_2 writes now as as polynomial on u and from $B_2 = 0$ we conclude

$$\begin{aligned} &a^2(u + c_2)^8 - 4(u + c_2)^6 + 2a^2c_2^4\Big(1 - c_2^2(\lambda^2 + \mu^2)\Big)(u + c_2)^4 \\ &+ a^2c_2^4\Big(1 - c_2^2(\lambda^2 + \mu^2)\Big)^2 = 0. \end{aligned}$$

The leading coefficient must vanish, that is, $a^2 = 0$: contradiction.

1. Assume $-2f'^2r' + 2g'^2r' + rf'f'' - rg'g'' = 0$. From here, we obtain f'' and putting it into A_4, we have

$$A_4 = -\frac{a^2r^6(f'^2 + g'^2)^2}{8f'^2}(rg'' - 2g'r')^2.$$

Then $rg'' - 2g'r' = 0$ and we now are in the position of the above case (6) and this finishes the proof.

3.1.2. Case $c \neq 0$

The computation of A_8 and B_8 gives respectively:

$$\begin{aligned} A_8 &= -\frac{1}{32}c^2r^8(f'^8 - 28f'^6g'^2 + 70f'^2g'^6 + g'^8) \\ B_8 &= \frac{1}{4}c^2r^8f'g'(-f'^6 - 7f'^4g'^2 - 7f'^2g'^4 + g'^6) \end{aligned}$$

Since $\alpha(u) = (f(u), g(u), 0)$ is not a constant planar curve, we parametrize it by the arc-length, that is, $(f(u), g(u)) = (x(\phi(u), y(\phi(u)),$ where

$$f'(u) = \phi'(u)\cos(\phi(u)), \quad g'(u) = \phi'(u)\sin(\phi(u)), \quad \phi'^2 = f'^2 + g'^2.$$

With this change of variable, the functions A_8 and B_8 write now as:

$$A_8 = -\frac{1}{32}c^2r^8\phi'^8\cos(8\phi(u)), \quad B_8 = -\frac{1}{32}c^2r^8\phi'^8\sin(8\phi(u)).$$

As $c \neq 0$ and $r > 0$, we conclude that $\phi' = 0$ on some interval. Therefore $f'^2 + g'^2 = 0$, which means that α is a constant curve, obtaining a contradiction. This finishes the proof of Theorem 1.1 for the case that the planes are spacelike.

3.4. The planes are timelike

Let M be a linear Weingarten spacelike surface foliated by pieces of circles in parallel timelike planes. After a motion of $\mathbf{E}_1^3$, we suppose that these planes are parallel to the plane $x_1 = 0$. In this case we parametrize the surface by

(7) $$\mathbf{X}(u, v) = (u, f(u), g(u)) + r(u)(0, \sinh v, \cosh v),$$

where $r > 0, f$ and g are smooth functions. This means that M is formed by a uniparametric family of vertical hyperbolas. In order to conclude that M is rotational it suffices to prove that f and g are constants.

3.2.1. Case $c = 0$

As in the case of spacelike planes, the reasoning is by contradiction. We assume that f or g is not constant, that is, $f' \neq 0$ or $g' \neq 0$.

1. We consider the case that one of the functions f or g is constant. For simplicity, we assume $f' = 0$ in some interval. Then A_4 writes as

$$A_4 = -\frac{1}{8}a^2r^6g'^2(-2r'g' + rg'')^2.$$

As $g' \neq 0$, from $A_4 = 0$ we have that $rg'' - 2r'g' = 0$. Then $g' = \mu r^2$, $\mu > 0$. Now

$$A_2 = -\frac{1}{2}\mu^2r^8(4r'^2 + a^2r^2A^2), \quad A_1 = -2\mu r^7r'(2r'' + a^2rA^2),$$

where $A = -1 + \mu^2r^4 - r'^2 + rr''$. As $a^2 > 0$, Equation $A_2 = 0$ implies that r is a constant function and $A = -1 + \mu^2r^4 = 0$. Then the computation of the coefficient E of the first fundamental form yields $E = 0$: contradiction.

2. We assume that both $f', g' \neq 0$. The coefficient B_4 yields:

$$(-4f'g'r' + rg'f'' + rf'g'')(-2f'^2r' - 2g'^2r' + rf'f'' + rg'g'') = 0.$$

We distinguish two cases.

1. If $-4f'g'r' + rg'f'' + rf'g'' = 0$, then

$$f'' = \frac{f'(4f'g'r' - rg'')}{rb'}. \tag{8}$$

Now $A_4 = 0$ gives

$$A_4 = \frac{a^2r^6(f'^2 - g'^2)^2(-2g'r' + rg'')^2}{8g'^2}.$$

(a1) If $f'^2 - g'^2 = 0$ then $g' = \pm f'$. Let $g' = f'$ (the case $g' = -f'$ is similar). Then $g = f + c_1$, $c_1 \in \mathbb{R}$. Putting it into A_4 and B_4, we obtain

$$A_4 = -a^2r^6f'^2(-2f'r' + rf''), \quad B_4 = a^2r^6f'^2(-2f'r' + rf'')^2.$$

As $f' \neq 0$, then $2f'r' = rf''$ and so, $f' = \lambda r^2$, $\lambda > 0$. The computation of A_2 and B_1 gives

$$A_2 = \lambda^2r^8(4r'^2 + a^2r^2A^2), \quad B_1 = 2\lambda r^7r'(2r'' + a^2rA^2),$$

where $A = 1 - r'^2 + rr''$. From Equation $A_2 = 0$ and the value of A, we discard the case that r is constant function. The combination of $A_2 = 0$ and $B_1 = 0$ implies that the function r satisfies $2r'^2 - rr'' = 0$. Then

$$r(u) = \frac{c_2}{u + c_1}, \quad c_1, c_2 \in \mathbb{R}.$$

But then $A_2 = 0$ gives a polynomial on u given by

$$4(u + c_1)^6 + a^2\Big((u + c_1)^4 + c_2^2\Big)^2 = 0,$$

whose leading coefficient is a^2: contradiction.

(a2) If $rg'' = 2g'r'$ then $g' = \mu r^2$ with $\mu > 0$. Using (8), $f' = \lambda r^2$, for some $\lambda > 0$. The computation of A_2 and A_1 leads to

$$A_2 = -\frac{1}{2}(\lambda^2 + \mu^2)r^8(a^2r^2A^2 + 4r'^2), \quad A_1 = 2\lambda r^7r'(2r'' + a^2rA^2)$$

where the value of A is now $A = -1 + (-\lambda^2 + \mu^2)r^4 + r'^2 - rr''$. Equation $A_2 = 0$ implies that r is a constant function and $(\lambda^2 - \mu^2)r^4 = -1$. This gives $E = 0$: contradiction.

2 If $-2f'^2r' - 2g'^2r' + rf'f'' + rg'g'' = 0$, we obtain f'' and hence we get

$$A_4 = \frac{a^2r^6(f'^2 - g'^2)^2(-2g'r' + rg'')^2}{8g'^2}.$$

(b1) If $f'^2 - g'^2 = 0$ then $g' = \pm f'$. Then we are as in the above case (a1).

(b2) If $rg'' = 2g'r'$ then $g' = \lambda r^2$ with $\lambda > 0$. Now we are in the position of the case (a2).

3.2.2. Case $c \neq 0$

The computations of A_8 and B_8 give:

$$\begin{aligned} A_8 &= -\frac{1}{32}c^2 r^8 (f'^8 + 28 f'^6 g'^2 + 70 f'^2 g'^6 + g'^8) \\ B_8 &= \frac{1}{4} c^2 r^8 f' g' (f'^6 + 7 f'^4 g'^2 + 7 f'^2 g'^4 + g'^6) \end{aligned}$$

Since $\alpha(u) = (f(u), g(u))$ is not a constant planar curve, we parametrize it by the arc-length, that is, $(f(u), g(u)) = (x(\phi(u), y(\phi(u)))$, where

$$f'(u) = \phi'(u)\cosh(\phi(u)), \quad g'(u) = \phi'(u)\sinh(\phi(u)), \quad \phi'^2 = f'^2 - g'^2.$$

With this change of variable, the functions A_8 and B_8 write now as:

$$A_8 = -\frac{1}{32}c^2 r^8 \phi'^8 \cosh(8\phi(u)), \qquad B_8 = \frac{1}{32} c^2 r^8 \phi'^8 \sinh(8\phi(u)).$$

As $c \neq 0$ and $r > 0$, we conclude that $\phi' = 0$ on some interval, that is, α is a constant curve, obtaining a contradiction. This finishes the Theorem for the case that the foliation planes are timelike.

3.3. The planes are lightlike

After a motion in $\mathbf{E}_1^3$, we parametrize the surface by

$$\mathbf{X}(u,v) = (f(u), g(u) + u, g(u) - u) + (v, r(u)\frac{v^2}{2}, r(u)\frac{v^2}{2}), \tag{9}$$

where $r > 0$, f and g are smooth functions. In such case, M is rotational if f is a constant function.

3.3.1. Case $c = 0$

We compute (1) and we take $4b^2 = 1$ again. With our parametrization, and we obtain

$$\sum_{j=0}^{6} A_j(u) v^n = 0, \tag{10}$$

for some functions A_j. As a consequence, all coefficients A_j vanish. Then

$$A_6 = -2a^2(2r^2 - r')(-4rr' + r'')^2.$$

1. If $2r^2 - r' = 0$ then r is given by

$$r(u) = \frac{1}{-2u-\lambda}, \quad \lambda \in \mathbb{R}.$$

Now

$$A_3 = \frac{16a^2 f'(-4f' + (2u+\lambda)f'')^2}{(2u+\lambda)^5}.$$

From $A_3 = 0$ we have

   (a) If $f' = 0$ then f is constant and M is rotational.
   (b) If $-4f' + (2u+\lambda)f'' = 0$ then $f'' = \frac{4f'}{(2u+\lambda)}$. Then $A_2 = -256f'^2$, which implies that $f' = 0$ and M is rotational again.

2. Assume $-4rr' + r'' = 0$. The coefficient A_4 gives $a^2(rf'' + 2r'f')^2 + 2r'^2(2r^2 - r') = 0$. A first integral of $-4rr' + r'' = 0$ is $2r^2 - r' = k$, for some constant $k \neq 0$. Then $A_4 = 0$ writes

$$a^2(rf'' + 2r'f')^2 + 2r'^2 k = 0.$$

   (a) If $k > 0$, then r is constant and $f'' = 0$. In particular, $f(u) = \lambda u + \mu$. Now $A_2 = 0$ implies $-16a^2r^2(\lambda^2 r + 4rg' + g'')^2 = 0$. Solving for g, we obtain $g(u) = -\lambda^2 u/4 - e^{-4ruc_1}/(4r) + c_2$, $c_1, c_2 \in \mathbb{R}$. Hence, (10) writes $-256c_1 r^4 e^{-8ru} = 0$: contradiction.
   (b) Assume $k = -\lambda < 0$. Then $A_4 = 0$ implies

$$rf'' + 2r'f' = \pm\sqrt{2\lambda/a^2}\,r'.$$

   Here we obtain f'', which it is substituted in A_3 to obtain g'' in terms of f' and g'. Substituting into A_2, we get that $A_1 = 0$ is equivalent to $(2r^2 + \kappa)/(\kappa - 2r^2) = 0$. Thus, the only possibility is that r is a constant function. But then $r' = 2r^2 + \lambda$ gets a contradiction.

3.3.2. c$\neq 0$

If we compute the Weingarten relation (1) with our parametrization, we obtain

$$\sum_{j=0}^{8} B_j(u)v^n = 0, \tag{11}$$

for some functions B_j. As a consequence, all coefficients B_j vanish. The coefficient B_8 is $B_8 = -64c^2(-2r^2 + r')^4$. Thus $-2r^2 + r' = 0$ and r is given by

$$r(u) = \frac{1}{-2u-\mu}, \ \mu \in \mathbb{R}.$$

Now $B_3 = \frac{1024c^2 f'^4}{(2u+\mu)^5}$. From $B_3 = 0$ we have $f' = 0$ and thus M is rotational.

Remark 3.1. Non-rotational spacelike surfaces in $\mathbf{E}_1^3$ with $H = 0$ and $K = 0$ are determined by the computation of (5). In the case $H = 0$ and if the planes of the foliations are spacelike or timelike, the functions f, g and r in the parametrizations (2) and (7) satisfy

$$f' = \lambda r^2, \quad g' = \mu r^2, \quad 1 - (\epsilon\lambda^2 + \mu^2)r^4 - r'^2 + rr'' = 0,$$

with $\lambda, \mu \in \mathbb{R}$ and $\epsilon = 1$ or $\epsilon = -1$ depending if the planes are spacelike or timelike, respectively. The solutions are given in terms of elliptic equations. If the planes are lightlike, then $H = 0$ means that, up a constant, $r = \tan(2u)$ and

$$\begin{aligned} f(u) &= \lambda(u + \frac{1}{2}\cot(2u)), \ g(u) \\ &= \frac{1}{32}\Big(4(4\mu - 3\lambda^2)u - 4\lambda^2\cot(2u) - (\lambda^2 - 4\mu)\sin(4u)\Big), \quad \lambda, \mu \in \mathbb{R}. \end{aligned}$$

If $K = 0$, then the functions satisfy $f'' = g'' = r'' = 0$ if the foliation planes are spacelike or timelike, and $f'' = g'' = 0$ and $r(u) = \lambda/(u + \mu)$ if the planes are lightlike.

4. PROOF OF THEOREM 1.2

Let M be a linear Weingarten spacelike surface foliated by a uniparametric family of circles. Consider a real interval $I \subset \mathbb{R}$ and $u \in I$ the parameter of each plane of the foliation that defines M. Let $\mathbf{G}(u)$ be a smooth unit vector field orthogonal to each u-plane. Assume that the u-planes are not parallel and we will conclude that M is a pseudohyperbolic surface. Then $\mathbf{G}'(u) \neq 0$ in some real interval. Without loss of generality, we assume that in that interval, the planes containing the circles of M have the same causal character. Consider an integral curve Γ of the vector field $\mathbf{G}$. Then Γ is not a straight-line. This allows to define a Frenet frame of Γ $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. We have to distinguish three cases according to the causal character of the foliation planes. If the planes are spacelike or timelike, the reasoning is similar. For this reason, we shall only consider that the foliation planes are spacelike and lightlike.

4.1. The planes are spacelike

Let $\{\mathbf{e}_1(u), \mathbf{e}_2(u)\}$ be an orthonormal basis in each u-plane. Then M parametrizes as

$$\mathbf{X}(u, v) = \mathbf{c}(u) + r(u)(\cos(v)\mathbf{e}_1(u) + \sin(v)\mathbf{e}_2(u))$$

where $r(u) > 0$ and $\mathbf{c}(u)$ are differentiable functions on u. Then $\mathbf{t} = \mathbf{G}$ is the unit tangent vector to Γ and the Frenet equations are

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n} \\ \mathbf{n}' &= \kappa\mathbf{t} + \sigma\mathbf{b} \\ \mathbf{b}' &= -\sigma\mathbf{n} \end{aligned}$$

A change of coordinates allows to write M as

$$\mathbf{X}(u,v)=\mathbf{c}(u)+r(u)(\cos(v)\mathbf{n}(u)+\sin(v)\mathbf{b}(u)).$$

Set $\mathbf{c}'=\alpha\mathbf{t}+\beta\mathbf{n}+\gamma\mathbf{b}$, where α,β y γ are smooth functions on u. Here $\mathbf{t}$ is a timelike unit vector and $\mathbf{n}$ and $\mathbf{b}$ are spacelike unit vectors. Because Γ is not a straight-line, $\kappa\neq 0$. By using $\mathbf{c}'$ and the Frenet equations, the expression (1) is a trigonometric polynomial on $\cos(jv)$ and $\sin(jv)$:

$$\sum_{j=0}^{8}A_j(u)\cos(jv)+B_j(u)\sin(jv)=0,$$

where A_j and B_j are smooth functions on u.

4.1.1. Case $c=0$ in the relation $aH+bK=c$

Without loss of generality, we assume that $4b^2=1$. The coefficient B_8 implies

$$\beta\gamma\Big(2a^2(3\beta^4-10\beta^2\gamma^2+3\gamma^4)+\kappa^2(1+12a^2r^2)(\gamma^2-\beta^2)+r^2\kappa^4(1+6a^2r^2)\Big)=0.$$

We discuss three cases.

1. Case $\beta=0$ in a sub-interval of I. Then $A_8=0$ writes as

$$(\gamma^2+r^2\kappa^2)^2\Big(4a^2\gamma^2+(1+4a^2r^2)\kappa^2\Big)=0.$$

This implies $r\kappa=0$: contradiction.

2. Case $\gamma=0$ in a sub-interval of I. Equation $A_8=0$ writes as

$$(\beta^2-r^2\kappa^2)^2\Big((1+4a^2r^2)\kappa^2-4a^2\beta^2\Big)=0.$$

If $\beta^2=r^2\kappa^2$, it follows that $A_6=-\frac{9}{32}\kappa^6r^{10}(\alpha-r')^2$. Then $A_6=0$ yields $\alpha=r'$. A computation of W gives $W=0$: contradiction. As a consequence, we assume $4\beta^2=(1+4a^2r^2)\kappa^2$. The computation of the coefficient A_7 leads to $\alpha^2(1+4a^2r^2)=4a^2r^2r'^2$. From the expression of the center curve $\mathbf{c}$, we have

$$\mathbf{c}'=\frac{rr'}{\sqrt{\frac{1}{4a^2}+r^2}}\,\mathbf{t}+\kappa\sqrt{\frac{1}{4a^2}+r^2}\,\mathbf{n}=(\sqrt{\frac{1}{4a^2}+r^2}\,\mathbf{t})'.$$

In particular, there exists $\mathbf{c}_0\in\mathbf{E}_1^3$ such that $\mathbf{c}=\mathbf{c}_0+\sqrt{\frac{1}{4a^2}+r^2}\,\mathbf{t}$. The parametrization $\mathbf{X}$ of the surface is now

$$\mathbf{X}(u,v)=\mathbf{c}_0+\sqrt{\frac{1}{4a^2}+r(u)^2}\,\mathbf{t}+r(u)(\cos(v)\mathbf{n}+\sin(v)\mathbf{b}).$$

Then

$$\langle \mathbf{X} - \mathbf{c}_0, \mathbf{X} - \mathbf{c}_0 \rangle = \frac{1}{4a^2}.$$

This means that the surface is a pseudohyperbolic surface.

3. Case $\beta\gamma \neq 0$. From $B_8 = 0$ we calculate β^2:

$$\beta^2 = \frac{1}{12a^2}\Big(20a^2\gamma^2 + (1 + 12a^2r^2)\kappa^2 \pm A\Big),$$

where $A = \sqrt{256a^2\gamma^4 + 16a^2\gamma^2\kappa^2 + 192a^4r^2\gamma^2\kappa^2 + \kappa^4}$. We consider the sign '+' in the value of β^2 (similarly with the minus sign). Placing β^2 into A_8 and taking in account that $\kappa \neq 0$, we obtain the following identity

$$\begin{aligned} & 26624a^6\gamma^6 + \kappa^6 + 1536a^2\gamma^4\kappa^2(1 + 2a^2r^2) + 72a^2\gamma^2\kappa^4 \\ = \; & -\Big(1792a^2\kappa^4 + \kappa^4 + 64a^2\gamma^2\kappa^2(1 + 12a^2r^2)\Big)A. \end{aligned}$$

Squaring both sides and after some manipulations, we obtain

$$(\gamma^2 + \kappa^2r^2)\Big((16a^2\gamma^2 + \kappa^2)^2 + 256a^2r^2\gamma^2\kappa^2\Big) = 0.$$

Hence we conclude $\kappa r = 0$, which it is a contradiction.

4.1.2. Case $c \neq 0$ in the relation $aH + bK = c$

Without loss of generality, we shall assume that $c = 1$. The computations of the coefficients A_8 and B_8 give

$$A_8 = -\frac{1}{32}r^8x_1, \qquad B_8 = \frac{1}{16}\beta\gamma r^8x_2,$$

where

$$\begin{aligned} x_1 = \; & \beta^8 - (28\gamma^2 + \kappa^2(a^2 + 2b + 4r^2))\beta^6 \\ & +(70\gamma^4 + 15\gamma^2\kappa^2(^2(a^2 + 2b + 4r^2) + \kappa^4(b^2 + 3(a^2 + 2b)r^2 + 6r^4))\beta^4 \\ & +(-28\gamma^6 - 15\gamma^4\kappa^2(a^2 + 2b + 4r^2) - \kappa^6r^2(2b^2 + 3(a^2 + 2b)r^2 + 4r^4) \\ & -6\gamma^2\kappa^4(b^2 + 3(a^2 + 2b)r^2 + 6r^4))\beta^2 \\ & +(\gamma^2 + r^2\kappa^2)^2(\gamma^4 + \gamma^2\kappa^2(a^2 + 2b + 2r^2) + \kappa^4(b^2 + (a^2 + 2b)r^2 + r^4)). \\ x_2 = \; & -4\beta^6 + (28\gamma^2 + 3\kappa^2(a^2 + 2b + 4r^2))\beta^4 \\ & -2(14\gamma^4 + 5\gamma^2\kappa^2(a^2 + 2b + 4r^2) + \kappa^4(b^2 + 3(a^2 + 2b)r^2 + 6r^4)\beta^2 \\ & +(\gamma^2 + r^2\kappa^2)(4\gamma^4 + \gamma^2\kappa^2(3a^2 + 6b + 8r^2) + \kappa^4(2b^2 + 3(a^2 + 2b)r^2 + 4r^4). \end{aligned}$$

From $B_8 = 0$, we discuss three cases:

1. Case $\gamma = 0$ in some sub-interval of I. Then $A_8 = 0$ is

$$(\beta^2 - r^2\kappa^2)^2\Big(\beta^4 - (a^2+2b+2r^2)\beta^2\kappa^2 + (b^2+(a^2+2b)r^2+r^4)\kappa^4\Big) = 0.$$

(a) Suppose $\beta^2 = \kappa^2 r^2$. Without loss of generality we assume that $\beta = \kappa r$. Now $A_6 = -\frac{9}{8}b^2\kappa^6 r^{10}(\alpha - r')^2$. Then $\alpha = r'$ and this gives $W = 0$: contradiction.

(b) Then $\beta^4 - (a^2+2b+2r^2)\beta^2\kappa^2 + (b^2+(a^2+2b)r^2+r^4)\kappa^4 = 0$. If we look this expression as a polynomial on β^2, the computation of the discriminant concludes that $a^2+4b \geq 0$. If $a^2+4b = 0$, $\beta^2 = (a^2+4r^2)\kappa^2/4$. Then $B_5 = 0$ gives

$$B_5 = \frac{1}{128}a^4\kappa^5 r^7\sigma\sqrt{a^2+4r^2}(\alpha\sqrt{a^2+4r^2} - 2rr')^2 = 0.$$

If $\sigma = 0$, $A_5 = 0$ implies $\alpha\sqrt{a^2+4r^2} - 2rr' = 0$ again. Therefore, in both cases, and from the value of α, we can write

$$\mathbf{c}' = \left(\frac{\sqrt{a^2+4r^2}}{2}\,\mathbf{t}\right)'$$

and so, there exists $\mathbf{c}_0 \in \mathbf{E}^3_1$ such that $\mathbf{c} = \mathbf{c}_0 + \frac{\sqrt{a^2+4r^2}}{2}\,\mathbf{t}$. As a consequence, we have

$$\mathbf{X}(u,v) = \mathbf{c}_0 + \frac{1}{2}\sqrt{a^2+4r(u)^2}\,\mathbf{t} + r(u)\Big(\cos(v)\mathbf{n} + \sin(v)\mathbf{b}\Big),$$

for some $\mathbf{c}_0 \in \mathbf{E}^3_1$. Therefore $\langle \mathbf{X}-\mathbf{c}_0, \mathbf{X}-\mathbf{c}_0\rangle = -a^2/4$, and the surface is a pseudohyperbolic surface.

Assume then $a^2+4b > 0$. The coefficient A_7 is

$$A_7 = \frac{1}{64}aAB\kappa^5 r^9(\alpha\kappa^2 - \kappa\beta' + \kappa'\beta) = 0,$$

with

$$A = 2b + a(a+\sqrt{a^2+4b}), \qquad B = a^3 + 4ab + (a^2+2b)\sqrt{a^2+4b}.$$

Then number A does not vanish and $B = 0$ holds only if $a^2+4b = 0$. From $A_7 = 0$ we conclude that $\alpha\kappa^2 - \kappa\beta' + \kappa'\beta = 0$, that is,

$$\alpha = \left(\frac{\beta}{\kappa}\right)',$$

which implies $\mathbf{c} = \mathbf{c}_0 + \beta/\kappa\mathbf{t}$ for some $\mathbf{c}_0 \in \mathbf{E}_1^3$. The derivative of the curve $\mathbf{c}$ is

$$\mathbf{c}' = \left(\frac{\beta}{\kappa}\right)'\mathbf{t} + \beta\mathbf{n} = \left(\frac{\beta}{\kappa}\mathbf{t}\right)'.$$

The expression of $\mathbf{X}(u, v)$ is

$$\mathbf{X}(u, v) = \mathbf{c}_0 + \frac{\beta}{\kappa}\mathbf{t} + r(\cos(v)\mathbf{n} + \sin(v)\mathbf{b}).$$

Using the value of β^2, we have,

$$\langle \mathbf{X} - \mathbf{c}_0, \mathbf{X} - \mathbf{c}_0\rangle = -\frac{\beta^2}{\kappa^2} + r^2 = -\left(\frac{a^2}{2} + b + \frac{a}{2}\sqrt{a^2 + 4b}\right).$$

This means that the surface is a pseudohyperbolic surface.

2. Case $\beta = 0$ in some sub-interval of I. Then

$$A_8 = -\frac{1}{32}r^8(\gamma^2 + \kappa^2 r^2)y_1, \qquad A_7 = -\frac{1}{16}\alpha\kappa r^9(\gamma^2 + \kappa^2 r^2)z_1,$$

where

$$\begin{aligned} y_1 &= \gamma^4 + (a^2 + 2b + 2r^2)\kappa^2\gamma^2 + (b^2 + (a^2 + 2b)r^2 + r^4)\kappa^4 \\ z_1 &= 8\gamma^4 + (7(a^2 + 2b) + 16r^2)\kappa^2\gamma^2 + (6b^2 + 7(a^2 + 2b)r^2 + 8r^4)\kappa^4. \end{aligned}$$

Assume $\alpha \neq 0$. From $y_1 = 0$, we obtain γ^2, which it is substituted into $z_1 = 0$, obtaining $a\sqrt{a^2 + 4b} = \pm(a^2 + 2b)$. Then $a^2(a^2 + 4b) = (a^2 + 2b)^2$, which implies $b = 0$: contradiction. Therefore, $\alpha = 0$. From $y_1 = 0$,

$$\gamma^4 + (a^2 + 2b + 2r^2)\kappa^2\gamma^2 + (b^2 + (a^2 + 2b)r^2 + r^4)\kappa^4 = 0.$$

Then

$$\gamma^2 = \frac{1}{2}\Big(\pm a\sqrt{a^2 + 4b} - (a^2 + 2b + 2r^2)\Big)\kappa^2. \tag{12}$$

We prove that the parenthesis in (12) is non-positive, that is, $\pm a\sqrt{a^2 + 4b} - (a^2 + 2b + 2r^2) \leq 0$. Since this function on r is decreasing on a, we show that (taking $r = 0$) $\pm a\sqrt{a^2 + 4b} - (a^2 + 2b) \leq 0$. Depending on the sign of a, we have two possibilities. If $a > 0$, the inequality $\pm a\sqrt{a^2 + 4b} \leq a^2 + 2b$ is trivial. If $a < 0$, the inequality is trivial if $a^2 + 2b \geq 0$. The only case to consider is $\pm a\sqrt{a^2 + 4b} \leq a^2 + 2b < 0$ ($\Rightarrow b < 0$). But $a^2 + 2b < 0$ and $a^2 + 4b \geq 0$ are not compatible. As a consequence of this reasoning, we conclude from (12) that $\gamma = 0$ and this case was studied in the above subsection.

3. Case $\beta\gamma \neq 0$. In this case, the computations become very complicated and difficult. For this reason, we only give the proof outline and we omit the details. Let $x = \beta^2$, $y = \gamma^2$. From $x_1 = 0$, we obtain the value of $a^2 + 2b$, which is substituted into $x_2 = 0$, obtaining

$$\Big((x+y)^2+2(y-x)r^2\kappa^2+r^4\kappa^4\Big)^2\Big((x+y)^2+2(y-x)r^2\kappa^2-b^2\kappa^4+r^4\kappa^4\Big) = 0.$$

If we see $(x + y)^2 + 2(y - x)r^2\kappa^2 + r^4\kappa^4 = 0$ as polynomical equation on $r^2\kappa^2$, we find that the discriminant is negative, and so, this case is impossible. Thus

$$(x + y)^2 + 2(y - x)r^2\kappa^2 - b^2\kappa^4 + r^4\kappa^4 = 0. \tag{13}$$

Then $y = -x - r^2\kappa^2 + \kappa\sqrt{4xr^2 + b^2\kappa^2}$. Putting into $x_1 = 0$, we conclude

$$16x^2 - 8x(a^2 + 2b + 2r^2)\kappa^2 + (a^4 + 4a^2b)\kappa^4 = 0$$

or

$$256x^4 - 512x^3r^2\kappa^2 - 128x^2(b^2 - 2r^4)\kappa^4 + 64b^2xr^2\kappa^6 + 3b^4\kappa^8 = 0.$$

We analyse the first possibility (the second one is analogous). If $16x^2 - 8x(a^2 + 2b + 2r^2)\kappa^2 + (a^4 + 4a^2b)\kappa^4 = 0$, then

$$\begin{aligned}\beta^2 &= \frac{\kappa^2}{4}\Big(a^2 + 2b + 2r^2 \pm 2Q\Big) \\ &\quad \Big(4\sqrt{b^2 + r^2(a^2 + 2b + 2r^2 \pm 2Q)} - (a^2 + 2b + 6r^2) \mp 2Q\Big),\end{aligned} \tag{14}$$

$$\gamma^2 = \frac{\kappa^2}{4} \tag{15}$$

where

$$Q = \sqrt{a^2r^2 + (b + r^2)^2)}.$$

With these values obtained for β^2 and γ^2, Equation $x_1 = 0$ depends only on the function r. Exactly, $x_1 = 0$ is a rational expression on r and $\sqrt{b^2 + (a^2 + 2b)r^2 + r^4}$:

$$\mathcal{P}(r, \sqrt{b^2 + (a^2 + 2b)r^2 + r^4}) = 0.$$

In particular, $r(u)$ is a constant function. We do the change

$$p = x - y, \qquad q = (x - y)^2 - 4xy$$

that is, $x = (p + \sqrt{2p^2 - q})/2$ and $y = (-p + \sqrt{2p^2 - q})/2$. Equation (13) writes as

(16) $2p^2 - q - 2pr^2\kappa^2 + (r^4 - b^2)\kappa^4 = 0 \Rightarrow q = 2p^2 + 2pr^2\kappa^2 + (r^4 - b^2)\kappa^4.$

We calculate q by other way. From $x_1 = 0$, we get a value of q, which it is substituted into $x_2 = 0$, obtaining

$$(p - r^2\kappa^2)\Big(2p - (a^2 + 2(b + r^2))\kappa^2\Big)$$
$$\Big(4p^2 - 2p(a^2 + 2b + 4r^4)\kappa^2 + (b^2 + 2a^2r^2 + 4br^2 + 4r^4)\kappa^4\Big) = 0.$$

Hence and together (16) we obtain different values for p and q. On the other hand, the values obtained for β and γ in (14) and (15) allow to get a pair of values for p and q, which they must be equal. For each pair of these values, we obtain different values for r, which are substituted into the coefficients A_i and B_i. The computation of the coefficients A_7 and B_7 gives $\alpha = 0$ and using A_5 and B_5, we get $\sigma = 0$. Finally the coefficients A_4 and B_4 give $\kappa = 0$, obtaining a contradiction.

4.2. The planes are lightlike

The surface M can be locally written as

$$\mathbf{X}(u, v) = \mathbf{c}(u) + v\mathbf{n}(u) + r(u)v^2\mathbf{t}(u),$$

where $r(u) > 0$, and $\mathbf{t}$ and $\mathbf{n}$ are the tangent vector and normal vector of Γ respectively. Since the planes are lightlike, $\langle \mathbf{t}, \mathbf{t}\rangle = 0$ and $\langle \mathbf{n}, \mathbf{n}\rangle = 1$. The Frenet frame for Γ is $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, where $\mathbf{b}$ is the unique lightlike vector orthogonal to $\mathbf{n}$ such that $\langle \mathbf{t}, \mathbf{b}\rangle = 1$ and $\det(\mathbf{t}, \mathbf{n}, \mathbf{b}) = 1$. The Frenet equations are

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n} \\ \mathbf{n}' &= \sigma\mathbf{t} - \kappa\mathbf{b} \\ \mathbf{b}' &= -\sigma\mathbf{n} \end{aligned}$$

Again, we put $\mathbf{c}' = \alpha\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b}$. By using $\mathbf{c}'$ and the Frenet equations, the expression (1) is a trigonometric polynomial on v such as $\sum_{j=0}^{n} A_j(u)v^j = 0$ with $n = 11$ if $c = 0$ and $n = 12$ if $c \neq 0$.

4.2.1. Case $c = 0$ in the relation $aH + bK = c$

Without loss of generality, we assume that $b = 1/2$. Then $A_{11} = 98a^2r^2\kappa^5(2r^2\gamma - r')^2$. Then $r' = 2r^2\gamma$ and so

$$A_8 = -64r^2\kappa^5(\sigma - 2r\beta)^2(-4a^2r\beta + r^2\kappa + 2a^2\sigma) = 0.$$

If $\sigma = 2r\beta$, then $A_6 = -100r^4\alpha^2\kappa^6$. This yields $\alpha = 0$ and $W = 0$: contradiction. Thus $2a^2\sigma = 4a^2r\beta - r^2\kappa$. Now $A_7 = 0$ gives $2a^2\alpha = r^2\gamma$. Then

$$\mathbf{c}' = \frac{r'}{4a^2}\mathbf{t} + \beta\mathbf{n} + \frac{r'}{2r^2}\mathbf{b} = \Big(\frac{r}{4a^2}\mathbf{t} - \frac{1}{2r}\mathbf{b}\Big)'.$$

Therefore, there exists $\mathbf{c}_0 \in \mathbf{E}_1^3$ such that

$$\mathbf{X}(u,v) = \mathbf{c}_0 + \Big(\frac{r}{4a^2}\mathbf{t} - \frac{1}{2r}\mathbf{b}\Big) + v\mathbf{n}(u) + rv^2\mathbf{t}(u).$$

In particular,

$$\langle \mathbf{X}(u,v) - \mathbf{c}_0, \mathbf{X}(u,v) - \mathbf{c}_0\rangle = -\frac{1}{4a^2},$$

which shows that the surface is a pseudohyperbolic surface.

4.2.2. Case $c \neq 0$ in the relation $aH + bK = c$

We assume that $c = 1$. Then $A_{12} = -64\kappa^4(r' - 2r^2\gamma)^4$. As above, $A_8 = 0$ gives two possibilities about the value of σ. In the first case, $\sigma = 2r\beta$ and $A_6 = 0$ yields $\alpha = 0$. This implies $W = 0$: contradiction. The other case for σ is

$$\sigma^2 + 2r\sigma\Big(-2\beta + (a^2+2b)r\kappa\Big) + 4\Big(\beta^2 - (a^2+2b)r\beta\kappa + b^2r^2\kappa^2\Big)r^4 = 0.$$

Then

$$\sigma = 2r\beta - a^2r^2\kappa - 2br^2\kappa \pm ar^2\kappa\sqrt{a^2+4b}.$$

In particular, $a^2 + 4b \geq 0$. We assume the choice '+' in the above identity (similar for the minus sign). From Equation $A_7 = 0$, we obtain

$$\alpha = \frac{1}{2}(a^2 + 2b - a\sqrt{a^2+4b})r'.$$

As in the case $c = 0$, we conclude the existence of $\mathbf{c}_0 \in \mathbf{E}_1^3$ such that

$$\mathbf{c} = \mathbf{c}_0 + \frac{1}{2}(a^2 + 2b - a\sqrt{a^2+4b})r\mathbf{t} - \frac{1}{2r}\mathbf{b}.$$

Now

$$\mathbf{X}(u,v) = \mathbf{c}_0 + \frac{1}{2}(a^2 + 2b - a\sqrt{a^2+4b})r\mathbf{t} - \frac{1}{2r}\mathbf{b} + v\mathbf{n}(u) + rv^2\mathbf{t}(u),$$

and

$$\langle \mathbf{X}(u,v) - \mathbf{c}_0, \mathbf{X}(u,v) - \mathbf{c}_0\rangle = -\frac{1}{2}(a^2 + 2b - a\sqrt{a^2+4b}),$$

showing that M is a pseudohyperbolic surface again.

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