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PENALIZED GENERALIZED FISCHER-BURMEISTER FUNCTION FOR SOCCP

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Abstract. Recently, Pan et al. [11] developed the merit function method for SOCCP based on the generalized Fischer-Burmeister (FB) function. This note is along the same line. Indeed, we study a penalized version of the generalized FB function and provide a basic theoretical property that the level set of the merit function induced by the penalized version of the generalized FB function is bounded under suitable assumptions. The proof relies on trace inequalities.

1. Introduction

The *symmetric cone complementarity problem* (for short, SCCP) on a symmetric cone Ω in a Euclidean Jordan algebra V (see Section 2 for details) is defined to be the problem finding vectors $x, y \in V$ such that

$$x \in \overline{\Omega}, \quad y \in \overline{\Omega}, \quad \langle x, y \rangle = 0, \quad y = F(x) \tag{1.1}$$

where $F : V \to V$ is a continuously differentiable mapping. This is equivalent to the form:

$$\text{Find } x \succeq 0, \quad \text{such that} \quad F(x) \succeq 0 \quad \text{and} \quad \langle x, F(x) \rangle = 0 \tag{1.2}$$

where $\preceq$ is the Lŏwner partial order on V defined by $x \preceq y \Longleftrightarrow y - x \in \overline{\Omega}$, and $x \prec y \Longleftrightarrow y - x \in \Omega$. In relation to (1.2), a function $\phi : V \times V \to V$ is called *a complementarity function* (C-function) (see [4, 1.5.1 Definition]) if

$$\phi(x, y) = 0 \text{ if and only if } \langle x, y \rangle = 0, \; x \succeq 0, \; y \succeq 0. \tag{1.3}$$

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It is well known that the function defined by

$$\phi(x, y) = x + y - (x^2 + y^2)^{1/2}$$

is a C-function [6], called the Fischer-Burmeister function.

The need for studying SCCP in optimization mostly comes from *second-order cone complementarity problems* (SOCCP) and *semidefinite complementarity problems* (SDCP). As SOCCP is concerned with, as is well-explained in [3], an important special case of SOCCP corresponds to the KKT optimality conditions of the convex second-order cone program (CSOCP):

$$\begin{array}{ll} \text{minimize} & g(x) \\ \text{subject to} & Ax = b, \quad x \in \mathcal{K}, \end{array} \tag{1.4}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable convex function, $A \in \mathbb{R}^{m \times n}$ has full row rank, $b \in \mathbb{R}^m$, and

$$\mathcal{K} := \left\{ (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1 \right\} \tag{1.5}$$

is the second-order cone (SOC). The convex SOCP has numerous applications in engineering design, finance, robust optimization, and convex quadratically constrained quadratic programs; see [1, 9]. So many methodologies to solve (SOCP) and (SOCCP) have introduced (see [3]). Especially *the merit function method* based on the Fischer-Burmeister (in short, FB) function was proposed by Chen and Tseng [3]. This is an approach based on reformulating CSOCP and SOCCP as an unconstrained smooth minimization problem. In fact, a function $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is called *a merit function* if

$$\psi(x, y) = 0 \iff \langle x, y \rangle = 0, \ x \succeq 0, \ y \succeq 0. \tag{1.6}$$

Chen and Tseng [3] chose as a merit function the popular FB merit function for SOCCP defined by

$$\psi_{\mathrm{FB}}(x, y) \ := \ \frac{1}{2}\|\phi_{\mathrm{FB}}(x, y)\|^2, \tag{1.7}$$

where $\phi_{\mathrm{FB}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the FB function given by

$$\phi_{\mathrm{FB}}(x, y) = x + y - (x^2 + y^2)^{1/2} \tag{1.8}$$

with x^2 the Jordan product of x with itself, $x^{1/2}$ a vector such that $(x^{1/2})^2 = x$. With this choice, they successfully established a nice theory of the merit function method for SOCCP [3].

In particular, in a recent paper [11], the merit function method for SOCCP based on *the generalized FB function* is presented. The definition of the generalized FB function is as follow: Let $x, y \in \mathbb{R}^n$. For $p > 1$,

$$\phi_p(x,y) = x + y - (|x|^p + |y|^p)^{\frac{1}{p}} \tag{1.9}$$

is called *the generalized Fischer-Burmeister function* of SOCCP. Clearly, when $p = 2$, this function reduces to the FB function itself. This note is also along the same line as the previous one [11]. Indeed, we study a penalized version of the generalized FB function and provide a basic theoretical property that the level set of the merit function induced by the penalized version of the generalized FB function is bounded under suitable assumptions. This is a crucial step toward an entire development of the merit function theory for SOCCP based on the penalized version as a future research.

2. Euclidean Jordan Algebras

We recall certain basic notions and well-known facts concerning Jordan algebras from the book by Faraut and Korányi [5]. A *Jordan algebra* V with an identity element e over the field $\mathbb{R}$ or $\mathbb{C}$ is a commutative algebra satisfying $x^2(xy) = x(x^2y)$ for all $x, y \in V$. Every Jordan algebra is power associative which means that the algebra generated by x and e is associative. Denote $L(x)$ by the multiplication operator $L(x)y = xy$, and set $P(x) = 2L(x)^2 - L(x^2)$ for $x \in V$. An element $x \in V$ is said to be *invertible* if there exists an element y in the subalgebra generated by x and e such that $xy = e$.

A finite-dimensional real Jordan algebra V is called a *Euclidean Jordan algebra* if it carries an associative inner product $\langle \cdot, \cdot \rangle$ on V, namely $\langle xy, z\rangle = \langle y, xz\rangle$ for all $x, y, z \in V$. An element $c \in V$ is *idempotent* if $c^2 = c$, and two idempotents c and c' are *orthogonal* if $cc' = 0$. If an idempotent c cannot be written by a sum of two non-zero idempotents then c is called *primitive*. One says that $c_1, \ldots, c_k$ is a *complete system of orthogonal idempotents* if $e = \sum_{i=1}^k c_i, c_ic_j = \delta_{ij}c_i$. A *Jordan frame* is a complete system of orthogonal primitive idempotents. The following two theorems are fundamental in the theory of Euclidean Jordan algebra. Actually, we introduce more detailed statements in [2] rather than the original ones in [5] as follows:

Theorem 2.1. (Spectral theorem, first version [5, Theorem III.1.1]). *For an element x of a Euclidean Jordan algebra V there exist unique real numbers $\lambda_1 > \cdots > \lambda_k$ and a unique complete system of orthogonal idempotents $c_1, \ldots, c_k$ such that $x = \sum_{i=1}^k \lambda_i c_i$. The uniqueness is in the following sense: if there exist a complete system of orthogonal idempotents $\{e_1, , \ldots, e_s\}$ and distinct real numbers $\eta_1 > \cdots > \eta_s$ such that $x = \sum_{i=1}^s \eta_i e_i$, then $k = s$ and $\eta_i = \lambda_i$ and $e_i = c_i$ for all $1 \le i \le k$.*

Theorem 2.2. (Spectral theorem, second version [5, Theorem III.1.2]). *For an element x of a Euclidean Jordan algebra V there exist a Jordan frame $c_1, \ldots, c_r$ (r is fixed and called the rank of V) and real numbers $\lambda_1 \geq \cdots \geq \lambda_r$ such that $x = \sum_{i=1}^r \lambda_i c_i$. If there exist a Jordan frame $e_1, \ldots, e_r$ and real numbers $\eta_1 \geq \cdots \geq \eta_r$ such that $x = \sum_{i=1}^r \eta_i e_i$, then $\eta_i = \lambda_i$ for all i and $\sum_{\{j|\eta_j=\alpha\}} e_j = \sum_{\{j|\eta_j=\alpha\}} c_j$ for each real number α.*

Let $\mathrm{tr}(x) = \sum_{i=1}^r \lambda_i$, the trace of $x = \sum_{i=1}^r \lambda_i c_i$ in the second spectral theorem. Then the trace inner product $\mathrm{tr}(xy)$ is associative. Let Ω be the open convex cone of invertible squares of a Euclidean Jordan algebra V. Then Ω is a symmetric cone, that is, the group $G(\Omega) := \{g \in \mathrm{GL}(V) : g(\Omega) = \Omega\}$ acts transitively on it and Ω is a self-dual cone with respect to the trace inner product. Recall the Löwner partial order on V defined by $x \preceq y :\Longleftrightarrow y - x \in \overline{\Omega}$, and $x \prec y :\Longleftrightarrow y - x \in \Omega$.

Lemma 2.3. Let p be a positive real number.

(i) *Each element $x \succeq 0$ has a unique p-th root denoted by $x^{1/p}$ in $\overline{\Omega}$. If $x \in \overline{\Omega}$ has a spectral decomposition $x = \sum_{i=1}^r \lambda_i c_i$, then $x^{1/p} = \sum_{i=1}^r \lambda_i^{1/p} c_i$.*

(ii) (The Löwner-Heinz inequality, [8])

$$0 \preceq x \preceq y \Longrightarrow x^p \preceq y^p, \ 0 \leq p \leq 1.$$

For $x \in V$, we denote $|x|$ by $|x| = (x^2)^{1/2}$ and

$$x_+ = \frac{x + |x|}{2}, \quad x_- = \frac{|x| - x}{2}. \tag{2.10}$$

If x has a spectral decomposition $x = \sum_{i=1}^r \lambda_i c_i$ then

$$x_+ = \sum_{i=1}^r (\lambda_i)_+ c_i, \quad x_- = \sum_{i=1}^r (\lambda_i)_- c_i, \quad |x| = \sum_{i=1}^r |\lambda_i| c_i \tag{2.11}$$

where for any scalar λ, $\lambda_+ = \max\{0, \lambda\}, \lambda_- = \max\{0, -\lambda\}$. Since $x = x_+ - x_-$ and $\langle x_+, x_- \rangle = 0$, by the Moreau decomposition, x_+ and $-x_-$ are the projections of x onto $\overline{\Omega}$ and $-\overline{\Omega}$, respectively. Moreover, $x_+ x_- = 0$.

We close this section with two typical examples of Euclidean Jordan algebras:

Example 2.4. Let $\mathbb{S}^n$ be the algebra of $n \times n$ real symmetric matrices with the Jordan product defined by

$$X \circ Y = \frac{XY + YX}{2}$$

where XY is the usual matrix multiplication of X and Y. Then $\mathbb{S}^n$ is a Euclidean Jordan algebra equipped with the trace inner product

$$\langle X, Y \rangle = \mathrm{tr}(XY).$$

In this case, Ω is the set of all positive definite matrices.

Example 2.5. Let $\mathbb{R}^n$ be the Euclidean space with the Jordan product defined by

$$x \circ y = (\langle x, y\rangle,\ x_1y_2 + y_1x_2)$$

where $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\langle x, y\rangle$ is the usual inner product in $\mathbb{R}^n$. Then $\mathbb{R}^n$ is a Euclidean Jordan algebra equipped with the standard inner product $\langle\cdot,\cdot\rangle$. In this case, Ω is the set $\operatorname{int}\mathcal{K} = \{(x_1, x_2) \in \mathbb{R}\times\mathbb{R}^{n-1} \ \mid\ \|x_2\| < x_1\}$.

For further definitions, terminologies and facts concerning a Euclidean Jordan algebra V and second-order cone $\mathcal{K}$, readers may refer to [5, 3, 7].

3. Main Results

We begin with quite elementary observations:

Lemma 3.1. *For two nonzero vectors $x,\ y \in \mathbb{R}^n$, we have*

$$\|x + y\| \geq \|x\| - \|y\|.$$

The equality holds if and only if $x = \alpha y$ for some $\alpha \leq -1$.

Proof. The above inequality is nothing but the triangle inequality. If the equality holds, then x and y are linearly dependent, so that $x = \alpha y$ for some $\alpha \in \mathbb{R}$. Then we can easily deduce from the equality $\|x + y\| = \|x\| - \|y\|$ that $\alpha \leq -1$. The converse is trivial. ∎

Lemma 3.2. *For nonnegative real numbers t and s, the followings hold:*

(i) $(t + s)^p \geq t^p + s^p$ for $p > 1$. The equality holds if and only if $ts = 0$.
(ii) $(t + s)^p \leq t^p + s^p$ for $0 \leq p < 1$. The equality holds if and only if $ts = 0$.

Lemma 3.3. *Let $\kappa \geq 0$. Let $f(x) = (\kappa - x)^p + (\kappa + x)^p$ for $-\kappa \leq x \leq \kappa$.*

(i) For $p > 1$, f is an even function which is strictly increasing on $0 \leq x \leq \kappa$.
(ii) For $0 \leq p < 1$, f is an even function which is strictly decreasing on $0 \leq x \leq \kappa$.

Theorem 3.4. *Let $a = (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $b = (s, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$. For $p > 1$, the generalized Fischer-Burmeister function*

$$\phi_p(a, b) = a + b - (|a|^p + |b|^p)^{\frac{1}{p}}$$

is a C-function of SOCCP.

Proof. Assume that $a \geq 0,\ b \geq 0$ and $ab = 0$. By means of Gowda et al. [6, Proposition 6], $[L(a), L(b)] = L(a)L(b) - L(b)L(a) = 0$. Due to Faraut and Korányi [5, Lemma X.2.2], there exists a common Jordan frame $c_1, \ldots, c_r$ such that

$a = \sum_{i=1}^{r} \lambda_i c_i$ and $b = \sum_{i=1}^{r} \mu_i c_i$ with nonnegative real numbers $\lambda_j,\ \mu_j$. Thus we have

$$(a+b)^p = (\lambda_1 + \mu_1)^p c_1 + \cdots + (\lambda_r + \mu_r)^p c_r$$

$$a^p + b^p = (\lambda_1^p + \mu_1^p) c_1 + \cdots + (\lambda_r^p + \mu_r^p) c_r$$

$$ab = (\lambda_1 \mu_1) c_1 + \cdots + (\lambda_r \mu_r) c_r = 0.$$

Hence $\lambda_1\mu_1 = \cdots = \lambda_r\mu_r = 0$, which means by Lemma 3.2 (i) that $(a+b)^p = a^p + b^p$, that is, $\phi_p(a,b) = 0$. Conversely, suppose that $\phi_p(a,b) = 0$, namely, $a+b = (|a|^p + |b|^p)^{1/p}$. Setting $w = (|a|^p + |b|^p)^{1/p}$, we have $w^p = |a|^p + |b|^p \geq |a|^p$ and $w^p = |a|^p + |b|^p \geq |b|^p$. By Lemma 2.3 (ii) (the Löwner-Heinz inequality), $w \geq |a|$ and $w \geq |b|$. Since $|a| \geq a$ and $|b| \geq b$, we then have

$$a = w - b \geq w - |b| \geq 0, \quad b = w - a \geq w - |a| \geq 0.$$

Thus $|a| = a$ and $|b| = b$, so $(a+b)^p = |a|^p + |b|^p = a^p + b^p$. Hence we have $\mathrm{tr}[(a+b)^p] = \mathrm{tr}(a^p) + \mathrm{tr}(b^p)$, that is,

$$\begin{aligned} &(t+s-\|x+y\|)^p + (t+s+\|x+y\|)^p \\ &= (t-\|x\|)^p + (t+\|x\|)^p + (s-\|y\|)^p + (s+\|y\|)^p. \end{aligned} \tag{3.12}$$

To show that $ab = 0$, it suffices to verify that $\langle a, b\rangle = 0$ because $a \geq 0$ and $b \geq 0$. Since $a \geq 0$ and $b \geq 0$, we have $t \geq \|x\|$ and $s \geq \|y\|$ so that $t + s \geq \|x+y\| \geq \|x\| - \|y\|$ (we may assume $\|x\| \geq \|y\|$ by the symmetry of the inequality in Lemma 3.1). Moreover,

$$\begin{aligned} &(t+s-\|x+y\|)^p + (t+s+\|x+y\|)^p \\ &\geq (t+s-\|x\|+\|y\|)^p + (t+s+\|x\|-\|y\|)^p \\ &\geq (t-\|x\|)^p + (s+\|y\|)^p + (t+\|x\|)^p + (s-\|y\|)^p. \end{aligned}$$

The first inequality follows from Lemma 3.3, and the second comes from Lemma 3.2. By (3.12), we get

$$\begin{aligned} &(t+s-\|x+y\|)^p + (t+s+\|x+y\|)^p \\ &= (t+s-\|x\|+\|y\|)^p + (t+s+\|x\|-\|y\|)^p \\ &= (t-\|x\|)^p + (s+\|y\|)^p + (t+\|x\|)^p + (s-\|y\|)^p. \end{aligned}$$

We first assume that two vectors $x,\ y$ are nonzero. From the first equality above, we obtain $x = \alpha y$ for some $\alpha \leq -1$ by Lemmas 3.1 and 3.3. Moreover, it can be easily checked from the last equality that either $t = 0$, or $s = 0$, or ($t = \|x\|$ and $s = \|y\|$) by Lemma 3.2. Since $x,\ y$ are nonzero, the only possible case is when $t = \|x\|$ and $s = \|y\|$. So we get

$$\langle x, y\rangle = \langle \alpha y, y\rangle = \alpha\|y\|^2 = -\|\alpha y\|\|y\| = -\|x\|\|y\| = -ts.$$

Thus, $\langle a, b\rangle = ts + \langle x, y\rangle = 0$. When either $x = 0$ or $y = 0$, we see again from the last equality that either $t = 0$ or $s = 0$ which clearly entails either $a = 0$ or $b = 0$, hence $\langle a, b\rangle = 0$. Therefore, we always have $\langle a, b\rangle = 0$. This completes the proof. ■

Similarly, we show that a penalized version of the generalized FB function is also a C-function of SOCCP. For $p > 1$, define a function

$$\psi_p(a, b) = a + b - (|a|^p + |b|^p)^{\frac{1}{p}} + a_+ b_+$$

and call it *the penalized version of the generalized FB function* ϕ_p (1.9) where a_+ denotes the orthogonal projection of a onto $\mathcal{K}$.

Theorem 3.5. Let $a = (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $b = (s, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$. For $p > 1$, the penalized version of the generalized FB function

$$\psi_p(a, b) = a + b - (|a|^p + |b|^p)^{\frac{1}{p}} + a_+ b_+$$

is still a C-function of SOCCP.

Proof. Assume that $a \geq 0$, $b \geq 0$ and $ab = 0$. Then $a_+ b_+ = ab = 0$ and hence $\psi_p(a, b) = a + b - (a^p + b^p)^{1/p} = \phi_p(a, b) = 0$ from Theorem 3.4. Conversely, suppose that $\psi_p(a, b) = a + b - (|a|^p + |b|^p)^{1/p} + a_+ b_+ = 0$. We can decompose a as $a = a_+ - a_-$ where $a_- = (-a)_+$. Multiplying by a_- both sides, we have

$$a_-\{(a + b - (|a|^p + |b|^p)^{1/p}) + a_+ b_+\} = 0$$

or from $a_+ a_- = 0$

$$\{-(a_-)^2 - a_-[(|a|^p + |b|^p)^{1/p} - b]\} + a_-(a_+ b_+) = 0.$$

Applying the trace operator to both sides yields that

$$-\{\mathrm{tr}((a_-)^2) + \mathrm{tr}(a_-[(|a|^p + |b|^p)^{1/p} - b])\} + \mathrm{tr}(a_-(a_+ b_+)) = 0.$$

Since both $(|a|^p + |b|^p)^{1/p} - b$ (by the Löwner-Heinz inequality) and a_- belong to $\mathcal{K}$, and the trace operator is associative, we see that

$$\mathrm{tr}((a_-)^2) = 0,$$

which implies that $a_- = 0$, whence $a \geq 0$. Similarly we obtain $b \geq 0$. Thus $a + b - (a^p + b^p)^{1/p} + ab = 0$. So we have $(a + b + ab)^p = a^p + b^p$. Hence we get $\mathrm{tr}[(a + b + ab)^p] = \mathrm{tr}(a^p) + \mathrm{tr}(b^p)$, that is,

$$\begin{aligned} &(t + s + ts + \langle x, y\rangle - \|(s + 1)x + (t + 1)y\|)^p \\ &+(t + s + ts + \langle x, y\rangle + \|(s + 1)x + (t + 1)y\|)^p \\ &= (t - \|x\|)^p + (t + \|x\|)^p + (s - \|y\|)^p + (s + \|y\|)^p. \end{aligned} \tag{3.13}$$

Now we verify that $\langle a,b\rangle = 0$, i.e., $ab = 0$ by using a similar argument in the proof of Theorem 3.4. Note that $a+b+ab = (t+s+ts+\langle x,y\rangle,\ (s+1)x+(t+1)y) \in \mathcal{K}$, equivalently, $t+s+ts+\langle x,y\rangle \geq \|(s+1)x+(t+1)y\|$. It can be easily checked that

$$\begin{aligned}
&(t+s+ts+\langle x,y\rangle - \|(s+1)x+(t+1)y\|)^p\\
&\quad + (t+s+ts+\langle x,y\rangle + \|(s+1)x+(t+1)y\|)^p\\
&\geq (t+s+ts+\langle x,y\rangle - \|(s+1)x\| + \|(t+1)y\|)^p\\
&\quad + (t+s+ts+\langle x,y\rangle + \|(s+1)x\| - \|(t+1)y\|)^p\\
&\geq (s+1)^p(t-\|x\|)^p + (s+\langle x,y\rangle + (t+1)\|y\|)^p\\
&\quad + (t+1)^p(s-\|y\|)^p + (t+\langle x,y\rangle + (s+1)\|x\|)^p\\
&\geq (t-\|x\|)^p + (s+\|y\|)^p + (s-\|y\|)^p + (t+\|x\|)^p.
\end{aligned}$$

The first inequality follows from Lemma 3.3, and the second one comes from Lemma 3.2. By (3.13), we get

$$\begin{aligned}
&(t+s+ts+\langle x,y\rangle - \|(s+1)x+(t+1)y\|)^p\\
&\quad + (t+s+ts+\langle x,y\rangle + \|(s+1)x+(t+1)y\|)^p\\
&= (t+s+ts+\langle x,y\rangle - \|(s+1)x\| + \|(t+1)y\|)^p\\
&\quad + (t+s+ts+\langle x,y\rangle + \|(s+1)x\| - \|(t+1)y\|)^p\\
&= (s+1)^p(t-\|x\|)^p + (s+\langle x,y\rangle + (t+1)\|y\|)^p\\
&\quad + (t+1)^p(s-\|y\|)^p + (t+\langle x,y\rangle + (s+1)\|x\|)^p\\
&= (t-\|x\|)^p + (s+\|y\|)^p + (s-\|y\|)^p + (t+\|x\|)^p.
\end{aligned}$$

We first assume that two vectors $x,\ y$ are nonzero. From the first equality above, we obtain $(s+1)x = \alpha(t+1)y$ for some $\alpha \leq -1$ by Lemmas 3.1 and 3.3. Moreover, it follows directly from the last equality that $t = \|x\|$ and $s = \|y\|$. So we get

$$\langle x,y\rangle = \langle \beta y, y\rangle = \beta\|y\|^2 = -\|\beta y\|\|y\| = -\|x\|\|y\| = -ts$$

where $\beta = \alpha\frac{t+1}{s+1} < 0$. Thus, $\langle a,b\rangle = ts+\langle x,y\rangle = 0$. When either $x = 0$ or $y = 0$, we see again from the last equality that either $t = \|x\| = 0$ or $s = \|y\| = 0$ which clearly entails either $a = 0$ or $b = 0$, hence $\langle a,b\rangle = 0$. This completes the proof. ■

Proposition 3.6. Let $a = (t,x) \in \mathbb{R}\times\mathbb{R}^{n-1}$ and $b = (s,y) \in \mathbb{R}\times\mathbb{R}^{n-1}$. For $p \geq 1$, we have

$$\text{tr}[(|a|^p + |b|^p)^{1/p}] \leq \text{tr}(|a|) + \text{tr}(|b|). \tag{3.14}$$

Proof. The inequality (3.14) is equivalent to the following:

$$\text{tr}[(a+b)^\alpha] \leq \text{tr}(a^\alpha) + \text{tr}(b^\alpha) \quad \text{for all} \quad a \geq 0 \text{ and } b \geq 0 \text{ where } \alpha = 1/p. \tag{3.15}$$

When $p = 1$, (3.14) is trivial so we may assume that $p > 1$.

$$\begin{aligned}\text{tr}[(a+b)^\alpha] &= (t+s-\|x+y\|)^\alpha + (t+s+\|x+y\|)^\alpha \\ &\le (t+s-\|x\|+\|y\|)^\alpha + (t+s+\|x\|-\|y\|)^\alpha \\ &\le (t-\|x\|)^\alpha + (s+\|y\|)^\alpha + (t+\|x\|)^\alpha + (s-\|y\|)^\alpha \\ &= \text{tr}(a^\alpha) + \text{tr}(b^\alpha).\end{aligned}$$

The first inequality comes from Lemma 3.3, and the second inequality follows from Lemma 3.2. ∎

Definition. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be $R_{01} - function$ if for any sequence $\{x^k\}$ such that

$$\|x^k\| \to \infty, \quad \frac{(-x^k)_+}{\|x^k\|} \to 0, \quad \frac{(-F(x^k))_+}{\|x^k\|} \to 0, \tag{3.16}$$

we have

$$\liminf_{k\to\infty} \frac{\langle x^k,\, F(x^k)\rangle}{\|x^k\|^2} > 0.$$

Proposition 3.7. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a R_{01}-function. For any sequence $\{x^k\}$ satisfying $\|x^k\| \to \infty,\ \limsup_{k\to\infty} \|x^k_-\| < \infty$ and $\limsup_{k\to\infty} \|F(x^k)_-\| < \infty$, we have

$$\langle x^k_+,\, F(x^k)_+\rangle \to \infty.$$

Proof. Since $(-x)_+ = x_-$ for every $x \in \mathbb{R}^n$, (3.16) is satisfied for the given sequence $\{x^k\}$. As F is a R_{01}-function, we have

$$\liminf_{k\to\infty} \frac{\langle x^k,\, F(x^k)\rangle}{\|x^k\|^2} > 0.$$

Hence

$$\langle x^k,\, F(x^k)\rangle \to \infty.$$

Decomposing $x^k = x^k_+ - x^k_-$ and $F(x^k) = F(x^k)_+ - F(x^k)_-$ yields that

$$\langle x^k,\, F(x^k)\rangle = \langle x^k_+,\, F(x^k)_+\rangle + \langle x^k_-,\, F(x^k)_-\rangle - \langle x^k_+,\, F(x^k)_-\rangle - \langle x^k_-,\, F(x^k)_+\rangle.$$

Therefore we conclude that $\langle x^k_+,\, F(x^k)_+\rangle \to \infty$. This completes the proof. ∎

Proposition 3.8. The penalized version of the generalized FB function

$$\psi_p(a,b) = a + b - (|a|^p + |b|^p)^{\frac{1}{p}} + a_+ b_+$$

satisfies

$$\|\psi_p(a,b)\| \ge \max\{\|a_-\|, \|b_-\|\}, \quad \forall a, b \in \mathbb{R}^n.$$

Proof. By taking the inner product with $-a_-$, we have

$$\langle -a_-,\ a+b-(|a|^p+|b|^p)^{1/p}+a_+b_+\rangle = \|a_-\|^2+\langle a_-,\ (|a|^p+|b|^p)^{1/p}-b\rangle \geq \|a_-\|^2$$

because both $(|a|^p+|b|^p)^{1/p}-b$ and a_- belong to $\overline{\Omega}$, $a_+a_- = 0$, and the inner product is associative with respect to the Jordan product. By the Cauchy-Schwarz inequality, we get

$$\|a_-\|\|\psi_p(a,b)\| \geq \langle -a_-,\ a+b-(|a|^p+|b|^p)^{1/p}+a_+b_+\rangle \geq \|a_-\|^2,$$

and hence $\|\psi_p(a,b)\| \geq \|a_-\|$. Similarly, we have $\|\psi_p(a,b)\| \geq \|b_-\|$. ∎

Proposition 3.9. Let $a, b \in \mathbb{R}^n$. If $\max\{\|a_-\|, \|b_-\|\} < C_0$, then

$$\mathrm{tr}(a+b-(|a|^p+|b|^p)^{\frac{1}{p}}) > -8C_0.$$

Proof. Observe that $\mathrm{tr}(a_-) \leq 2\|a_-\| < 2C_0$, $\ \mathrm{tr}(b_-) \leq 2\|b_-\| < 2C_0$. The trace inequality (3.14) yields

$$\begin{aligned}\mathrm{tr}(a+b-(|a|^p+|b|^p)^{1/p}) &= \mathrm{tr}(a+b)-\mathrm{tr}(|a|^p+|b|^p)^{1/p}\\ &\geq \mathrm{tr}(a+b)-\mathrm{tr}(|a|+|b|)\\ &= -2\mathrm{tr}(a_-)-2\mathrm{tr}(b_-) > -8C_0.\end{aligned}$$ ∎

Now the boundedness of level sets of the merit function $\|\Psi_p(x)\| = \|\psi_p(x, F(x))\|$ is proved, which is the main result of this paper.

Theorem 3.10. Let $p > 1$. The level set $M = \{x \in V \mid \|\Psi_p(x)\| \leq C\}$ is bounded provided that F is R_{01}-function.

Proof. It is sufficient to prove that $\|\Psi_p(x^k)\| \to \infty$ as $\|x^k\| \to \infty$. If $||x^k_-|| \to \infty$ or $||F(x^k)_-|| \to \infty$, the result holds by Proposition 3.8. Suppose that $\limsup_{k\to\infty} ||x^k_-|| < \infty$ and $\limsup_{k\to\infty} ||F(x^k)_-|| < \infty$. So there is a C_0 with $\max\{||x^k_-||, ||F(x^k)_-||\} < C_0$. By Proposition 3.7, we obtain

$$\mathrm{tr}(x^k_+F(x^k)_+) = 2\langle x^k_+,\ F(x^k)_+\rangle \to \infty.$$

Then we have

$$\begin{aligned}||\Psi_p(x^k)|| &\geq \frac{1}{2}\mathrm{tr}\{x^k+F(x^k)-(|x^k|^p+|F(x^k)|^p)^{1/p}+x^k_+F(x^k)_+\}\\ &= \frac{1}{2}\mathrm{tr}\{x^k+F(x^k)-(|x^k|^p+|F(x^k)|^p)^{1/p}\}+\frac{1}{2}\mathrm{tr}(x^k_+F(x^k)_+).\end{aligned}$$

It follows from Proposition 3.9 that

$$\liminf_{n\to\infty}\ \mathrm{tr}\{x^k+F(x^k)-(|x^k|^p+|F(x^k)|^p)^{1/p}\} > -\infty.$$

Therefore we conclude $||\Psi_p(x^k)|| \to \infty$ as $k \to \infty$. ∎

Remark. Using a similar argument, we can show that Theorem 3.10 still holds true under the strict feasibility condition on F.

4. Concluding Remarks

The results of the previous section in a sense only begin the theoretical study of the merit function for SOCCP based on the penalized version of the generalized Fischer-Bermeister function. So the next logical step in future research is to analyse semismoothness or differentiablity of the merit function, and to report its numerical performances as in [10]. Also there is an enough motivation to develop the corresponding extensions to symmetric cones.

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