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ALMOST SURE AND COMPLETE CONVERGENCE OF RANDOMLY WEIGHTED SUMS OF INDEPENDENT RANDOM ELEMENTS IN BANACH SPACES

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Abstract. This work develops almost sure and complete convergence of randomly weighted sums of independent random elements in a real separable Banach space. Sufficient conditions for almost sure convergence and complete convergence in the sense defined by Hsu and Robbins are provided. Examples showing that the conditions cannot be removed or weakened are given. It is also demonstrated that some of the known theorems in the literature are special cases of our results.

1. INTRODUCTION

According to Hsu and Robbins [9], a sequence of real-valued random variables $\{X_n, n \ge 1\}$ *converges completely* to 0 if,

$$\sum_{n=1}^{\infty} P\big(|X_n| > \varepsilon\big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{1.1}$$

Based on this definition, Hsu and Robbins [9] and Erdös [7] derived a necessary and sufficient condition for a sequence of independent identically distributed (i.i.d.) random variables $\{X_n,\, n \ge 1\}$. The assertion is: $EX_1 = 0$ and $E|X_1|^2 < \infty$ if and only if

$$\sum_{n=1}^{\infty} P\Big(\big|\sum_{i=1}^{n} X_i\big| > \varepsilon n\Big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{1.2}$$

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Their theorem was extended by Baum and Katz [4] to establish a rate of convergence in the sense of Marcinkiewicz-Zygmund's strong law of large numbers, which reads: Let $\alpha > 1/2$, $p \geq 1$, and $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. Then $EX_1 = 0$ and $E|X_1|^p < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(|\sum_{i=1}^{n} X_i| > \varepsilon n^{\alpha}\Big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{1.3}$$

The results have been generalized and extended in several directions. A host of researchers considered the problem in a Banach space setting; see Ahmed et al. [3], Hu et al. [11], Sung [21], Hernández et al. [8], Sung et al. [19, 22], and Wang [25] among others.

Recall that a sequence of random elements $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ is *stochastically dominated* by a random element V if for some constant $C < \infty$,

$$P\{\|V_{ni}\| > t\} \leq CP\{\|V\| > t\}, t \geq 0, n \geq 1, 1 \leq i \leq n. \tag{1.4}$$

Using stochastic dominance, Hu et al. [10] treated triangular arrays of row-wise independent random variables and obtained the following complete convergence result with a Marcinkiewicz-Zygmund type normalization. Let $\{X_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of row-wise independent mean 0 random variables that is stochastically dominated by a random variable X, that is, condition (1.4) is satisfied. If $E|X|^{2t} < \infty$ where $1 \leq t < 2$, then

$$\sum_{n=1}^{\infty} P\Big(|\sum_{i=1}^{n} X_{ni}| > \varepsilon n^{1/t}\Big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{1.5}$$

Subsequently, Wang et al. [25] considered triangular arrays of row-wise independent random elements and obtained the following convergence rate. Let $1/2 < \alpha \leq 1$ and $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of row-wise independent random elements, which is stochastically dominated by a random element V in the sense of condition (1.4) being satisfied. If $E\|V\|^p < \infty$ for some $p > 1$ and

$$\max_{k \leq n} P\Big(\|\sum_{i=1}^{k} V_{ni}\| > \varepsilon n^{\alpha}\Big) \to 0 \text{ as } n \to \infty \quad \text{for all} \quad \varepsilon > 0, \tag{1.6}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(|\sum_{i=1}^{n} V_{ni}| > \varepsilon n^{\alpha}\Big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{1.7}$$

Later, Sung established a complete convergence for weighted sums of random elements in [20]. Let $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of row-wise independent random elements that is stochastically dominated by a random element V. Let $p \geq 1$ and $\{b_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of constants satisfying

(1.8) $$\max_{1\le i\le n} |b_{ni}| = O(n^{-1/p})$$

and

(1.9) $$\sum_{i=1}^{n} b_{ni}^2 = o\Big(\frac{1}{\log n}\Big).$$

If $E\|V\|^{2p} < \infty$ and

(1.10) $$\sum_{i=1}^{n} b_{ni}V_{ni} \to 0 \text{ in probability as } n \to \infty,$$

then

(1.11) $$\sum_{n=1}^{\infty} P\Big(\|\sum_{i=1}^{n} b_{ni}V_{ni}\| > \varepsilon\Big) < \infty \text{ for all } \varepsilon > 0.$$

A number of researchers considered the weighted sums of random elements where the weights are constants; see Ahmed et al. [3], Hu et al. [11], and Sung [21]. In [16], Li et al. improved the main result of Thrum [24]; their result reads: Let $\{V_i, i \ge 1\}$ be a sequence of independent identically distributed random elements with $E\|V_1\|^p < \infty$ for some $p \ge 1$. Let $\{a_{ni}, n \ge 1, 1 \le i \le n\}$ be an array of constants such that $\sup_{n,i} |a_{ni}| < \infty$ and

(1.12) $$\sum_{i=1}^{n} a_{ni}^2 = O(n^{\mu}) \text{ for some } 0 < \mu < \min\{1, 2/p\}.$$

If

(1.13) $$\frac{1}{n^{1/p}} \sum_{i=1}^{n} a_{ni}V_i \to 0 \text{ in probability,}$$

then

(1.14) $$\frac{1}{n^{1/p}} \sum_{i=1}^{n} a_{ni}V_i \to 0 \text{ a.s.}$$

In this paper, we obtain the almost sure convergence and complete convergence for randomly weighted sums of arrays of row-wise independent random elements. Although Cuzick [5], Li and Tomkins [18] and Li et al. [17] also considered almost sure convergence with random weights, to the best of our knowledge, there has yet been any result on the complete convergence of the case in which the weights are random variables. Moreover, the results in the aforementioned papers do not imply our results even in the case the weights are non-random. Most of the results of the aforementioned work in the literature can be covered and improved by our results.

As a motivation, we describe a problem arising in systems theory. Recall that a transfer function is commonly used in the analysis of single-input single-output systems, in signal processing, communication theory, and control theory. Consider a continuous-time, linear, time-invariant system. Then a transfer function is the ratio of the Laplace transforms of the output and input. It gives input/output characteristics in the frequency domain. In certain applications, the underlying systems need to be identified, which can be carried out by estimating the transfer functions using observable data. Suppose that we wish to identify an unknown transfer function $K(\cdot)$. The input is given by $z(\cdot)$, sampling interval is Δ, and output (at sampling time $n\Delta$) is y_n, where

$$y_n = \int_0^1 K(u)z(n\Delta - u)du + \psi_n,$$

where $\{\psi_n\}$ is a sequence of observation noise, which is a sequence of i.i.d. random variables with 0 mean and finite variance and is independent of $z(\cdot)$. To identify or to estimate $K(\cdot)$, we treat this as a stochastic optimization problem in $L_2[0,1]$, which is the Hilbert space consisting of square integral functions defined on the interval $[0,1]$. Let $\langle x, y\rangle$ and $||x|| = [\langle x, x\rangle]^{1/2}$ denote the inner product and the norm on $L_2[0,1]$, respectively. To carry out the desired task, we construct algorithm of the form

$$\widehat{K}_{n+1}(t) = \widehat{K}_n(t) + \varepsilon_n z(n\Delta t)\Big[y_n - \int_0^1 \widehat{K}_n(u)z(n\Delta - u)du\Big]. \tag{1.15}$$

The above algorithm is stochastic approximation type analogs to its finite dimensional counterpart [14]. Define $\delta\widehat{K}_n(t) = \widehat{K}_n(t) - K(t)$, and use $\widehat{K}_n$, $\delta\widehat{K}_n$, K, and z_n denote the random variables whose point values are $\widehat{K}(t)$, $\delta\widehat{K}_n(t)$, $K(t)$, and $z_n(n\Delta - t)$, resp.

$$\widehat{K}_{n+1}(t) = \widehat{K}_n(t) + \varepsilon_n z_n[\langle \delta\widehat{K}_n, z_n\rangle + \psi_n]. \tag{1.16}$$

In the implementation, one often wishes to use a projection algorithm

$$\widehat{K}_{n+1}(t) = \pi\Big[\widehat{K}_n(t) + \varepsilon_n z_n[\langle \delta\widehat{K}_n, z_n\rangle + \psi_n]\Big], \tag{1.17}$$

where $\pi(\cdot)$ is the projection to the unit ball in $L_2[0,1]$. As explained in [13, p. 782], this can be further rewritten as

$$\widehat{K}_{n+1}(t) = \widehat{K}_n(t) + \varepsilon_n z_n[\langle \delta\widehat{K}_n, z_n\rangle + \psi_n] - \varepsilon_n c_n\widehat{K}_n + O_n, \tag{1.18}$$

where c_n satisfies $\varepsilon_n c_n < 1$, and O_n satisfies $\sum_{j=0}^{m(T)} O_j \to 0$, for any $0 < T < \infty$ with $m(t) = \max\{n : t_n \le t\}$ and $t_n = \sum_{j=0}^{n-1} \varepsilon_j$. In analyzing such algorithms, one often needs to deal with a term (see [13, p. 783])

$$\sum_{k=0}^{n} \Big[\prod_{j=k+1}^{n} (1 - \varepsilon_j c_j)\Big]\varepsilon_k z_k(\langle z_k, \delta\widehat{K}_k\rangle + \psi_k).$$

As can be seen that this term is of the form a weighted sum of random variables. Our work in this paper provides a sufficient condition for almost sure convergence and complete convergence of such a sequence.

We also note that many linear models in statistics based on a random sample involve weighted sums of dependent random variables. Examples include least-squares estimators, nonparametric regression function estimators, and jackknife estimates, among others. In this respect, the study of convergence for these weighted sums has impact on statistics, probability, and their applications.

The rest of the paper is arranged as follows. Section 2 presents some definitions and lemmas to be used in the proof of our results. Section 3 provided conditions for almost sure convergence for randomly weighted sums. Our results in this section improve the results by Li et al. [16]. Section 4 establishes complete convergence for randomly weighted sums of arrays of row-wise independent random elements in a real separable Banach space. In the main results, Theorem 3.1, Theorem 3.3, and Theorem 4.1, no assumptions are made concerning the geometry of the underlying Banach space. Finally, in Section 5, some corollaries and examples are presented to illustrate the results. It demonstrates that our conditions are sharp and our results lead to better convergence rates. Throughout this paper, the symbol C denotes a generic positive constant ($0 < C < \infty$) whose values may be different for different appearances.

2. Preliminaries

We begin with definitions, notation, and preliminary results prior to establishing the main results. Let $\mathcal{X}$ be a real separable Banach space with norm $||\cdot||$. A random element in $\mathcal{X}$ will be denoted by V or by V_n, W_n, etc.

The *expected value* or *mean* of a random element V, denoted EV, is defined to be the *Pettis integral* provided it exists. That is, V has expected value $EV \in \mathcal{X}$ if $h(EV) = E(h(V))$ for every $h \in \mathcal{X}^*$, where $\mathcal{X}^*$ denotes the (*dual*) space of all continuous linear functionals on $\mathcal{X}$. If $E||V|| < \infty$, then (see, e.g., Taylor [23, p. 40]) V has an expected value. But the expected value can exist when $E||V|| = \infty$. For an example, see Taylor [23, p. 41].

Lemma 2.1 below is a Marcinkiewicz type inequality for some of independent random elements; see de Acosta [1] for a proof.

Lemma 2.1. *Let $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of row-wise independent random elements. Then for every $r \geq 1$, there is a positive constant C_r depending only on r such that for all $n \geq 1$,*

$$E\Big|\|\sum_{i=1}^{n} V_{ni}\| - E\|\sum_{i=1}^{n} V_{ni}\|\Big|^r \leq C_r \sum_{i=1}^{n} E\|V_{ni}\|^r, \text{ for } 1 \leq r \leq 2, \tag{2.19}$$

and

$$\begin{aligned} &E\Big|\|\sum_{i=1}^{n}V_{ni}\|-E\|\sum_{i=1}^{n}V_{ni}\|\Big|^{r} \\ &\le C_r\Big(\big(\sum_{i=1}^{n}E\|V_{ni}\|^2\big)^{r/2}+\sum_{i=1}^{n}E\|V_{ni}\|^r\Big),\ \text{for } r>2. \end{aligned} \tag{2.20}$$

The following lemma is a Banach space version of Lemma 2.1 of Li at al. [16]. The proof is similar to that of Lemma 2.2 in Hu at al. [11].

Lemma 2.2. *Let $r \ge 1$, and $\{V_n, n \ge 1\}$ be a sequence of random elements and $\{V_n^{'}, n \ge 1\}$ an independent copy of $\{V_n, n \ge 1\}$. If $V_n \to 0$ in probability, then*

(i) $V_n \to 0$ *a.s. if and only if* $V_n - V_n^{'} \to 0$ *a.s.*

(ii) $\sum_{n=1}^{\infty} n^{r-2}P\{\|V_n\| \ge \varepsilon\} < \infty$ *for every* $\varepsilon > 0$ *if and only if* $\sum_{n=1}^{\infty} n^{r-2}P\{\|V_n - V_n^{'}\| \ge \varepsilon\} < \infty$ *for every* $\varepsilon > 0$.

The third lemma due to Hu at al. [11] concerning the relationship between convergence in probability and mean convergence for sums of independent bounded random elements.

Lemma 2.3. *Let $\{V_{ni}, n \ge 1, 1 \le i \le n\}$ be an array of row-wise independent symmetric random elements and suppose there exists $\delta > 0$ such that $\|V_{ni}\| \le \delta$ a.s. for all $n \ge 1$, $1 \le i \le n$. If $\sum_{i=1}^{n} V_{ni} \to 0$ in probability as $n \to \infty$, then $E\big\|\sum_{i=1}^{n} V_{ni}\big\| \to 0$ as $n \to \infty$.*

The following lemma is a result of Adler at al. [2] concerning the expectation of the truncation random elements that are stochastically dominated by a random element V.

Lemma 2.4. *Let $\{V_{ni}, n \ge 1, 1 \le i \le n\}$ be an array of random elements that is stochastically dominated by a random element V. Then for all $p > 0$*

$$\begin{aligned} &E\Big(\|V_{ni}\|^p I(\|V_{ni}\| \le t)\Big) \\ &\le Ct^p P(\|V\|>t)+CE\Big(\|V\|^p I(\|V\|\le t)\Big), t \ge 0, n \ge 1, 1 \le i \le n, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} &E\Big(\|V_{ni}\| I(\|V_{ni}\| > t)\Big) \\ &\le CE\Big(\|V\| I(\|V\|>t)\Big), t\ge 0, n \ge 1, 1 \le i \le n. \end{aligned} \tag{2.22}$$

Finally, we present two exponential inequalities for independent Banach valued random elements. See Sung [20, Lemma 1] for a proof of Lemma 2.5 (see also [15]). The proof of Lemma 2.6 is also similar to that of Sung [20, Lemma 1].

Lemma 2.5. *Let $\{V_i, 1 \le i \le n\}$ be independent random elements such that $\|V_i\| \le A$ a.s. for some constant $A > 0$. Then for all $t > 0$,*

$$E\Big(\exp(t\|\sum_{i=1}^n V_i\|)\Big) \le \exp\Big(tE\|\sum_{i=1}^n V_i\| + 2t^2 \sum_{i=1}^n E\|V_i\|^2 \exp(2tA)\Big).$$

Lemma 2.6. *Let $\{V_i, 1 \le i \le n\}$ be independent random elements such that for each $1 \le i \le n$ $\|V_i\| \le A_i$ a.s. for some constants $A_i > 0$. Then for all $t > 0$,*

$$E\Big(\exp(t\|\sum_{i=1}^n V_i\|)\Big) \le \exp\Big(tE\|\sum_{i=1}^n V_i\| + 4t^2 \sum_{i=1}^n E\|V_i\|^2 g(2tA_i)\Big),$$

where $g(x) = x^{-2}(e^x - 1 - x)$.

3. Almost Sure Convergence

In this section, we establish the almost sure convergence of randomly weighted sums of independent random elements. The first theorem is a new result even when A_{ni} are constants. This theorem improves the cited result of Li et al. [16].

Theorem 3.1. *Let $p \ge 1$, and $\{V_i, i \ge 1\}$ be a sequence of independent random elements that are stochastically dominated by a random element V with $E\|V\|^p < \infty$. Let $\{A_{ni}, n \ge 1, 1 \le i \le n\}$ be an array of row-wise independent random variables such that*

(3.1) $$\sup_{n\ge1, 1\le i\le n} |A_{ni}| \le K \text{ a.s. for some positive constant } K.$$

Assume that for all $n \ge 1$, the random variables $\{A_{ni}, 1 \le i \le n\}$ are independent of $\{V_i, i \ge 1\}$, and

(3.2) $$\lim_{n\to\infty} \frac{1}{n^{1/p}} E\|\sum_{i=1}^n A_{ni}V_i I(\|A_{ni}V_i\| \le n^{1/p}\log^{-1} n)\| = 0.$$

(1) *If $p > 2$ and*

(3.3) $$\sum_{i=1}^n E(|A_{ni}|^2) = o(n^{2/p}\log^{-1} n),$$

then

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} A_{ni} V_i = 0 \text{ a.s.} \tag{3.4}$$

(2) *If* $1 \le p \le 2$ *and*

$$\sum_{i=1}^{n} E(|A_{ni}|^p) = O(n^{\mu}) \text{ for some } 0 < \mu < 1, \tag{3.5}$$

then (3.4) *holds.*

(3) *If* $1 \le p \le 2$, *and* (3.1) *and* (3.2) *are respectively replaced by*

$$\begin{aligned} &\max_{1\le i\le n} |A_{ni}| \\ &\le K/\log n \text{ a.s. for some contant } K \text{ and} \\ &\lim_{n\to\infty} \frac{1}{n^{1/p}} E\|\sum_{i=1}^{n} A_{ni} V_i I(\|V_i\| \le n^{1/p})\| = 0, \end{aligned} \tag{3.6}$$

then (3.4) *holds.*

Proof. For $\varepsilon > 0$, we choose a positive integer N (to be specialized later), and let

$$X_{ni} = V_i I(\|A_{ni} V_i\| \le n^{1/p} \log^{-1} n),$$

$$Y_{ni} = V_i I(n^{1/p} \log^{-1} n < \|A_{ni} V_i\| \le n^{1/p} \frac{\varepsilon}{N}),$$

and

$$Z_{ni} = V_i I(\|A_{ni} V_i\| > n^{1/p} \frac{\varepsilon}{N}).$$

Proof of part (1) . For all $t > 0$, by Lemma 2.5

$$\begin{aligned} &P(\|\sum_{i=1}^{n} A_{ni} X_{ni}\| > n^{1/p}\varepsilon) \\ &\le \exp(-tn^{1/p}\varepsilon) E\Big(\exp(t\|\sum_{i=1}^{n} A_{ni} X_{ni}\|)\Big) \quad \text{(by the Markov inequality)} \\ &\le \exp(-tn^{1/p}\varepsilon) \\ &\quad \exp\Big(tE\|\sum_{i=1}^{n} A_{ni} X_{ni}\| + 2t^2 \sum_{i=1}^{n} E\|A_{ni} X_{ni}\|^2 \exp(2tn^{1/p} \log^{-1} n)\Big). \end{aligned} \tag{3.7}$$

Since $E\|V\|^p < \infty$, $p > 2$, we have from (3.1) and (3.3) that

$$\sum_{i=1}^{n} E\|A_{ni} X_{ni}\|^2 \le C \sum_{i=1}^{n} E|A_{ni}|^2 = o(n^{2/p} \log^{-1} n). \tag{3.8}$$

Putting $t = \dfrac{2n^{-1/p}\log n}{\varepsilon}$, it follows from (3.2), (3.7) and (3.8) that there exists $1 < \alpha < 2$ such that

$$P(\|\sum_{i=1}^{n} A_{ni}X_{ni}\| > n^{1/p}\varepsilon) \le C\frac{1}{n^{\alpha}} \text{ and for all large } n. \tag{3.9}$$

Therefore

$$\sum_{n=1}^{\infty} P(\|\sum_{i=1}^{n} A_{ni}X_{ni}\| > n^{1/p}\varepsilon) < \infty. \tag{3.10}$$

Now, we use the method in [12]. From the definition of Y_{ni},

$$\begin{aligned}
&P(\sum_{i=1}^{n} \|A_{ni}Y_{ni}\| > n^{1/p}\varepsilon)\\
&\le P(\text{there are at least } N \text{ such } i\text{'s that } \|A_{ni}V_i\| > n^{1/p}\log^{-1} n)\\
&\le \sum_{1\le i_1<\cdots<i_N\le n} P\Big(\|A_{ni_1}V_{i_1}\| > n^{1/p}\log^{-1} n, \ldots,\\
&\qquad \|A_{ni_N}V_{i_N}\| > n^{1/p}\log^{-1} n\Big)\\
&\le \Big(\sum_{i=1}^{n} P(\|A_{ni}V_i\| > n^{1/p}\log^{-1} n)\Big)^N\\
&\le \Big(\sum_{i=1}^{n} \frac{E\|A_{ni}V_i\|^p}{n}\log^p n\Big)^N\\
&\le \Big(K^{p-2}\sum_{i=1}^{n} \frac{E|A_{ni}|^2 E\|V_i\|^p}{n}\log^p n\Big)^N\\
&\le \Big(C\sum_{i=1}^{n} \frac{E|A_{ni}|^2}{n}\log^p n\Big)^N\\
&\le C\Big(\frac{\log^p n}{n^{1-2/p}}\Big)^N.
\end{aligned} \tag{3.11}$$

Choose N large enough such that $(1-2/p)N > 1$, we get from (3.11) that

$$\sum_{n=1}^{\infty} P(\|\sum_{i=1}^{n} A_{ni}Y_{ni}\| > n^{1/p}\varepsilon) < \infty. \tag{3.12}$$

Now, we note that $E\|V\|^p < \infty$ implies $\sum_{i=1}^{\infty} P(\|V_i\| > i^{1/p}) < \infty$, so by the Borel-Cantelli lemma,

$$\sum_{i=1}^{n} \|Z_{ni}\| = O(1) \text{ a.s.} \tag{3.13}$$

It follows from (3.1) and (3.13) that

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \Big\| \sum_{i=1}^{n} A_{ni} Z_{ni} \Big\| \le \lim_{n\to\infty} \frac{K}{n^{1/p}} \sum_{i=1}^{n} \|Z_{ni}\| = 0 \text{ a.s.} \tag{3.14}$$

The conclusion (3.4) follows from (3.10), (3.12), and (3.14).

Proof of part (2). We see that (3.8) remains true since $\mu < 1$ and

$$\begin{aligned} &\sum_{i=1}^{n} E\|A_{ni}X_{ni}\|^2 \\ &\le \Big(\sum_{i=1}^{n} E\|A_{ni}X_{ni}\|^p\Big) n^{-1+2/p} \log^{p-2} n \\ &\le C\Big(\sum_{i=1}^{n} E|A_{ni}|^p\Big) n^{-1+2/p} \log^{p-2} n \\ &\le C n^{\mu-1+2/p} \log^{p-2} n. \end{aligned} \tag{3.15}$$

Therefore (3.9) remains true. The rest of the argument is similar to that of (3.12) and (3.14).

Proof of part (3). Let $g(x) = x^{-2}(e^x - 1 - x)$ and

$$U_i = V_i I(\|V_i\| \le i^{1/p}), \ W_i = V_i - U_i.$$

For all $t > 0$, by Lemma 2.6 and the first part of (3.6),

$$\begin{aligned} &E\Big(\exp(t\|\sum_{i=1}^{n} A_{ni}U_i\|)\Big) \\ &\le \exp\Big(tE\|\sum_{i=1}^{n} A_{ni}U_i\| + 4t^2 \sum_{i=1}^{n} E\|A_{ni}U_i\|^2 g(2tn^{1/p}K/\log n)\Big) \\ &\le \exp\Big(tE\|\sum_{i=1}^{n} A_{ni}U_i\| + 4t^2 g(2tn^{1/p}K/\log n) \\ &\quad \sum_{i=1}^{n} E|A_{ni}|^2 E\|U_i\|^p n^{2/p-1}\Big) \\ &\le \exp\Big(tE\|\sum_{i=1}^{n} A_{ni}U_i\| + Ct^2 n^{2/p}(K/\log n)^2 g(2tn^{1/p}K/\log n)\Big) \\ &\le \exp\Big(tE\|\sum_{i=1}^{n} A_{ni}U_i\| + C\exp(2tn^{1/p}K/\log n)\Big) \\ &\quad (\text{since } x^2 g(x) < e^x \text{ for } x > 0). \end{aligned} \tag{3.16}$$

For all $\varepsilon > 0$, it follows from (3.16) and the Markov inequality that

$$\begin{aligned} &P(\|\sum_{i=1}^{n} A_{ni}U_i\| > n^{1/p}\varepsilon) \\ &\le \exp(-tn^{1/p}\varepsilon) \exp\Big(tE\|\sum_{i=1}^{n} A_{ni}U_i\| + C\exp(2Ktn^{1/p}/\log n)\Big). \end{aligned} \tag{3.17}$$

Putting $t = \dfrac{3n^{-1/p}\log n}{\varepsilon}$, it follows from (3.17) and the second part of (3.6) that

$$P(\|\sum_{i=1}^{n} A_{ni}U_i\| > n^{1/p}\varepsilon) \le C\frac{1}{n^2} \text{ and for all large } n. \tag{3.18}$$

Therefore

$$\sum_{n=1}^{\infty} P(\|\sum_{i=1}^{n} A_{ni}U_i\| > n^{1/p}\varepsilon) < \infty. \tag{3.19}$$

By the Borel-Cantelli lemma,

$$n^{-1/p}\|\sum_{i=1}^{n} A_{ni}U_i\| \to 0 \text{ a.s. as } n \to \infty. \tag{3.20}$$

By the argument similar to that of (3.13), we have $\sum_{i=1}^{n}\|W_i\|$ is bounded a.s. This implies

$$n^{-1/p}\|\sum_{i=1}^{n} A_{ni}W_i\| \le \frac{K}{n^{1/p}\log n}\sum_{i=1}^{n}\|W_i\| \to 0 \text{ a.s. as } n \to \infty. \tag{3.21}$$

Conclusion (3.4) follows from (3.20) and (3.21). ∎

Remark 3.2. We note:

(i) When $A_{ni} \equiv a_{ni}$ a.s., where $\{a_{ni}, n \ge 1, 1 \le i \le n\}$ is a bounded array of constants, then both condition (3.2) and the second part of (3.6) can be replaced by

$$n^{-1/p}\sum_{i=1}^{n} a_{ni}V_i \to 0 \text{ in probability as } n \to \infty. \tag{3.22}$$

The proof is standard (see Remark 4.3).

(ii) When $p > 2$, the condition (1.12) of Li et al. [16] is stronger than (3.3). So, Theorem 3.1 improves the cited result of Li et al. [16] even when the non-randomly weighted case.

(iii) We now compare Part (1) of Theorem 3.1 with result of Sung [20] cited in Section (when $p > 2$). When $A_{ni} \equiv a_{ni} \equiv n^{1/p}b_{ni}$, (3.1), (3.3) and (3.22) coincide with (1.8), (1.9), (1.10), respectively. Sung [20] obtained the complete convergence of $\sum_{i=1}^{n} b_{ni}V_{ni}$ under condition $E\|V\|^{2p} < \infty$ while in Theorem 3.1, we obtain the almost sure convergence $\sum_{i=1}^{n} b_{ni}V_i$ under condition $E\|V\|^p < \infty$.

(iv) Condition (3.5) is strong, but as we can see in Part (3), when (3.1) is strengthened to $\max_{1\le i\le n}|A_{ni}| \le K/\log n$ a.s., (3.5) can be removed. According to a result by Cuzick [5, Corollary 2.1 and Lemma 2.1], Part (3) does not hold if $\max_{1\le i\le n}|A_{ni}| \le K/\log n$ a.s. is weakened to (3.1).

The proof of the following theorem is similar to that of Part (1) of Theorem 3.1.

Theorem 3.3. *Let $p > 2$, and $\{V_i,\, i \ge 1\}$ be a sequence of independent random elements that are stochastically dominated by a random element V with $E\|V\|^p < \infty$. Let $\{A_{ni},\, n \ge 1, 1 \le i \le n\}$ be an array of row-wise independent random variables such that* (3.1) *holds. Assume that for all $n \ge 1$, the random variables $\{A_{ni},\, 1 \le i \le n\}$ are independent of $\{V_i,\, i \ge 1\}$, and* (3.2) *holds. If*

$$\sum_{i=1}^{n} E(|A_{ni}|^2) = O(n^{2/p}\log^{-1} n), \tag{3.23}$$

then

$$\limsup_{n\to\infty} \frac{1}{n^{1/p}} \|\sum_{i=1}^{n} A_{ni}V_i\| < \infty \text{ a.s.} \tag{3.24}$$

4. Complete Convergence

The following theorem provides conditions for complete convergence of randomly weighted sums of row-wise independent random elements in Banach spaces. Examples are provided in Section 5 to show that these conditions cannot be dispensed with or weakened. No assumptions are made concerning the geometry of the underlying Banach space.

Theorem 4.1. *Let $\alpha > 1/2$, $1 \le p < 2$, $\{V_{ni},\, n \ge 1, 1 \le i \le n\}$ be an array of row-wise independent random elements that are stochastically dominated by a random element V with $E\|V\|^p < \infty$. Let $\{A_{ni},\, n \ge 1, 1 \le i \le n\}$ be an array of row-wise independent random variables. Assume that for all $n \ge 1$, the random variables $\{A_{ni},\, 1 \le i \le n\}$ are independent of $\{V_{ni},\, 1 \le i \le n\}$. If*

$$\sum_{i=1}^{n} E(|A_{ni}|^2) = O(n), \tag{4.1}$$

and

$$\lim_{n\to\infty} n^{-\alpha} E\Big\|\sum_{i=1}^{n} A_{ni}V_{ni} I(\|V_{ni}\| \le n^{\alpha})\Big\| = 0, \tag{4.2}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\|\sum_{i=1}^{n} A_{ni}V_{ni}\| > \varepsilon n^{\alpha}\Big) < \infty \quad \text{for all } \varepsilon > 0. \tag{4.3}$$

Proof. Noting that (4.3) is automatic if $\alpha p < 1$. So, we assume that $\alpha p \geq 1$. For $n \geq 1$, set

$$W_{ni} = V_{ni} I(\|V_{ni}\| \leq n^{\alpha}),\ 1 \leq i \leq n.$$

For all $\varepsilon > 0$,

$$\begin{aligned} &P\Big(\|\sum_{i=1}^{n} A_{ni} V_{ni}\| > \varepsilon n^{\alpha}\Big) \\ &\leq P\Big(\max_{1\leq i\leq n} \|V_{ni}\| > n^{\alpha}\Big) + P\Big(\|\sum_{i=1}^{n} A_{ni} W_{ni}\| > \varepsilon n^{\alpha}\Big). \end{aligned} \tag{4.4}$$

To obtain (4.3), it remains to show that

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\Big(\max_{1\leq i\leq n} \|V_{ni}\| > n^{\alpha}\Big) < \infty, \tag{4.5}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\Big(\|\sum_{i=1}^{n} A_{ni} W_{ni}\| > \varepsilon n^{\alpha}\Big) < \infty. \tag{4.6}$$

First, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leq i\leq n} \|V_{ni}\| > n^{\alpha}\Big) \\ &\leq \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} P\Big(\|V_{ni}\| > n^{\alpha}\Big) \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1} P(\|V\| > n^{\alpha}) \\ &= C \sum_{n=1}^{\infty} n^{\alpha p-1} \sum_{i=n}^{\infty} P(i^{\alpha} < \|V\| \leq (i+1)^{\alpha}) \\ &= C \sum_{i=1}^{\infty} \sum_{n=1}^{i} n^{\alpha p-1} P(i^{\alpha} < \|V\| \leq (i+1)^{\alpha}) \\ &\leq C \sum_{i=1}^{\infty} i^{\alpha p} P(i^{\alpha} < \|V\| \leq (i+1)^{\alpha}) \\ &\leq C E\|V\|^{p} < \infty. \end{aligned} \tag{4.7}$$

This proves (4.5). Note that it follows from (4.2) that for all large n,

$$n^{-\alpha} E\Big\|\sum_{i=1}^{n} A_{ni} V_{ni} I(\|V_{ni}\| \leq n^{\alpha})\Big\| < \varepsilon/2. \tag{4.8}$$

So, to prove (4.6), it suffice to show that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\Big|\|\sum_{i=1}^{n} A_{ni}W_{ni}\| - E\|\sum_{i=1}^{n} A_{ni}W_{ni}\|\Big| > n^{\alpha}\varepsilon/2\Big) < \infty. \tag{4.9}$$

We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\Big|\|\sum_{i=1}^{n} A_{ni}W_{ni}\| - E\|\sum_{i=1}^{n} A_{ni}W_{ni}\|\Big| > n^{\alpha}\varepsilon/2\Big)\\
&\le \frac{4}{\varepsilon^2}\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} E\Big(\Big|\|\sum_{i=1}^{n} A_{ni}W_{ni}\| - E\|\sum_{i=1}^{n} A_{ni}W_{ni}\|\Big|^2\Big)\\
&\qquad \text{(by Chebyshev's inequality)}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\sum_{i=1}^{n} E\Big\|A_{ni}W_{ni}\Big\|^2 \quad \text{(by Lemma 2.1 with } r=2)\\
&= C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\sum_{i=1}^{n} E|A_{ni}|^2 E\|W_{ni}\|^2 \quad \text{(since } A_{ni} \text{ is independent of } V_{ni})\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\Big(\sum_{i=1}^{n} E|A_{ni}|^2\Big)\Big(n^{2\alpha}P(\|V\|>n^{\alpha}) + E\|VI(\|V\|\le n^{\alpha})\|^2\Big)\\
&\qquad \text{(by (2.21))}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} E\|VI(\|V\|\le n^{\alpha})\|^2 + C\sum_{n=1}^{\infty} n^{\alpha p-1}P(\|V\|>n^{\alpha}) \text{ (by (4.1))}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha}\sum_{i=1}^{n} E\Big(\|V\|^2 I\big((i-1)^{\alpha} < \|V\| \le i^{\alpha}\big)\Big) + CE\|V\|^p \text{ (by (4.7))}\\
&= C\sum_{i=1}^{\infty}\sum_{n=i}^{\infty} n^{\alpha p-1-2\alpha} E\Big(\|V\|^2 I\big((i-1)^{\alpha} < \|V\| \le i^{\alpha}\big)\Big) + CE\|V\|^p\\
&\le C\sum_{i=1}^{\infty} i^{\alpha p-2\alpha} E\big(\|V\|^2 I((i-1)^{\alpha} < \|V\| \le i^{\alpha})\big) + CE\|V\|^p\\
&\le C\sum_{i=1}^{\infty} i^{\alpha p} P\big((i-1)^{\alpha} < \|V\| \le i^{\alpha}\big) + CE\|V\|^p\\
&\le CE\|V\|^p < \infty.
\end{aligned}$$

This proves (4.9). The proof of the theorem is completed. ∎

Remark 4.2.

(i) When $\alpha > 1$, condition (4.2) is automatically satisfied. To see this, for all $n \ge 1$,

$$\Big(\sum_{i=1}^{n} E|A_{ni}|\Big)^2 \le \Big(\sum_{i=1}^{n}(EA_{ni}^2)^{1/2}\Big)^2 \text{ (by Jensen's inequality)}$$
$$\le n\sum_{i=1}^{n} EA_{ni}^2.$$

So, it follows from (4.1) that

$$\sum_{i=1}^{n} E|A_{ni}| = O(n). \tag{4.10}$$

Hence

$$0 \le \lim_{n\to\infty} n^{-\alpha} E\Big\|\sum_{i=1}^{n} A_{ni}V_{ni}I(\|V_{ni}\| \le n^{\alpha})\Big\|$$
$$\le \lim_{n\to\infty} n^{-\alpha}\sum_{i=1}^{n} E\|A_{ni}V_{ni}I(\|V_{ni}\| \le n^{\alpha})\|$$
$$\le \lim_{n\to\infty} Cn^{-\alpha}\Big(\sum_{i=1}^{n} E|A_{ni}|\Big)\Big(E\|VI(\|V\| \le n^{\alpha})\| + n^{\alpha}P(\|V\| > n^{\alpha})\Big)$$
(by (2.21))
$$\le \lim_{n\to\infty} Cn^{1-\alpha}E\|V\| \text{ (by (4.10))}$$
$$= 0 \text{ (since } \alpha > 1 \text{ and } E\|V\| < \infty).$$

(ii) When $\mathcal{X}$ is a real separable Hilbert space and $EV_{ni} \equiv 0$, by similar method (using the identity $E(\sum_{i=1}^{n} V_{ni})^2 = \sum_{i=1}^{n} EV_{ni}^2$ instead of Lemma 2.1) we can prove that Theorem 4.1 still holds without (4.2).

Remark 4.3. When $A_{ni} \equiv a_{ni}$ a.s., where $\{a_{ni}, n \ge 1, 1 \le i \le n\}$ is an array of constants satisfying

$$\sup_{n\ge1,\ i\ge1} |a_{ni}| < \infty, \tag{4.11}$$

then the condition (4.1) is satisfied automatically. Moreover, the condition (4.2) can be replaced by

$$n^{-\alpha}\sum_{i=1}^{n} a_{ni}V_{ni} \to 0 \text{ in probability as } n \to \infty. \tag{4.12}$$

To see this, by Lemma 2.2 and (4.12), we can assume that for each $n \ge 1$, $\{V_{ni}, 1 \le i \le n\}$ are symmetric random elements. By Remark 4.2, we need only prove (4.2) when $0 < \alpha \le 1$. We have

$$
\begin{aligned}
0 &\le \lim_{n\to\infty} n^{-\alpha} E\Big\|\sum_{i=1}^{n} a_{ni}V_{ni}I(\|V_{ni}\|>n^{\alpha})\Big\| \\
&\le \lim_{n\to\infty} n^{-\alpha}\sum_{i=1}^{n} E\|a_{ni}V_{ni}I(\|V_{ni}\|>n^{\alpha})\| \\
&\le \lim_{n\to\infty} Cn^{-\alpha}\Big(\sum_{i=1}^{n}|a_{ni}|\Big)E\|VI(\|V\|>n^{\alpha})\| \text{ (by (2.22))} \\
&\le \lim_{n\to\infty} Cn^{1-\alpha}E\|VI(\|V\|>n^{\alpha})\| \text{ (by (4.11))} \\
&\le \lim_{n\to\infty} CE\Big(\|V\|^{1/\alpha}I(\|V\|>n^{\alpha})\Big) \\
&\le \lim_{n\to\infty} C\Big(E\big(\|V\|^{p}I(\|V\|>n^{\alpha})\big)\Big)^{1/\alpha p} \\
&\qquad \text{(by Jensen's inequality and noting that } \alpha p\ge 1) \\
&= 0 \text{ (since } E\|V\|^{p}<\infty).
\end{aligned}
$$

It follows that

$$
n^{-\alpha}\sum_{i=1}^{n} a_{ni}V_{ni}I(\|V_{ni}\|>n^{\alpha}) \to 0 \text{ in probability as } n\to\infty. \tag{4.13}
$$

Combining (4.12) and (4.13), we have

$$
n^{-\alpha}\sum_{i=1}^{n} a_{ni}V_{ni}I(\|V_{ni}\|\le n^{\alpha}) \to 0 \text{ in probability as } n\to\infty. \tag{4.14}
$$

Now, (4.11) implies that there exists a constant C such that

$$
\Big\|n^{-\alpha}a_{ni}V_{ni}I(\|V_{ni}\|\le n^{\alpha})\Big\| \le |a_{ni}| \le C \text{ a.s. for all } n\ge 1,\ 1\le i\le n.
$$

Hence by Lemma 2.3 and (4.14), (4.2) follows.

Remark 4.4. If $p\ge 2$, then Theorem 4.1 still hold if (4.1) is replaced by

$$
\sum_{i=1}^{n} E(|A_{ni}|^{q}) = O(n) \text{ for some } q > 2(\alpha p-1)/(2\alpha-1). \tag{4.15}
$$

The proof is similar to that of Theorem 4.1, by applying Lemma 2.1 with $r=q/2$. The verbatim argument is omitted.

Now, if we consider $V_{ni}\to a_{ni}V_{ni}$ as a linear operator $f_{ni}:\mathcal{X}\to\mathcal{X}$, then we have the following theorem. The proof is exactly as that of Theorem 4.1 and Remark 4.4.

Theorem 4.5. *Let $\alpha > 1/2$, $p \geq 1$, $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of row-wise independent random elements that are stochastically dominated by a random element V with $E\|V\|^p < \infty$. Let $\{f_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of continuous linear operators from $\mathcal{X}$ to $\mathcal{X}$. If*

$$\lim_{n\to\infty} n^{-\alpha} E\Big\|\sum_{i=1}^{n} f_{ni}(V_{ni}) I(\|V_{ni}\| \leq n^{\alpha})\Big\| = 0, \tag{4.16}$$

and if either

$$1 \leq p < 2, \ \sum_{i=1}^{n} \|f_{ni}\|^2 = O(n), \tag{4.17}$$

or

$$p \geq 2, \ \sum_{i=1}^{n} \|f_{ni}\|^q = O(n) \ \text{ for some } q > 2(\alpha p - 1)/(2\alpha - 1), \tag{4.18}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\|\sum_{i=1}^{n} f_{ni}(V_{ni})\| > \varepsilon n^{\alpha}\Big) < \infty \ \text{ for all } \ \varepsilon > 0. \tag{4.19}$$

By applying the fact that

$$E \max_{1\leq k\leq n} \Big\|\sum_{i=1}^{k} V_{ni}\Big\| \leq \sum_{i=1}^{n} E\|V_{ni}\| \text{ for all arbitrary array } \{V_{ni}, n \geq 1, 1 \leq i \leq n\},$$

we obtain the following theorem. No row-wise independence conditions are needed.

Theorem 4.6. *Let $\alpha > 0$, $0 < p < 1$, $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of random elements that are stochastically dominated by a random element V with $E\|V\|^p < \infty$. Let $\{A_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of random variables. Assume that for all $n \geq 1$, $1 \leq i \leq n$, the random variable A_{ni} is independent of V_{ni}. If*

$$\sum_{i=1}^{n} E(|A_{ni}|) = O(n), \tag{4.20}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leq k\leq n} \|\sum_{i=1}^{k} A_{ni} V_{ni}\| > \varepsilon n^{\alpha}\Big) < \infty \ \text{ for all } \ \varepsilon > 0. \tag{4.21}$$

Proof. Let W_{ni} be as in the proof of Theorem 4.1. We need only prove the theorem for $\alpha p \geq 1$. For all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leq k\leq n} \| \sum_{i=1}^{k} A_{ni}W_{ni}\| > n^{\alpha}\varepsilon \Big)$$

$$\leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big(\max_{1\leq k\leq n} \| \sum_{i=1}^{k} A_{ni}W_{ni}\| \Big) \text{ (by Chebyshev's inequality)}$$

$$\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} E\Big\| A_{ni}W_{ni}\Big\|$$

$$= C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} E|A_{ni}| E\|W_{ni}\| \quad \text{(since } A_{ni} \text{ is independent of } V_{ni}\text{)}$$

$$\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \Big(\sum_{i=1}^{n} E|A_{ni}| \Big) \Big(n^{\alpha} P(\|V\| > n^{\alpha}) + E\|VI(\|V\| \leq n^{\alpha})\| \Big)$$

(by (2.21))

$$\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} E\|VI(\|V\| \leq n^{\alpha})\|^2 + C \sum_{n=1}^{\infty} n^{\alpha p-1} P(\|V\| > n^{\alpha}) \tag{4.22}$$

(by (4.20))

$$\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{i=1}^{n} E\Big(\|V\| I\big((i-1)^{\alpha} < \|V\| \leq i^{\alpha}\big) \Big) + CE\|V\|^p$$

$$= C \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} n^{\alpha p-1-\alpha} E\Big(\|V\| I\big((i-1)^{\alpha} < \|V\| \leq i^{\alpha}\big) \Big) + CE\|V\|^p$$

$$\leq C \sum_{i=1}^{\infty} i^{\alpha p-\alpha} E\big(\|V\| I((i-1)^{\alpha} < \|V\| \leq i^{\alpha}) \big) + CE\|V\|^p$$

$$\leq C \sum_{i=1}^{\infty} i^{\alpha p} P\big((i-1)^{\alpha} < \|V\| \leq i^{\alpha} \big) + CE\|V\|^p$$

$$\leq CE\|V\|^p < \infty.$$

Similar to (4.7), we also have

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max\nolimits_{1\leq i\leq n} \|V_{ni}\| > n^{\alpha} \Big) \leq CE\|V\|^p < \infty. \tag{4.23}$$

Since

$$P\Big(\max_{1\leq k\leq n} \Big\| \sum_{i=1}^{k} A_{ni}V_{ni} \Big\| > \varepsilon n^{\alpha} \Big)$$

$$\leq P\Big(\max_{1\leq i\leq n} \|V_{ni}\| > n^{\alpha} \Big) + P\Big(\max_{1\leq k\leq n} \| \sum_{i=1}^{k} A_{ni}W_{ni}\| > \varepsilon n^{\alpha} \Big),$$

the conclusion (4.21) follows from (4.22) and (4.23). ∎

5. Corollaries and Examples

In this section, we present several corollaries and examples. The first corollary extends the result of Wang et al. [25] by including larger range for α and p.

Corollary 5.1. *Let $\alpha > 1/2$, $p \geq 1$, $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of row-wise independent random elements that are stochastically dominated by a random element V with $E\|V\|^p < \infty$. Let $\{a_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of constants satisfying*

$$\sup_{n\geq 1} \sup_{1\leq i\leq n} |a_{ni}| < \infty. \tag{5.1}$$

If

$$n^{-\alpha}\Big\| \sum_{i=1}^{n} a_{ni}V_{ni}\Big\| \to 0 \text{ in probability as } n \to \infty, \tag{5.2}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\| \sum_{i=1}^{n} a_{ni}V_{ni}\| > \varepsilon n^{\alpha}\Big) < \infty \quad \text{for all } \ \varepsilon > 0. \tag{5.3}$$

Proof. Set $A_{ni} \equiv a_{ni}$ a.s. Since (4.1) and (4.15) are immediately consequences of (5.1), Corollary 5.1 follows from Theorem 4.1, Remark 4.3 and Remark 4.4 ∎

Remark 5.2. Corollary 5.1 extends the result of Wang et al. [25] and provides better convergence rates.

(i) We only assume (5.2), whereas Wang et al. [25] assume (1.6).

(ii) We allow $p \geq 1$, whereas Wang et al. [25] require that $p > 1$.

Corollary 5.3. *Let $1/2 \leq t < 2$ and let $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of row-wise independent random elements that are stochastically dominated by a random element V with $E\|V\|^{2t} < \infty$. Let $\{a_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of constants satisfying*

$$\sup_{n\geq 1} \sup_{1\leq i\leq n} |a_{ni}| < \infty. \tag{5.4}$$

If

$$n^{-1/t}\Big\| \sum_{i=1}^{n} a_{ni}V_{ni}\Big\| \to 0 \text{ in probability as } n \to \infty, \tag{5.5}$$

then

$$\sum_{n=1}^{\infty} P\Big(\| \sum_{i=1}^{n} a_{ni}V_{ni}\| > \varepsilon n^{1/t}\Big) < \infty \quad \text{for all } \ \varepsilon > 0. \tag{5.6}$$

Proof. Let $p = 2t$, $\alpha = 2/p = 1/t$. Noting that $1/2 \leq t < 2$ implies $\alpha > 1/2$. Therefore Corollary 5.3 follows from Corollary 5.1. ∎

Remark 5.4. Let $a_{ni} \equiv n^{1/t} b_{ni}$. Then Corollary 5.3 implies the "$p < 2$ part" of Sung [20]. If the Banach space is real valued and $EV_{ni} = 0$, $n \geq 1$, $1 \leq i \leq n$, then by Remark 4.2 (ii), Corollary 5.3 also extends result of Hu et al. [10].

Next we consider two illustrative examples. Recall that ℓ_p (where $p \geq 1$) is the real separable Banach space of absolute p^{th}-power summable real sequences $v = \{v_i, i \geq 1\}$ with norm $||v|| = (\sum_{i=1}^{\infty} |v_i|^p)^{1/p}$. The element of ℓ_p having 1 in its n^{th} position and 0 elsewhere will be denoted by $v^{(n)}$, $n \geq 1$. Define a sequence $\{W_n, n \geq 1\}$ of independent 0-mean random elements in ℓ_p by requiring the $\{V_n, n \geq 1\}$ to be independent with

$$P\{W_n = v^{(n)}\} = P\{W_n = -v^{(n)}\} = \frac{1}{2}, n \geq 1.$$

The random elements $\{W_n, n \geq 1\}$ will be used in the first example. This example shows that the condition (4.2) cannot be removed in Theorem 4.1.

Example 5.5. Let $1 \leq p < 2$, $\alpha = 1/p$ and consider the Banach space ℓ_p and the sequence $\{W_n, n \geq 1\}$ of independent 0-mean random elements in ℓ_p. Let $V_{ni} = W_i$ for all $n \geq 1$. Then $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ is stochastically dominated by W_1 that satisfies $E\|W_1\|^p < \infty$. Let $\{A_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of row-wise independent random variables such that the sequences $\{A_{ni}, i \geq 1\}$ and $\{V_{ni}, i \geq 1\}$ are independent for all $n \geq 1$, and for all $n \geq 1$ and $1 \leq i \leq n$,

$$P\{A_{ni} = -1\} = P\{A_{ni} = 1\} = 1/2. \tag{5.7}$$

Then $\|V_{ni}\| = 1$ a.s. for all $n \geq 1, 1 \leq i \leq n$ and the condition (4.1) is satisfied. Since

$$\Big\| \sum_{i=1}^{n} A_{ni} V_{ni} \Big\| = n^{1/p} \text{ a.s.}, \tag{5.8}$$

and so the (4.2) fails. It is easy to see that (4.3) also fails.

By applying Hölder's inequality, if

$$\sum_{i=1}^{n} E|A_{ni}|^q = O(n) \text{ for some } q > 0, \tag{5.9}$$

then

$$\sum_{i=1}^{n} E|A_{ni}|^r = O(n) \text{ for all } 0 < r < q. \tag{5.10}$$

The following example shows that in Theorem 4.1, we cannot replace (4.1) by the weaker condition

$$\sum_{i=1}^{n} E|A_{ni}|^p = O(n). \tag{5.11}$$

Example 5.6. Let $1 \leq p < 2$, $\alpha = 1/p$ and consider an array $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ of independent mean 0 real-valued random variables such that for all $n \geq 1$ and all $1 \leq i \leq n$,

$$\begin{aligned} &P\{V_{ni} = 0\} = 1 - 1/\log(n+1) \text{ and } P\{V_{ni} = -1\} \\ &= P\{V_{ni} = 1\} = 1/(2\log(n+1)). \end{aligned} \tag{5.12}$$

Then the array $\{V_{ni}, n \geq 1, 1 \leq i \leq n\}$ is stochastically dominated by V_{11}. Let $\{A_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of random variables such that for all $n \geq 1$

$$P\{A_{ni} = 0\} = 1 \text{ for all } 1 \leq i < n \text{ and } P\{A_{nn} = n^{1/p}\} = 1. \tag{5.13}$$

Then $\sum_{i=1}^{n} E|A_{ni}|^p = O(n)$ but (4.1) fails. We also see that

$$n^{-\alpha} E\Big|\sum_{i=1}^{n} A_{ni}V_{ni}I(|V_{ni}| \leq n^{\alpha})\Big| = E|V_{nn}| = 1/\log(n+1) \tag{5.14}$$

and so (4.2) holds. However, since

$$\begin{aligned} &P\Big(\Big|\sum_{i=1}^{n} A_{ni}V_{ni}\Big| > \varepsilon n^{\alpha}\Big) \\ &= P\big(|V_{nn}| > \varepsilon\big) = 1/\log(n+1) \text{ for all } 0 < \varepsilon < 1, \end{aligned} \tag{5.15}$$

conclusion (4.3) fails.

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