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SOME INEQUALITIES FOR DIFFERENTIABLE MAPPINGS AND APPLICATIONS TO FEJÉR INEQUALITY AND WEIGHTED TRAPEZOIDAL FORMULA

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Abstract. In this paper, we establish some inequalities for differentiable convex mappings whose derivatives in absolute value are convex. This result is connected with Fejér's inequality holding for convex mappings. Some applications for the weighted trapezoidal formula, $r-moment$, and the expectation of a symmetric and continuous random variable are given.

1. Introduction

Throughout this paper, let I be an interval on R and let $\|g\|_{[c,d],\infty} = \sup_{t\in[c,d]} |g(t)|$.

Let $f : I \to R$ be a convex mapping and let $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b f(s)\,ds \leq \frac{f(a)+f(b)}{2}(b-a) \tag{1.1}$$

is well known in the literature as Hermite-Hadamard inequality for convex mappings [8].

It follows easily from the midpoint and trapezoidal approximation to the middle term of (1.1).

For some results which generalize, improve, and extend Hermite-Hadamard inequality (1.1) and trapezoidal inequality, see [1]-[7] and[9]-[15].

In [4], Dragomir and Agarwal established the following results connected with the right-hand side of Hermite-Hadamard inequality (1.1).

Theorem 1. *Let $f : I^o \to R$ be differentiable on I^o and let $f' \in L[a,b]$ for $a, b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(s)\,ds\right| \leq \frac{(b-a)^2(|f'(a)|+|f'(b)|)}{8}. \tag{1.2}$$

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Theorem 2. *Let $f : I^o \to R$ be differentiable on I^o and let $f' \in L[a,b]$ for $a, b \in I^o$ with $a < b$. If $p > 1$ and $|f'|^{p/(p-1)}$ is convex on $[a,b]$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(s)\,ds \right| \leq \frac{(b-a)^2}{2(p+1)^{1/p}} \left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right]^{(p-1)/p}. \tag{1.3}$$

In [7], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 3. *Let $f : I \to R$ be convex on I and let $a, b \in I$ with $a < b$. Then the inequality*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)\,dx \tag{1.4}$$

holds, where $g : [a,b] \to R$ is nonnegative, integrable, and symmetric to $(a+b)/2$.

In this paper, we establish some weighted generalizations of Theorems 1-2 and give some applications connected with the right-hand side of Fejér inequality (1.4). We also give several applications for the weighted trapezoidal formula, $r-moment$, and the expectation of a symmetric, continuous random variable.

2. Main Results

In order to prove our results, we need the following lemma.

Lemma 4. *Let $f : I^o \to R$ be differentiable on $I^o$and $a, b \in I^o$ with $a < b$ and let $g : [a,b] \to R$. If $f', g \in L[a,b]$, then, for all $x \in [a,b]$, the following identity holds:*

$$f(a) \int_a^x g(s)\,ds + f(b) \int_x^b g(s)\,ds - \int_a^b f(s)\,g(s)\,ds = \int_a^b \left[\int_x^t g(s)\,ds \right] f'(t)\,dt. \tag{2.1}$$

Proof. By integration by parts, we have the following identity:

$$\begin{aligned} &\int_a^b \left[\int_x^t g(s)\,ds \right] f'(t)\,dt \\ &= \left[\int_x^t g(s)\,ds \right] f(t) \Big|_a^b - \int_a^b f(t)\,g(t)\,dt \\ &= f(a) \int_a^x g(s)\,ds + f(b) \int_x^b g(s)\,ds - \int_a^b f(s)\,g(s)\,ds. \end{aligned}$$

This completes the proof.

Remark 5. In Lemma 4, let g be symmetric to $(a+b)/2$ and let $x=(a+b)/2$. Then (2.1) can be written as

$$\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(s)\,g(s)\,ds=\int_a^b\left[\int_{\frac{a+b}{2}}^t g(s)\,ds\right]f'(t)\,dt. \tag{2.2}$$

Now, we are ready to state and prove our results.

Theorem 6. *Let I^o, a, b, f be defined as in Theorem 1 and let $g:[a,b]\to R$ be continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have*

$$\begin{aligned}
&\left|f(a)\int_a^x g(s)\,ds+f(b)\int_x^b g(s)\,ds-\int_a^b f(s)\,g(s)\,ds\right|\\
&\le\frac{(x-a)^2(3b-2a-x)\|g\|_{[a,x],\infty}+(b-x)^3\|g\|_{[x,b],\infty}}{6(b-a)}\left|f'(a)\right|\\
&\quad+\frac{(x-a)^3\|g\|_{[a,x],\infty}+(b-x)^2(2b-3a+x)\|g\|_{[x,b],\infty}}{6(b-a)}\left|f'(b)\right|\\
&\le\left(\frac{\left[(x-a)^2(3b-2a-x)+(b-x)^3\right]|f'(a)|}{6(b-a)}\right.\\
&\quad+\left.\frac{\left[(x-a)^3+(b-x)^2(2b-3a+x)\right]|f'(b)|}{6(b-a)}\right)\|g\|_{[a,b],\infty}.
\end{aligned} \tag{2.3}$$

Proof. Let $x\in[a,b]$. Using Lemma 4 and the convexity of $|f'|$, we have

$$\begin{aligned}
&\left|f(a)\int_a^x g(s)\,ds+f(b)\int_x^b g(s)\,ds-\int_a^b f(s)\,g(s)\,ds\right|\\
&\le\int_a^x\left|\int_x^t g(s)\,ds\right|\left|f'(t)\right|dt+\int_x^b\left|\int_x^t g(s)\,ds\right|\left|f'(t)\right|dt\\
&\le\|g\|_{[a,x],\infty}\int_a^x(x-t)\left|f'(t)\right|dt+\|g\|_{[x,b],\infty}\int_x^b(t-x)\left|f'(t)\right|dt\\
&=\|g\|_{[a,x],\infty}\int_a^x(x-t)\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|dt\\
&\quad+\|g\|_{[x,b],\infty}\int_x^b(t-x)\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|dt
\end{aligned} \tag{2.4}$$

$$\leq \frac{\|g\|_{[a,x],\infty}|f'(a)|}{b-a}\int_a^x (x-t)(b-t)\,dt$$
$$+\frac{\|g\|_{[a,x],\infty}|f'(b)|}{b-a}\int_a^x (x-t)(t-a)\,dt$$
$$+\frac{\|g\|_{[x,b],\infty}|f'(a)|}{b-a}\int_x^b (t-x)(b-t)\,dt$$
$$+\frac{\|g\|_{[x,b],\infty}|f'(b)|}{b-a}\int_x^b (t-x)(t-a)\,dt.$$

Since

$$\int_a^x (x-t)(b-t)\,dt = \frac{(x-a)^2(3b-2a-x)}{6},$$
$$\int_x^b (t-x)(t-a)\,dt = \frac{(b-x)^2(2b-3a+x)}{6},$$
$$\int_x^b (t-x)(b-t)\,dt = \frac{(b-x)^3}{6}$$

and

$$\int_a^x (x-t)(t-a)\,dt = \frac{(x-a)^3}{6},$$

using (2.4), we obtain (2.3). This completes the proof.

Using Theorem 6, we have the following corollaries which are connected with the right-hand side of Fejér inequality (1.4).

Corollary 7. *Let* $0 \leq \alpha \leq 1$ *and* $x = \alpha a + (1-\alpha)b$ *in Theorem 6. Then we have*

$$\left| f(a)\int_a^{\alpha a+(1-\alpha)b} g(s)\,ds + f(b)\int_{\alpha a+(1-\alpha)b}^b g(s)\,ds - \int_a^b f(s)g(s)\,ds \right|$$
$$\leq \frac{(b-a)^2}{6}\Big(\Big[(1-\alpha)^2(2+\alpha)\|g\|_{[a,\alpha a+(1-\alpha)b],\infty} + \alpha^3\|g\|_{[\alpha a+(1-\alpha)b,b],\infty}\Big]|f'(a)|$$
$$+\Big[(1-\alpha)^3\|g\|_{[a,\alpha a+(1-\alpha)b],\infty} + \alpha^2(3-\alpha)\|g\|_{[\alpha a+(1-\alpha)b,b],\infty}\Big]|f'(b)|\Big)$$
$$\leq \frac{(b-a)^2\|g\|_{[a,b],\infty}}{6}\left[\left(2-3\alpha+2\alpha^3\right)|f'(a)| + \left(1-3\alpha+6\alpha^2-2\alpha^3\right)|f'(b)|\right].$$

Corollary 8. *Let* $g : [a,b] \to R$ *be symmetric to* $(a+b)/2$ *and* $\alpha = 1/2$ *in Corollary 7. Then we have the inequality*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(s)\,g(s)\,ds\right| \\
&\le \frac{(b-a)^2}{48}\left[\left(5\,\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\right)\left|f'(a)\right|\right. \\
&\left.+\left(\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}+5\,\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\right)\left|f'(b)\right|\right] \\
&\le \frac{(b-a)^2\,\|g\|_{[a,b],\infty}\left(|f'(a)|+|f'(b)|\right)}{8}
\end{aligned}
\tag{2.5}
$$

*which is the "**weighted trapezoid**" inequality provided that $|f'|$ is convex on $[a,b]$.*

Proof. Using the symmetry of g, we have the following identity

$$
\begin{aligned}
&f(a)\int_a^{\frac{a+b}{2}} g(s)\,ds+f(b)\int_{\frac{a+b}{2}}^b g(s)\,ds-\int_a^b f(s)\,g(s)\,ds \\
=\;&\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(s)\,g(s)\,ds.
\end{aligned}
$$

From this identity and Corollary 7, we have the inequality (2.5). This completes the proof.

Remark 9. If we choose $g(t)\equiv 1$ on $[a,b]$, then the inequality (2.5) reduces to (1.2).

Remark 10. Let $f:[a,b]\to R$ be a convex and differentiable mapping on $[a,b]$ and let $g:[a,b]\to[0,\infty)$ is continuous and symmetric to $(a+b)/2$. Then (2.5) is an error bound of the second inequality in Fejér inequality (1.4)

$$
\begin{aligned}
0 &\le \frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(s)\,g(s)\,ds \\
&\le \frac{(b-a)^2}{48}\left[\left(5\,\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\right)\left|f'(a)\right|\right. \\
&\left.+\left(\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}+5\,\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\right)\left|f'(b)\right|\right] \\
&\le \frac{(b-a)^2\,\|g\|_{[a,b],\infty}\left(|f'(a)|+|f'(b)|\right)}{8}
\end{aligned}
$$

provided that $|f'|$ is convex on $[a,b]$ and $f'\in L(a,b)$.

Theorem 11. *Let I^o, a, b, f, p be defined as in Theorem 2 and let g be defined as in Theorem 6. Then, for all $x\in[a,b]$, the following inequality holds:*

$$
\begin{aligned}
&\left| f(a)\int_a^x g(s)\,ds + f(b)\int_x^b g(s)\,ds - \int_a^b f(s)\,g(s)\,ds \right| \\
&\leq \frac{\|g\|_{[a,b],\infty}}{(p+1)^{\frac{1}{p}}}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}} \\
&\quad\times\left[\left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right)(b-a)\right]^{\frac{p-1}{p}}.
\end{aligned}
\tag{2.6}
$$

Proof. Let $x \in [a,b]$. Using Lemma 4, Hölder's inequality, and the convexity of $|f'|^{\frac{p}{p-1}}$, it follows that

$$
\begin{aligned}
&\left| f(a)\int_a^x g(s)\,ds + f(b)\int_x^b g(s)\,ds - \int_a^b f(s)\,g(s)\,ds \right| \\
&\leq \left[\int_a^b \left|\int_x^t g(s)\,ds\right|^p dt\right]^{\frac{1}{p}} \left[\int_a^b \left|f'(t)\right|^{\frac{p}{p-1}} dt\right]^{\frac{p-1}{p}} \\
&\leq \|g\|_{[a,b],\infty}\left[\int_a^b |t-x|^p\,dt\right]^{\frac{1}{p}} \left[\int_a^b \left(\frac{b-t}{b-a}\left|f'(a)\right|^{\frac{p}{p-1}}+\frac{t-a}{b-a}\left|f'(b)\right|^{\frac{p}{p-1}}\right)dt\right]^{\frac{p-1}{p}} \\
&= \frac{\|g\|_{[a,b],\infty}}{(p+1)^{\frac{1}{p}}}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right)(b-a)\right]^{\frac{p-1}{p}}
\end{aligned}
$$

which is the inequality (2.6).

Using Theorem 11, we have the following corollary which is connected with the right-hand side of Fejér inequality (1.4).

Corollary 12. *Let* $0 \leq \alpha \leq 1$ *and* $x = \alpha a + (1-\alpha)b$ *in Theorem 11. Then we have the inequality*

$$
\begin{aligned}
&\left| f(a)\int_a^{\alpha a+(1-\alpha)b} g(s)\,ds + f(b)\int_{\alpha a+(1-\alpha)b}^b g(s)\,ds - \int_a^b f(s)\,g(s)\,ds \right| \\
&\leq \frac{(b-a)^2\,\|g\|_{[a,b],\infty}}{(p+1)^{\frac{1}{p}}}\left[(1-\alpha)^{p+1}+\alpha^{p+1}\right]^{\frac{1}{p}} \\
&\quad\times\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}.
\end{aligned}
\tag{2.7}
$$

Corollary 13. *Let* $g : [a,b] \to R$ *be symmetric to* $(a+b)/2$ *and* $\alpha = 1/2$ *in*

Corollary 12. Then we have the inequality

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(s)\,g(s)\,ds\right| \le \frac{(b-a)^2\|g\|_{[a,b],\infty}}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}} \tag{2.8}$$

*which is the "**weighted trapezoid**" inequality provided that* $|f'|^{p/(p-1)}$ *is convex on* $[a,b]$ *and* $f'\in L(a,b)$ *where* $p>1$.

Remark 14. If we choose $g(t)\equiv 1$ on $[a,b]$, then the inequality (2.8) reduces to (1.3).

Remark 15. Let $p>1$, $f:[a,b]\to R$ be a convex and differentiable mapping on $[a,b]$ and let $g:[a,b]\to[0,\infty)$ is continuous and symmetric to $(a+b)/2$. Using Theorem 3 and Corollary 13, we obtains an error bound of the second inequality in (1.4)

$$\begin{aligned} 0 &\le \frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(s)\,g(s)\,ds \\ &\le \frac{(b-a)^2\|g\|_{[a,b],\infty}}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}} \end{aligned}$$

provided that $|f'|^{p/(p-1)}$ *is convex on* $[a,b]$ and $f'\in L(a,b)$.

3. Applications to Weighted Trapezoided Formula

Throughout this section, let $g:[a,b]\to R$ be continuous, $f:[a,b]\to R$ be integrable and let $I_n: a=x_0<x_1<\cdots<x_n=b$ be a partition of $[a,b]$ and $l_i=x_{i+1}-x_i$ $(i=0,1,...,n-1)$. Define

$$\begin{aligned} T(f,I_n) &= \sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}l_i, \\ E(f,I_n) &= \int_a^b f(x)\,dx-T(f,I_n)\,, T(f,g,I_n) \\ &= \sum_{i=0}^{n-1}\left[f(x_i)\int_{x_i}^{\frac{x_i+x_{i+1}}{2}} g(s)\,ds+f(x_{i+1})\int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} g(s)\,ds\right] \end{aligned}$$

and

$$E(f,g,I_n)=\int_a^b f(x)\,g(x)\,dx-T(f,g,I_n)\,.$$

In [4], Dragomir and Agarwal also established the following two propositions:

Proposition 16. *Let f be defined as in Theorem 1. Then we have*

$$\begin{aligned} |E(f, I_n)| &\le \frac{1}{8}\sum_{i=0}^{n-1}\left(\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right) l_i^2 \\ &\le \frac{\max\{|f'(a)|, |f'(b)|\}}{4}\sum_{i=0}^{n-1} l_i^2. \end{aligned} \tag{3.1}$$

Proposition 17. *Let $p > 1$ and f be defined as in Theorem 2. Then we have*

$$\begin{aligned} |E(f, I_n)| &\le \frac{1}{2(p+1)^{\frac{1}{p}}}\sum_{i=0}^{n-1}\left(\frac{|f'(x_i)|^{\frac{p}{p-1}} + |f'(x_{i+1})|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}} \cdot l_i^2 \\ &\le \frac{\max\{|f'(a)|, |f'(b)|\}}{2(p+1)^{\frac{1}{p}}}\sum_{i=0}^{n-1} l_i^2. \end{aligned} \tag{3.2}$$

Using the generalized triangle inequality and Corollary 8, Corollary 13 on $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$, we have the following proposition which reduces Propositions 16-17 as $g(t) \equiv 1$ on $[a, b]$.

Proposition 18. *Let f be defined as in Corollary 8. Then we have*

$$\begin{aligned} |E(f, g, I_n)| &\le \frac{1}{8}\sum_{i=0}^{n-1}\left(\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right) l_i^2 \|g\|_{[x_i, x_{i+1}],\infty} \\ &\le \frac{\max\{|f'(a)|, |f'(b)|\}}{4}\sum_{i=0}^{n-1} l_i^2 \|g\|_{[x_i, x_{i+1}],\infty}. \end{aligned} \tag{3.3}$$

Proof. Apply Corollary 8 on $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ to get

$$\begin{aligned} &\left|f(x_i)\int_{x_i}^{\frac{x_i+x_{i+1}}{2}} g(s)\,ds + f(x_{i+1})\int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} g(s)\,ds - \int_{x_i}^{x_{i+1}} f(s)g(s)\,ds\right| \\ &\le \frac{l_i^2 \|g\|_{[x_i, x_{i+1}],\infty}}{8}\left(\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right). \end{aligned} \tag{3.4}$$

Using (3.4), the generalized triangle inequality and the convexity of $|f'|$, we have the inequality (3.3). This completes the proof.

Remark 19. If we choose $g(t) \equiv 1$ on $[a, b]$, then the inequality (3.3) reduces to (3.1).

Proposition 20. *Let $p > 1$ and f be defined as in Theorem 11. Then we have*

$$
\begin{aligned}
& |E(f, g, I_n)| \\
\text{(3.5)} \qquad & \leq \frac{1}{2(p+1)^{\frac{1}{p}}} \sum_{i=0}^{n-1} \left(\frac{|f'(x_i)|^{\frac{p}{p-1}} + |f'(x_{i+1})|^{\frac{p}{p-1}}}{2} \right)^{\frac{p-1}{p}} l_i^2 \|g\|_{[x_i, x_{i+1}], \infty} \\
& \leq \frac{\max\{|f'(x_i)|, |f'(x_{i+1})|\}}{2(p+1)^{\frac{1}{p}}} \sum_{i=0}^{n-1} l_i^2 \|g\|_{[x_i, x_{i+1}], \infty}.
\end{aligned}
$$

Proof. The proof uses Corollary 13 and similar to that of Proposition 18.

Remark 21. If we choose $g(t) \equiv 1$ on $[a, b]$, then the inequality (3.5) reduces to the inequality (3.2).

4. Some Inequalities for Random Variables

Throughout this section, let $0 < a < b$, $r \geq 1$, and let X be a continuous random variable, $g : [a, b] \to R$ be the continuous probability density function of X which is symmetric to $(a + b)/2$ and the $r - moment$

$$E_r(X) := \int_a^b s^r g(s)\, ds,$$

which is assumed to be finite.

Theorem 22. *The inequality*

$$\text{(4.1)} \qquad \left| E_r(X) - \frac{a^r + b^r}{2} \right| \leq \frac{r(b-a)^2 \|g\|_{[a,b],\infty} \left(a^{r-1} + b^{r-1}\right)}{8}$$

holds.

Proof. Let $f(s) = s^r$ $(s \in [a, b])$ in Corollary 8. Then we have the following identities

$$\int_a^b f(s) g(s)\, ds = E_r(X), \frac{f(a) + f(b)}{2} \int_a^b g(s)\, ds = \frac{a^r + b^r}{2},$$

and

$$\left|f'(a)\right| + \left|f'(b)\right| = r\left(a^{r-1} + b^{r-1}\right).$$

Using the above identities and the inequality (2.5), we have the inequality (4.1). This completes the proof.

If we choose $r = 1$ in Theorem 22, then we have the following remark:

Remark 23. The inequality

$$\left| E_r(X) - \frac{a+b}{2} \right| \leq \frac{(b-a)^2 \|g\|_{[a,b],\infty}}{8} \tag{4.2}$$

where $E(X)$ is the expectation of the random variable X.

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